

— ESENOTES.COM —

STRENGTH OF MATERIAL



BY JASPAL SIR

Strength of Material

By Jaspal Sir

- Properties of Material- 01
- Shear Force and Bending Moment- 87
- Transformation of Stress & Strain- 154
- Bending Stress- 227
- Shear Stress in Beam- 279
- Torsion of Circular Shaft- 315
- Deflection of Beam- 360
- Thick and Thin Vessels- 450
- Springs- 464
- Column- 476
- Moment of Inertia- 487





STRENGTH OF MATERIAL

★ Strength of material / Solid mechanics / Mechanics of solid / Mechanics of deformable bodies.

Properties of Material

- It is the branch of science which deals with study of the behaviour of material when it is subjected to external loading. (Here load can be in form of force/moment).
- Each member of structures [eg:- beam, column, foundation, dams, pavement] are made up of different material, which could be **rigid material** or **deformable material**.

★ Rigid material

- It is the type of material which doesn't undergo any change in its geometry (shape/size) upon application of load.

(or)

- A body/material is termed as rigid body/material if the distance between any two points

★ Deformable material

- It is the type of material which undergoes change in geometry (shape/size) upon application.

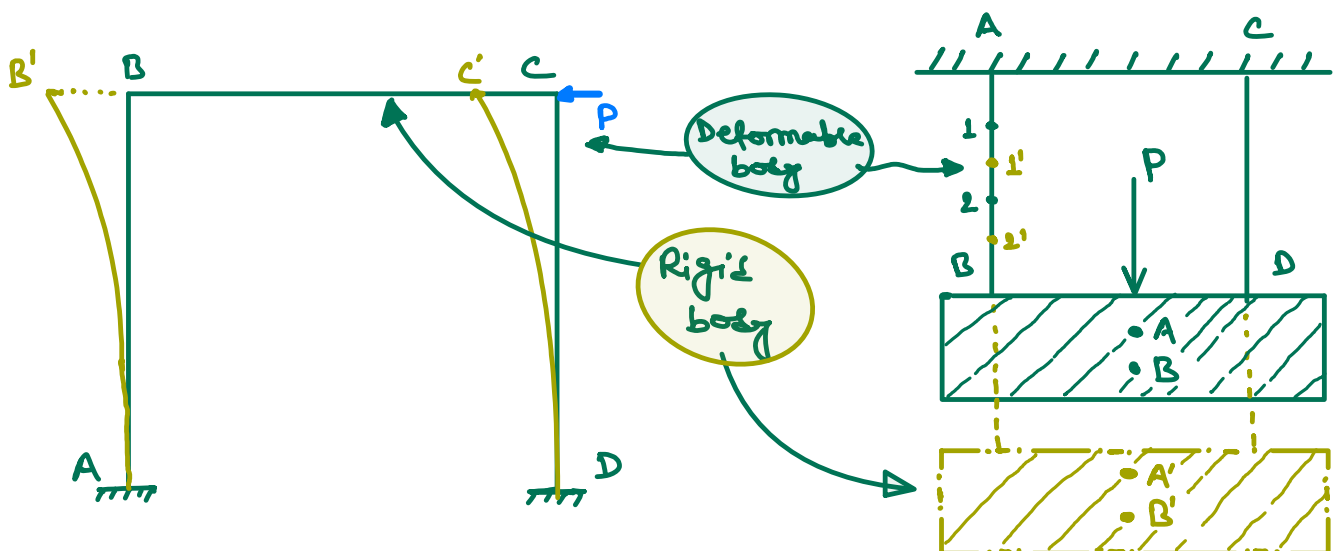
- This geometrical change in the body is termed as deformation/Strain.

(or)

- A body is termed as deformable when distance between any two point changes upon application of load.

Note:-

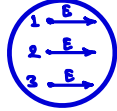
The concept of ideal rigid body is Hypothetical and this hypothesis is made to simplify the calculation.



★ Assumptions in strength of material

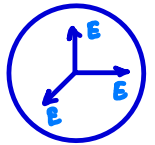
i.) Material is considered to be homogeneous

- material is termed to be homogeneous if it exhibit same elastic properties at any point in given direction.
- Hence properties of homogeneous material is independent of point.
- Eg:- Steel.



ii.) Material is considered to be isotropic.

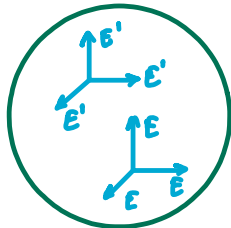
- material is termed to be isotropic if it same elastic properties in all directions at a given point
- Hence, properties of isotropic material is independent of direction.
- Eg:- Glass



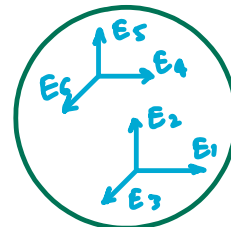
Note:-

- A homogeneous material need not be isotropic & an isotropic material need not be homogeneous.
- on ^{Atomic level.} macro level almost all materials are homogeneous & isotropic but on micro level all materials are Non-Homogeneous & Non-isotropic (here material is used for metals)

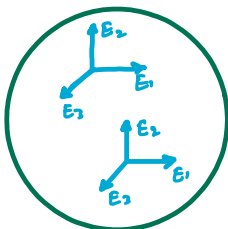
→ practically all combinations of these two properties are possible as follows:-



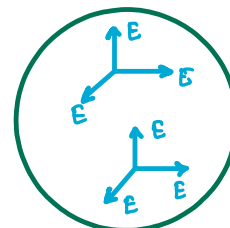
Non-Homogeneous
+
isotropic



Non-Homogeneous
+
Non-isotropic



Homogeneous + Non-isotropic



Homogeneous + isotropic

In actual every material is like this.

this is assumed for simple calculation

Load

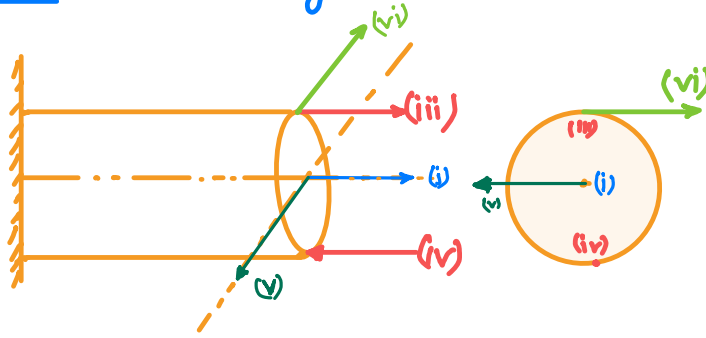
- It is an External force or moment experienced by the body.

Classification

a) Based on direction of loading

i) Axial/longitudinal load:- Load being applied parallel to the axis of member
 - May be tensile/compressive ^{(i, iii) vs (ii, iv)}
 - May be Eccentric/concentric ^{(i) (ii) vs (iii) (iv)} eg:- [(i), (ii), (iii), (iv)]

ii) Transverse load : Load being applied perpendicular to axis of member [(v), (vi)]

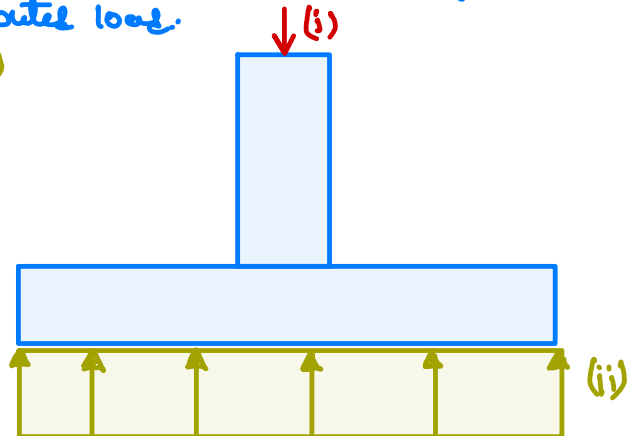


b) Based on Extent of loading

i) point load :- If the load acts comparatively on smaller area it is termed as point load. eg (i)

(ii) Distributed load :- If load is distributed over a larger area it is termed as Distributed load.

eg:- (ii)



c) Based on dimension

i) Body/volume load : It is the load which acts over the volume or Body.

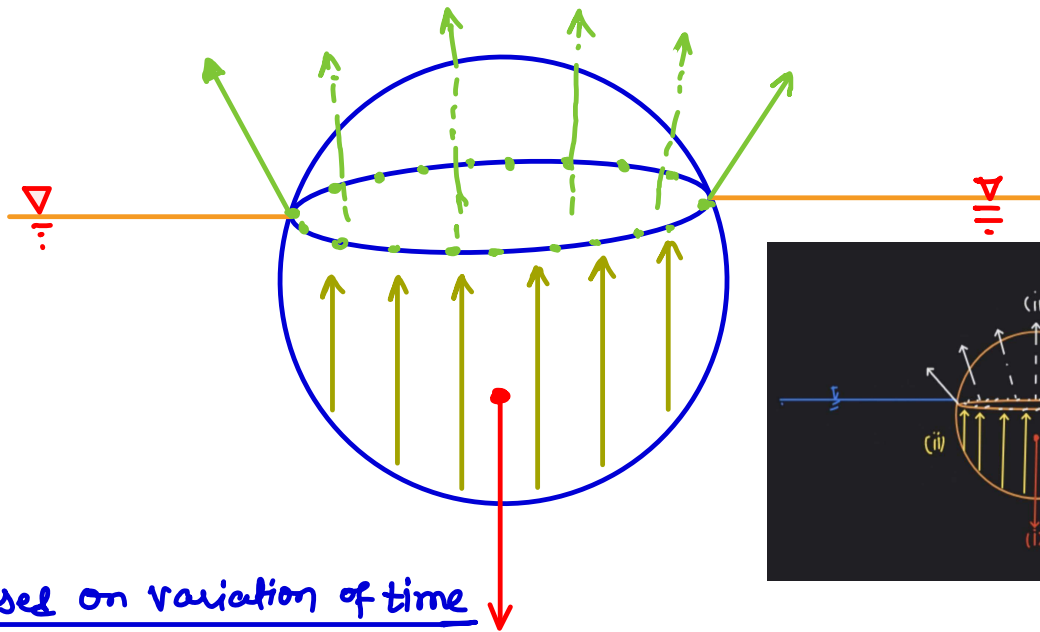
eg:- Self weight, Buoyant weight, Centrifugal force.

ii.) Surface load : load which acts over the Surface.

Eg:- Drag force, Shear force.

iii.) Line load : Load which acts over the length (one dimension).

eg:- Rail Road, Surface Tension



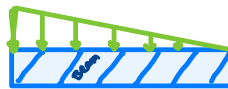
e.) Based on variation of time

i.) Static load : A load is said to be static if its magnitude, direction & point of application do not change wrt time.

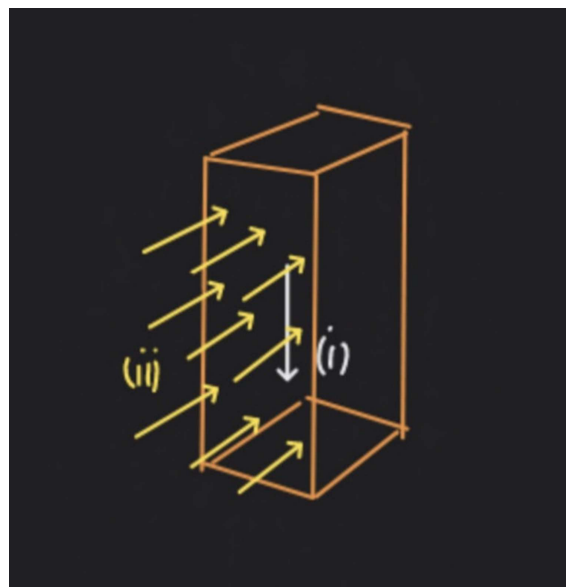
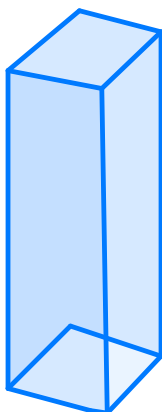
Eg:- Self weight (i)

ii.) Dynamic load : A load is said to be dynamic any of its magnitude, direction & point of application varies with respect to (ii) time.

Eg:-



Wind load, Seismic load, Wave load.



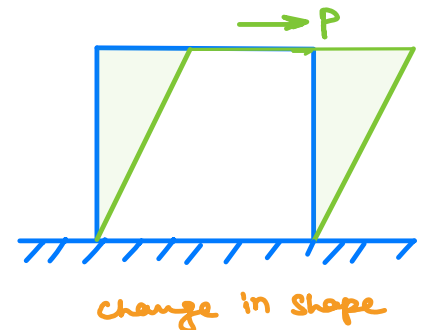
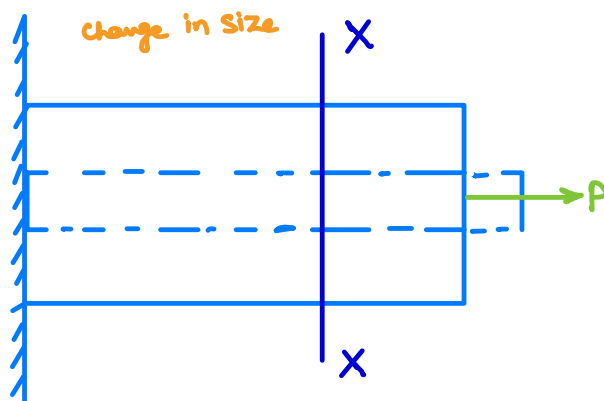
f.) Based on load Application time

- i.) Gradually applied load / Quasi load : It is type of load which reaches its maximum value in infinite time.
- ii.) Suddenly Applied load : Type of load which reaches its maximum value instantly.
- iii.) Impact Loading : In this type of loading, the time gap of application of load is more and relative velocity exist between loading and loading member.
- iv.) Shock loading : It is the type of load for which rate of application is very 2 time of application is very less.

★ Stress And Strain

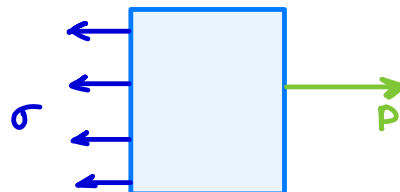
Stress : The internal resistance offered by material at a point against deformation caused by external load is termed as stress.

- It is always developed when body is restricted/constrained against deformation.



$$\sigma_{\text{longitudinal}} \neq 0$$

$$\sigma_{\text{lateral}} = 0$$



Note:-

- Hence, "Strain is the Cause of Stress"
- "The maximum value of stress which a body can resist without failure is termed as Strength."
- Strength is the inherent property of material & does not depend upon its shape size, area or load being applied on it.
- whereas Stress is not the property of material & is dependent upon load & area on which it is applied.

Note:-

- Stress may appear to be similar to pressure but is totally different from it as follows:-

- Stress is internal force per unit area whereas pressure is external force per unit area.
- Stress may not be normal to area, but pressure is always normal to area.
- Stress does not lead to pressure, but pressure lead to stress.
- Stress cannot be measured
- Pressure is a scalar quantity but stress is tensor quantity of 2nd degree or order.

Stress can be classified as follows:-

- Normal Stress.
- Shear Stress

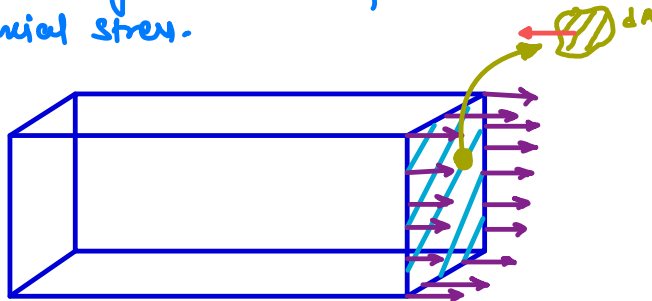
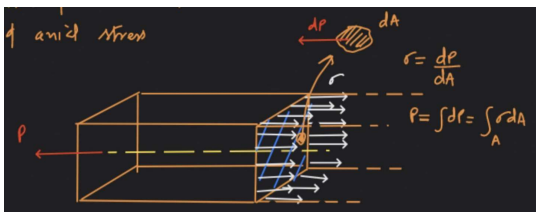
1) Normal Stress:-

- It can be tensile or compressive.

★ These are further classified as:-

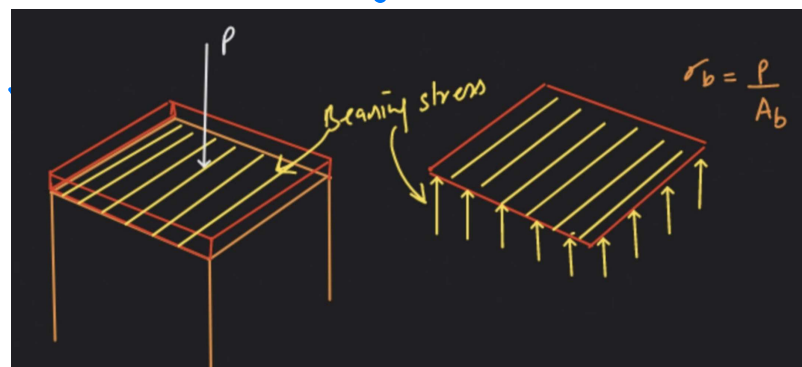
a) Axial Stress:-

- If the load is directed along the axis of member it would lead to the development of axial stress.

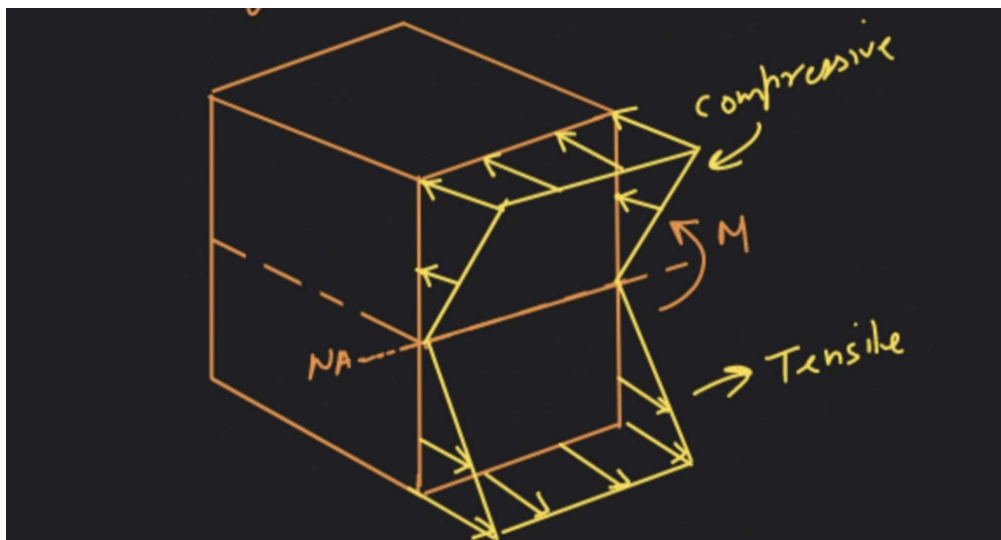


b) Bearing Stress:-

- It is a stress which develops when body is supported by another body.
- It is compressive in nature.

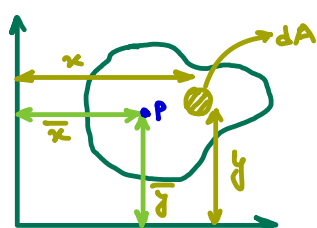


- c.) Bending Stress:- Stresses developed due to bending of the members.
 - It may be tensile or compressive in Nature.



Note:-

Line of action of axial force for uniform stress passes through C.G. of Section.



$$\text{Moment about } x\text{-Axis due to uniform stress} = \int \sigma dA y \quad \text{--- (i)}$$

$$\text{Moment about } y\text{-Axis due to uniform stress} = \int \sigma dA x \quad \text{--- (ii)}$$

$$\text{moment about } x\text{-axis} = P \bar{y} \quad \text{--- (iii)}$$

$$\text{Moment about } y\text{-axis} = P \bar{x} \quad \text{--- (iv)}$$

from (i) & (iii)

$$\int \sigma dA y = P \bar{y}$$

$$\int \frac{P}{A} dA y = P \bar{y}$$

$$\boxed{\bar{y} = \int y \frac{dA}{A}} \quad \text{--- (A)}$$

from (ii) & (iv)

$$\int \sigma dA x = P \bar{x}$$

$$\int \frac{P}{A} dA x = P \bar{x}$$

$$\boxed{\bar{x} = \int x \frac{dA}{A}} \quad \text{--- (B)}$$

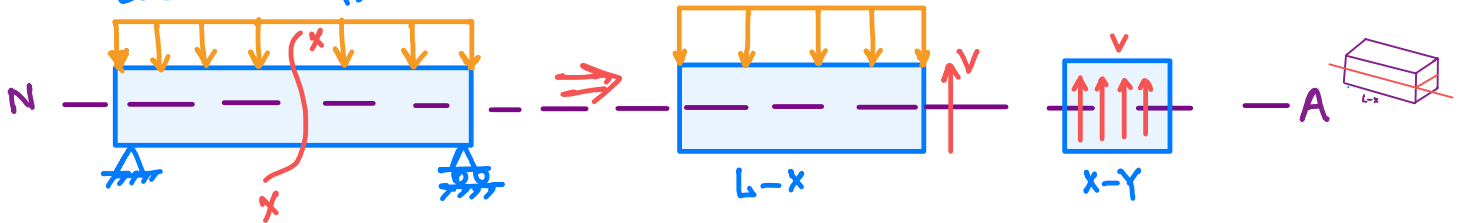
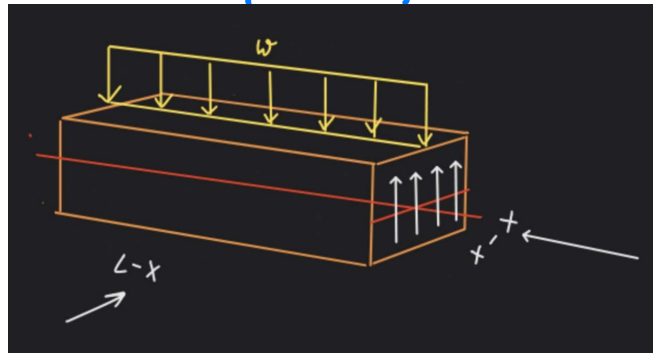
11) Shearing Stress

- It is the type of stress which acts in the plane of section.

- It is further classified as:-

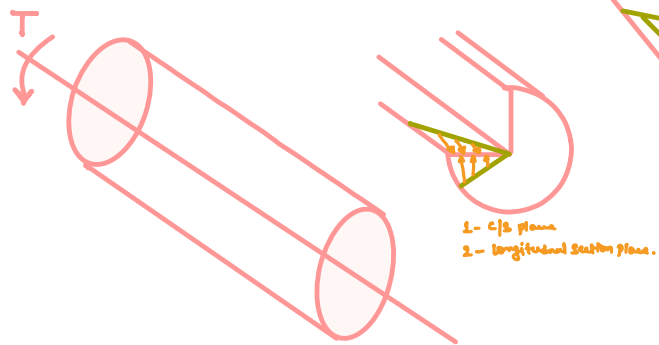
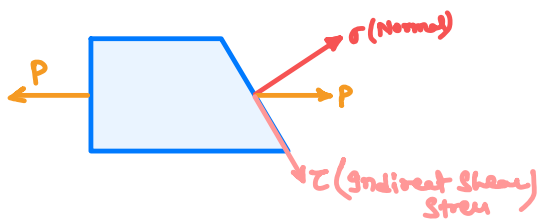
i) Direct Shear Stress

- The Shear Stress which is developed due to direct action of force trying to cut through the material.



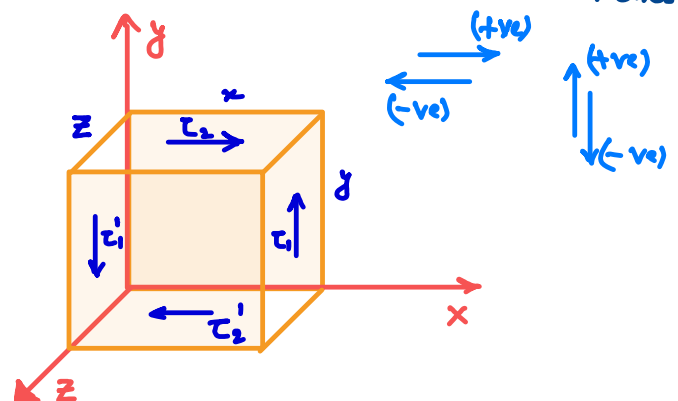
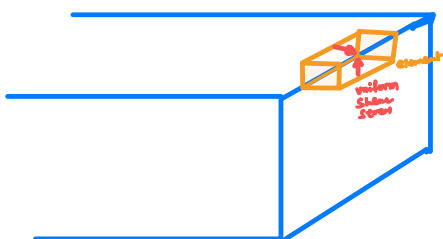
ii) Indirect Shear Stress

- It is developed due to either tension/compression/Torsion.



Note:-

- i) Shear Stress acting on opposite faces of the element are equal in magnitude but opposite in direction
- ii) Shear Stress acting on adjacent and perpendicular faces of the element are equal in magnitude. Such that both the stresses point towards each other or both point away from line of intersection of the faces. (These stresses are termed as Conjugate Stresses)



from force equilibrium:-

$$i) \sum F_y = 0$$

$$\tau_1 (yz) - \tau_1' (zy) = 0$$

$$\tau_1 = \tau_1'$$

$$ii) \sum F_x = 0$$

$$\tau_2 (xz) - \tau_2' (x'z) = 0$$

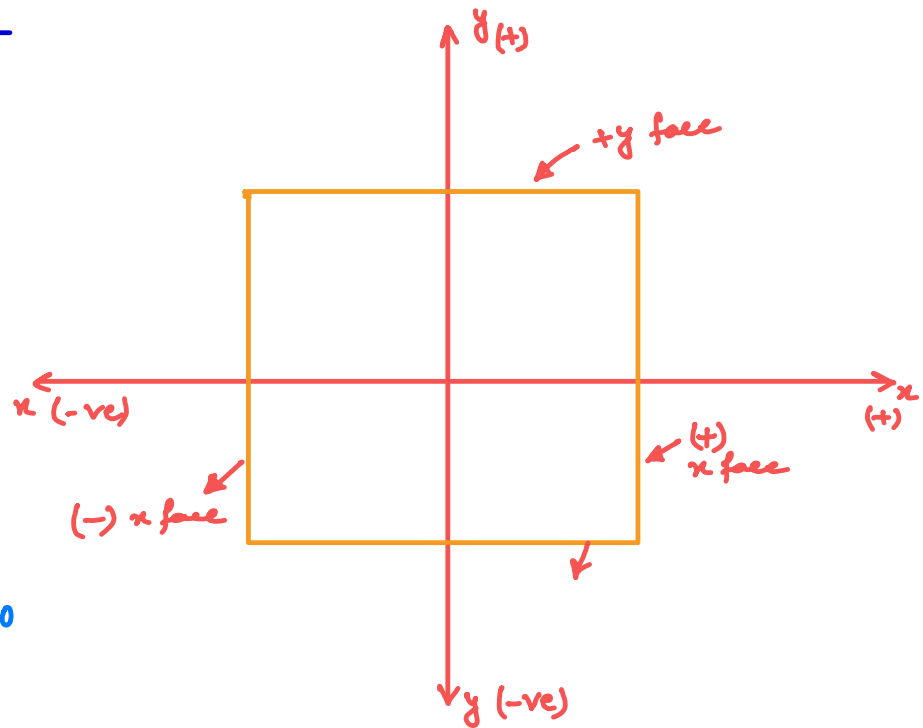
$$\tau_2 = \tau_2'$$

from moment equilibrium:-

$$iii) \sum M_z = 0$$

$$\tau_1 (yz) x - \tau_2 (xz) y = 0$$

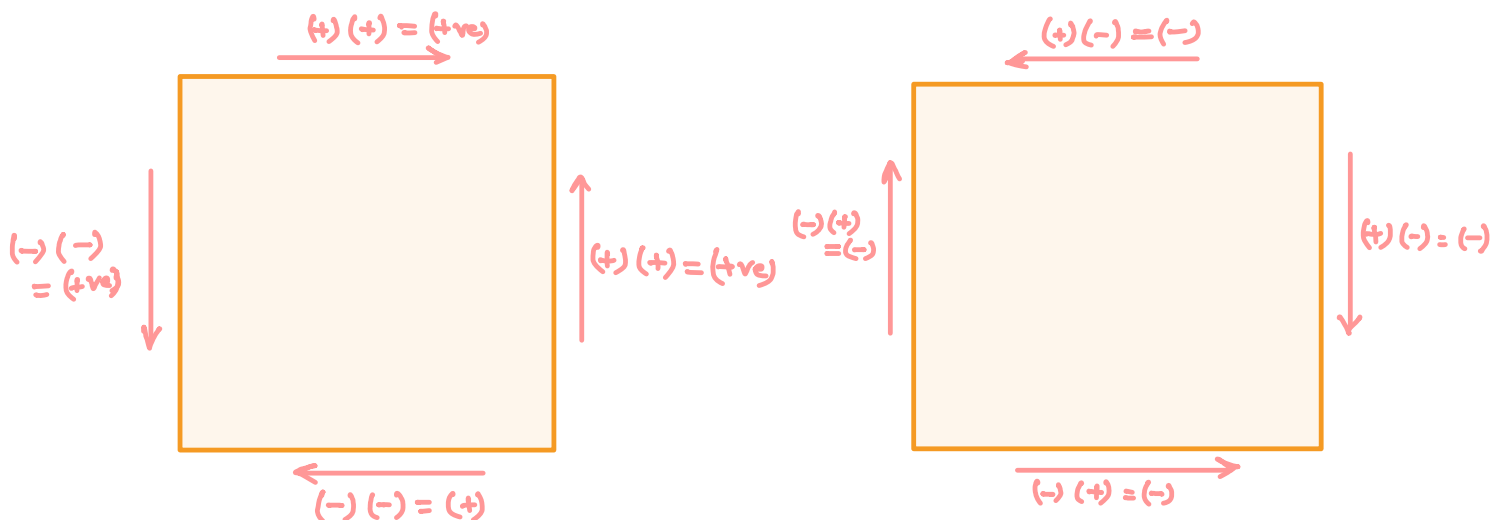
$$\tau_1 = \tau_2$$



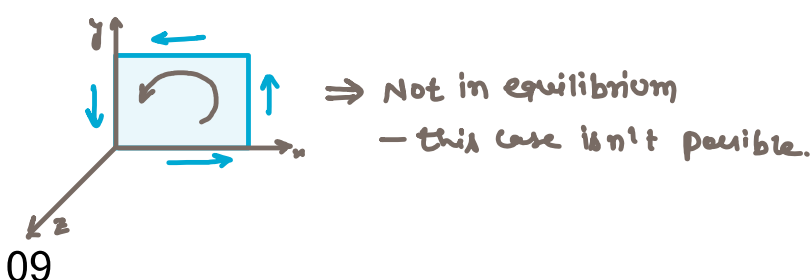
Note:-

Sign convention for Shear Stress:-

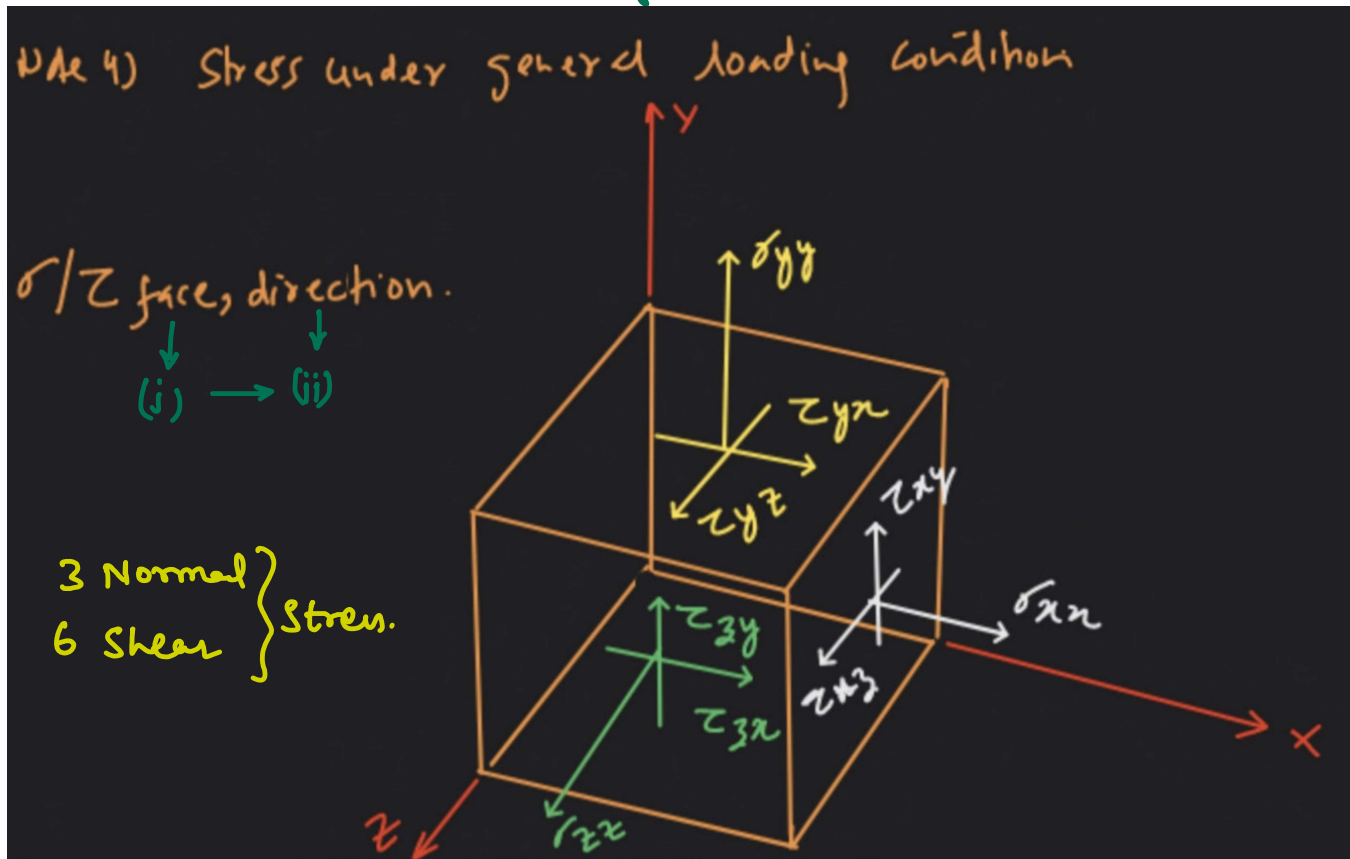
- a) A shear stress acting on positive face is (+ve) when it acts in positive co-ordinate direction. & (-ve) if it acts in (-ve) coordinate direction.
- b) A shear stress acting on Negative face is positive, if it acts in negative (-ve) coordinate direction. & Negative, if it acts in (+ve) positive coordinate direction.



Problem:-



Note:- Stress under general loading condition :-



— Stress tensor is as follows:-

$$\sigma \begin{matrix} \text{Stress} \\ \text{tensor} \end{matrix} = \begin{matrix} \begin{matrix} x & y & z \\ \text{direction} \end{matrix} & \begin{matrix} x & y & z \\ \text{face} \end{matrix} \\ \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \end{matrix}$$

- Hence, at any point Stress has 9 dimensions/components in which
- 3 are Normal Stresses ($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$)
 - 6 are Shear Stresses ($\tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx}, \tau_{yz}, \tau_{zy}$)
 - But Complementary Stress are equal in magnitude.
 - Hence only 6 dimensions/component are required to define the Stress at any given point.

→ 2D $\rightarrow \sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix} \Rightarrow$ only 3 are required to define in 2D.

Note:-

- for safe design, stresses in the member generated due to external loading must be less than allowable stress.

$$\text{Allowable Stress} = \frac{\text{Yield Stress}}{\text{FOS}}$$

$$A.S < \text{Stress}$$

- Here, yield stress is used only for ductile material and ultimate stress is used for brittle material.
- Here, Margin of Safety = FOS - 1

MOS indicates → Range
FOS → Min. limit | safe failure pt.

b. > Strain

- when the force is applied over the material, it may result in its change of shape & size. this change in shape & size is termed as deformation.
- This deformation of the material is represented in terms of Strain.

★ Strain is further classified as:-

i. > Normal Strain.

- It is defined as deformation (only in size) i.e. extension or contraction of the material per unit length.

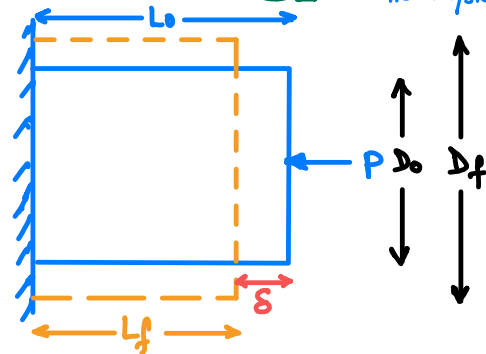
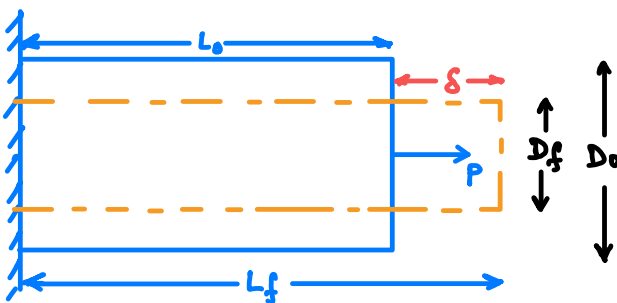
$$\epsilon = \frac{\delta}{L}$$

δ = change in length
 L = original length.

Note:-

The above expression would only give the average strain whereas actual strain is given by.

$$\epsilon = \frac{d\delta}{dL} \rightarrow \begin{matrix} \text{differential / infinitesimal / small deformation} \\ \text{differential / infinitesimal / small length} \end{matrix}$$

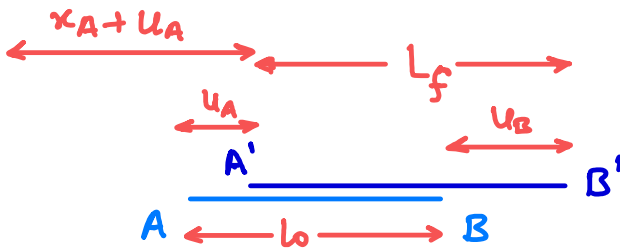


$$\epsilon = \frac{\delta}{L_0} = \frac{L_f - L_0}{L_0}$$

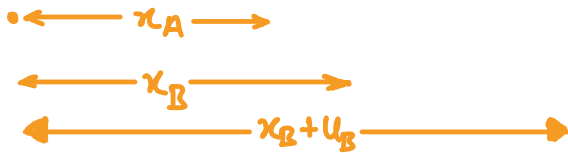
for extension, $\epsilon = (+ve)$
for contraction, $\epsilon = (-ve)$

$$\Rightarrow L_f = L_0(1 + \epsilon) \quad \text{ESE.}$$

for exact Analysis, consider [GATE]



Ref.



$$L_0 = x_B - x_A$$

$$L_f = (x_B + u_B) - (x_A + u_A)$$

$$L_f = x_B - x_A + u_B - u_A$$

$$L_f = L_0 + u_B - u_A$$

$$\epsilon = \frac{L_f - L_0}{L_0}$$

$$\epsilon = \frac{L_0 + u_B - u_A - L_0}{L_0}$$

$$\epsilon = \frac{u_B - u_A}{L_0} = \frac{\partial u}{\partial x}$$

Note

The Normal Strain perpendicular to the direction of loading is termed as Lateral Strain.

$$\epsilon_{\text{Lateral}} = \frac{D_f - D_0}{D_0}$$

- Normal Strain is measured with the help of Extensometer / wire strain gauge.
- If u, v, w are displacements in x, y, z directions respectively then Normal Strain in x, y & z directions are:-

$$\epsilon_x = \frac{du}{dx}, \quad \epsilon_y = \frac{dv}{dy}, \quad \epsilon_z = \frac{dw}{dz}$$

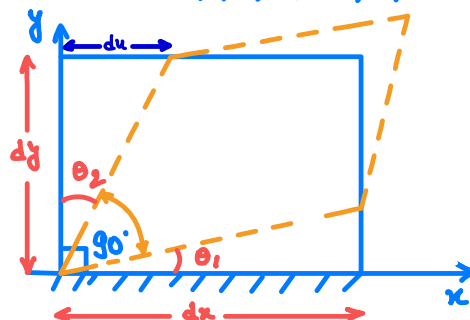
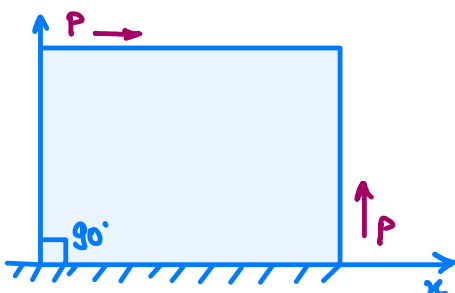
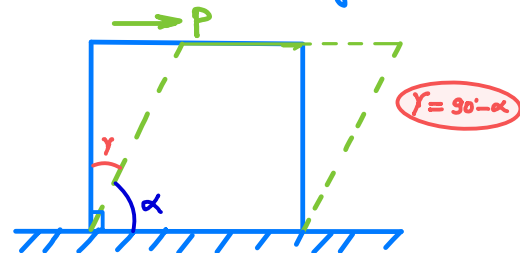
← GATE

ii-> Shear Strain

- It is defined as change in angle that occurs between the two line segments that were originally perpendicular to each other.

Note:-

Normal Strain leads to change in Size of the body whereas Shear strain (γ) leads to change in Shape of the Body.



$$\theta_1 + \theta_2 + \alpha = 90^\circ$$

$$\gamma = 90^\circ - \alpha = \theta_1 + \theta_2$$

$$\tan \theta_1 = \frac{dv}{dx}, \quad \tan \theta_2 = \frac{du}{dy}$$

If θ is small, $\theta \approx \tan \theta \approx \sin \theta$

$$\gamma_{xy} = \frac{dv}{dx} + \frac{du}{dy}$$

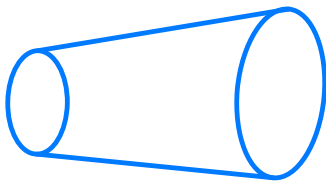
on the same line,

$$\gamma_{yz} = \frac{dw}{dy} + \frac{dv}{dz}$$

$$\gamma_{xz} = \frac{dw}{dx} + \frac{du}{dz}$$

Mathematical
Expression of
Shear Strain

Stress - Strain Curve



Non-prismatic



prismatic

Uniform
cross-section.

- If we consider prismatic bar (structural member having constant cross-section area throughout length) of different length and C.S.A (diameter) and plot the load deformation curve, it is observed to be different for all the bars.
- But Stress - Strain curve for all these bars is observed to be approximately same. Hence, it can be stated that Stress - Strain curve is characteristic property of material which don't depend upon the dimensions of material.

A. Stress - Strain curve for Mild/Low Carbon/Ductile Steel

Strain
Stress

Case - 1) Mild Steel in Tension.

- Tensile test of mild steel is standardised by ASTM, and is performed in UTM (Universal testing machine) in which a solid cylindrical mild steel specimen of gauge length 2" and diameter 0.5 inch is subjected to gradually applied load.
- Deformation/Elongation of Specimen is measured with the help of Extensometer which is used to calculate the strain for a particular loading of specimen.

Note:-

- Gauge length / Initial length is the length of specimen which is used for computation of strain.
- Either it is marked over the specimen or is taken to be 2".



*Significant points Regarding Stress - Strain curve of mild Steel in Tension:-

a> Region OA

- from point O to A Stress is proportional to Strain

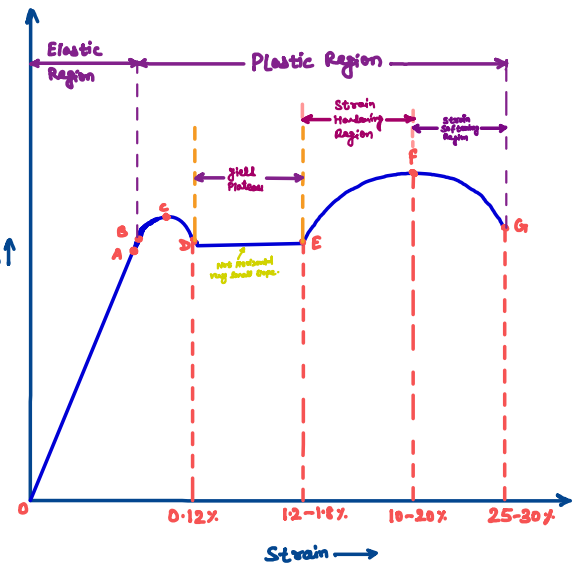
$$\text{Stress} \propto \text{Strain}$$

$$\text{Stress} = \text{Constant} \times \text{Strain}$$

$$\text{Constant} = \frac{\text{Stress}}{\text{Strain}} = \text{Slope of Curve OA} = E$$

$E = \text{Modulus of Elasticity}$

- Strain is infinitesimal.
- volume of specimen increases
- for most of the material, curve is straight line from OA, except concrete, Gray Cast iron, for which it comes out to be curved.
- But for designing point of view it is assumed to be straight.



- Here point A is termed as proportional limit i.e. the maximum stress at which stress is proportional to strain.

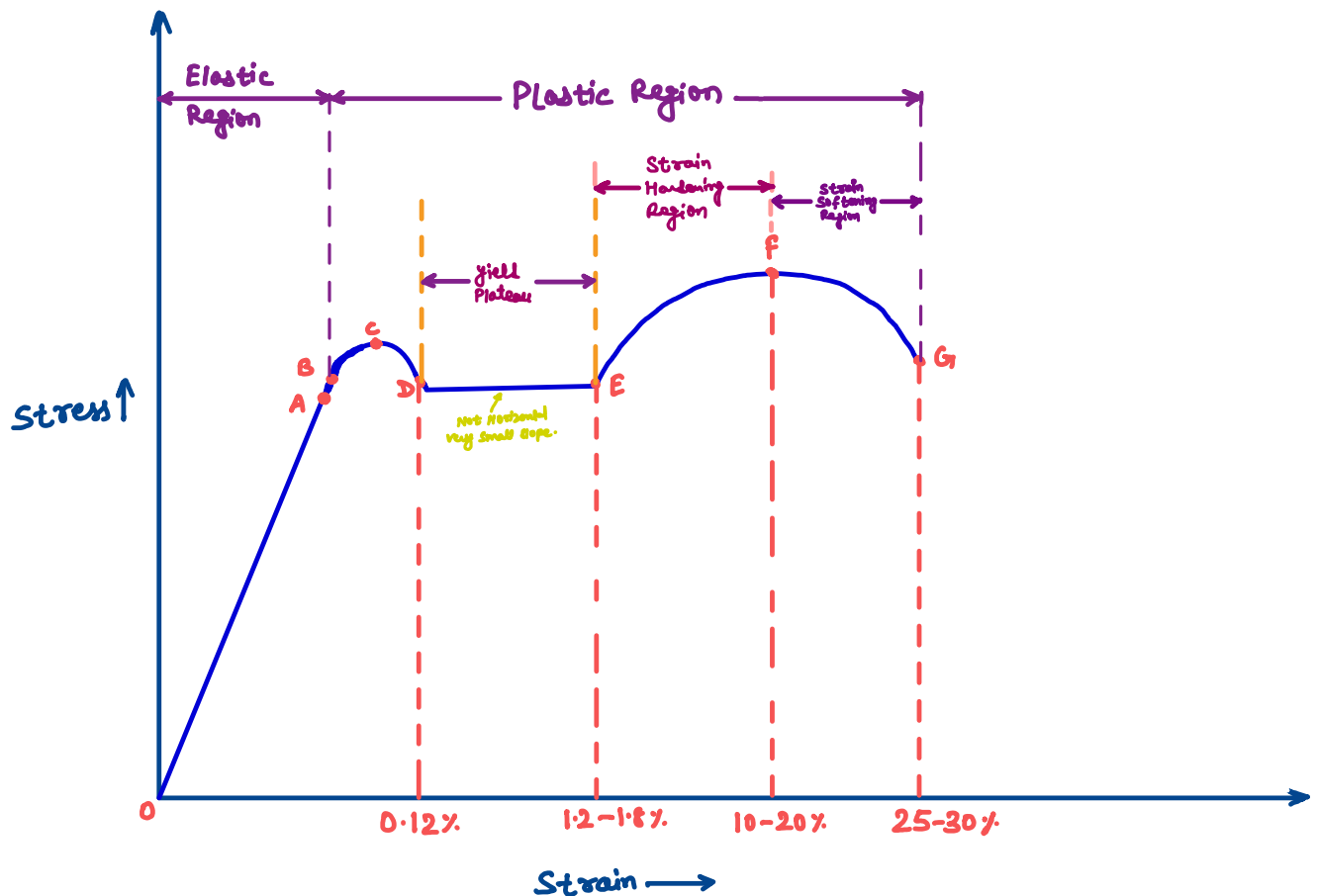
b> Region AD

- Beyond A, strain increases more rapidly as compared to stress.
- Here point B is termed as Elastic limit point i.e. the maximum stress upto which material behaves like elastic material.
- upto point B, if specimen is loaded, it regains its original geometry and unloading curve follows path BAD.
- In between point A & B, stress is not proportional to strain but for designing point of view it is assumed to be proportional.

- * point A & B are practically very close to each other.
- Upper yield point C, corresponds to load just before yielding of the specimen starts.
- Lower yield point D, corresponds to load required to maintain the yielding.
- Here yield point C, is transient (Not constant) and lower end point D is fixed hence it is used to determine the yield strength of material.
- Here difference between upper and lower yield point is observed due to slipping of carbon atoms in molecular structure of steel.
- Lower yield point D, is very close to elastic limit point B, hence for practical purposes yield stress is assumed to be equal to elastic stress.
- In this region, also volume of specimen increases.
- for practical purposes, A, B & D are assumed to be same.

c> Region DE → yield plateau

- once the yield stress attained material undergoes large deformation with the relatively small increase in load applied.
- This deformation is caused due to slippage of material along oblique surface ($\approx 45^\circ$) which is primarily due to shearing stresses.
- These deformations are permanent and no change in volume of specimen.



Concrete
Grey Cast iron

0-A → Curve
Proportional Limit

SP. wt. → Same
then → Vol. Same

Vol. increases → Due to Strain (Normal → Size & Volume)

A point → Proportional limit.

B point → Elastic limit → behaves like elastic material → Specimen → unloading → comes to red shape

Region A to B → Non-linear ⇒ Assumption → linear

Pre-dominant property.

* AD Region → Rate of gain of Strain (AD) region is more than OA region. ⇒ Vol. changes ↑

C → Upper yield point → Transient (Not final) → Strength variable

D → Lower yield point → Corresponds to load which is required to maintain the yield. because of slipping of C-Atom

upto B point → Elastic Region.

Beyond B point → Plastic Region

failure started → Strain increases rapidly

D point stress → Yield Stress → Material strength
No final knee k
like D point stress considered

* DE Region → Large deformation / yielding ⇒ strain ↑. No Vol. changes. → Bond breaks.

* EF Region → Resistance in material ↑ → Stress ↑ ⇒ Strain hardening → Process in EF.
Changes to crystalline structure

Point F → Ultimate Stress point. → local instability

* FG Region → Strain softening Region → Diameter drastically reduces

Necking

Point G → Cup-cone failure fracture point → material fails. { No relation b/n F & G point }

length ↑ → Dia ↓

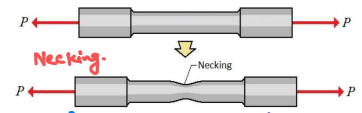
Ductile → failure due to shearing stress at 45°
At fracture point → Area of Mild Steel reduced by 50%.

Stress-strain curve is same for M.S. Except → temp ↑ → A ↓
temp ↓ → A ↑
• Rate of loading
• Manufacturing process of MS
• The No. of times our member is being loaded.

d) Region EF → Strain Hardening.

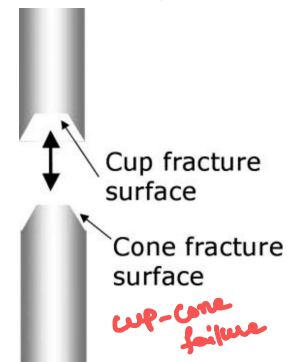
Crystalline → have definite geometry.
Amorphous → do not have definite shape & size

- After undergoing large strain, crystalline structure of the material changes resulting in increased resistance to deformation.
- Hence, more load is required to develop same strain in this region, thereby this region is termed as Strain Hardening region.
- This increase in stress takes place upto point F termed as ultimate stress point



e) Region FG / Strain Softening Region

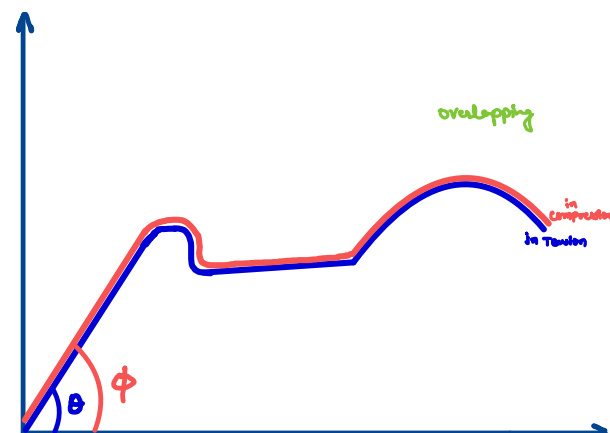
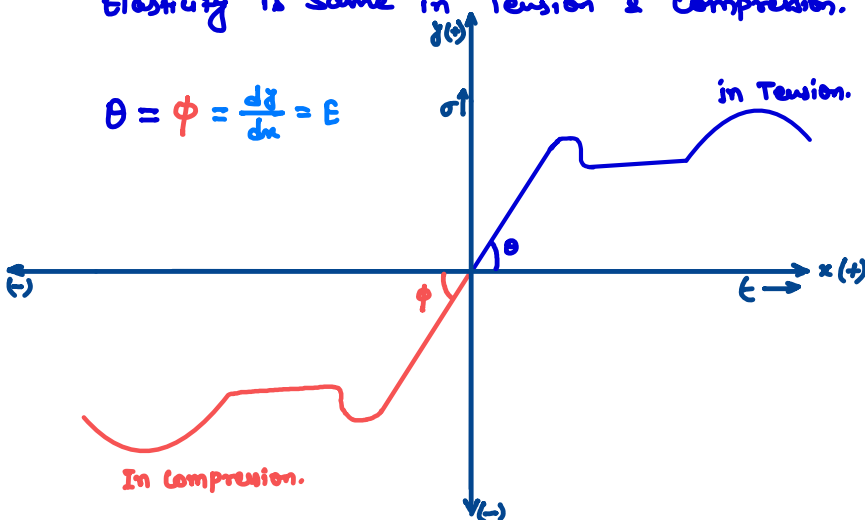
- After point F, diameter of particular section of the specimen starts decreasing (necking) because of local instability in structure of specimen.
- After necking has begun, requirement of the load reduces for same deformation upto it finally ruptures at point G termed as fracture point.
- Rupture occurs along the cone shaped surface at an angle of 45° with original surface of the specimen i.e. termed as cup-cone failure.
- Percentage reduction of area at fracture is approximately 50%.
- Rate of straining is in range of $6-60 \text{ N/mm}^2/\text{sec}$.
- Stress strain curve for a particular material depends upon :-
 - i) Temperature of specimen.
 - ii) Rate of loading
 - iii) Manufacturing process.
 - iv) Number of times specimen is loaded.



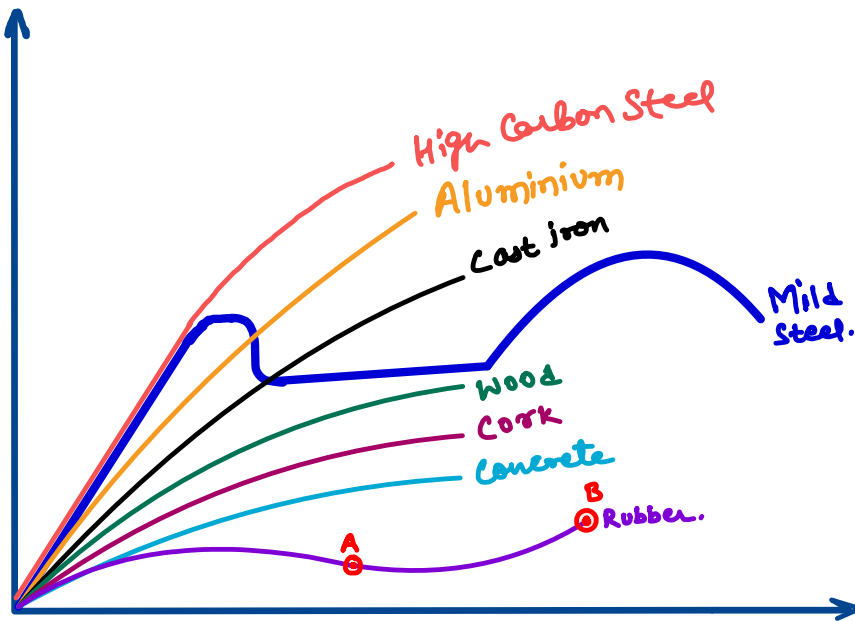
case -(ii) Mild Steel in Compression

- If mild steel is subjected to compression instead of tension, the stress-strain curve will essentially be same through its initial straight line portion followed by beginning of portion corresponding to yield and strain hardening.
- However, for large value of strain, this curve deviates.
- Here No Necking is observed in compression.
- Slope of curve in initial straight line portion is observed to be same i.e. Modulus of Elasticity is same in Tension & Compression.

$$\theta = \phi = \frac{d\sigma}{d\epsilon} = E$$



Stress - Strain curve for different Materials :-



$$E_{M.S} = 2.1 \times 10^5 \text{ MPa}$$

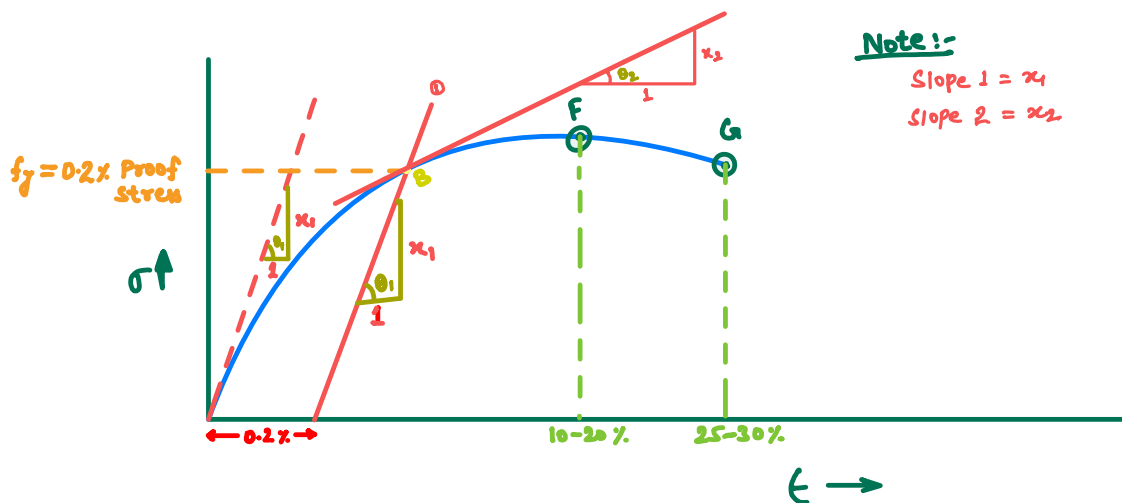
$$E_{Al} = \left(\frac{1}{2} - \frac{1}{3}\right) E_{M.S}$$

$$E_{C.I} = \frac{1}{3} E_{M.S}$$

$$E_{Concrete} = \left(\frac{1}{10} - \frac{1}{20}\right) E_{M.S} \quad \leftarrow \text{depends upon grade of concrete}$$

- for brittle material, small or no plastic or post-elastic deformation occurs and the material failure is very near to linear elastic region.
- Mild Steel is more elastic than rubber. (Elasticity is represented by its modulus of Elasticity).
- Rubber is Non-elastic material
- Rubber is brittle material as its post elastic strain is negligible.

Note:-



Note:-

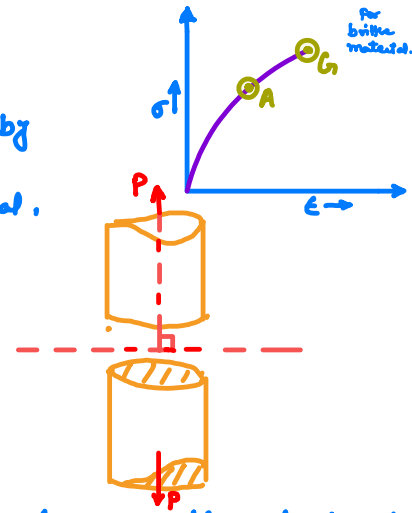
$$\text{Slope 1} = x_1$$

$$\text{Slope 2} = x_2$$

- Stress-strain diagram for most of the material is curved, hence there is no definite proportional limit
- In such cases, Stress-strain proportionality is taken at the stress at which rate of increase of strain is 50% greater than the rate of increase of strain shown by initial tangent i.e. $\frac{1}{x_2} = 1.5 \frac{1}{x_1}$
- If a line is drawn parallel to the initial tangent at an offset of 0.2% strain, it meets the curve at the point where strain increases at a rate of 50% greater than shown by initial tangent.

B. > Stress - Strain Diagram for Brittle material

- Brittle material like Cast iron, stone, glass etc. are characterised by sudden rupture without any prior change in rate of elongation.
- At rupture, strain is much lesser than compared to ductile material.
- No Necking is found in this case.
- Normal stresses are primarily responsible for failure
- Since, Normal stresses for uniaxial tensile loading is maximum at plane normal to the direction of loading
- failure of brittle material is also along the plane normal to the direction of loading.
- for most of the brittle material ultimate strength in compression is more than in tension.
- This is due to presence of inherent flaws in structure of brittle material i.e. cracks, which enhances due to tensile loading and are not affected by compressive loading.
- Hence, for some of brittle material Modulus of Elasticity in compression is more than tension.



$$E_{\text{compression}} > E_{\text{Tension}}$$

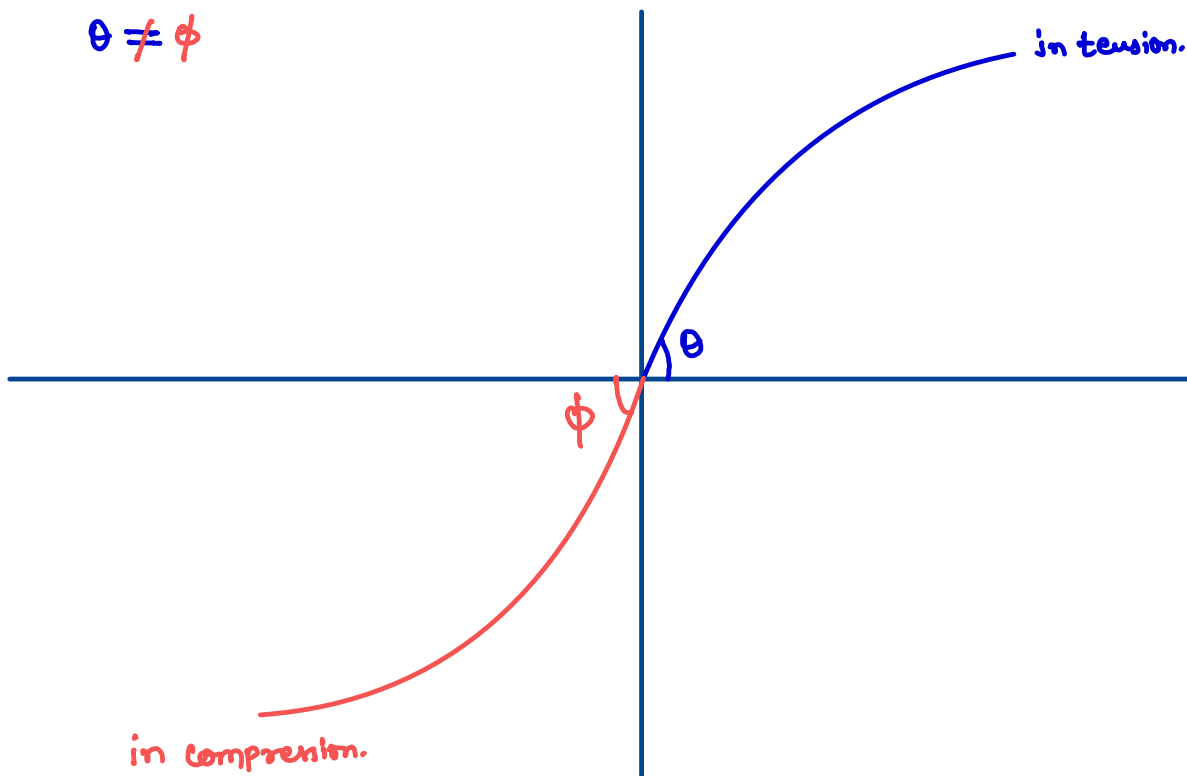
- A ductile material can show brittle fracture if temperature is reduced and brittle material can show ductile fracture if temperature is increased at same modulus of Elasticity (E).

Note:-

In case of torsion shear stress is maximum at plane normal to the direction of loading & normal stress is maximum at angle 45° with direction of loading.

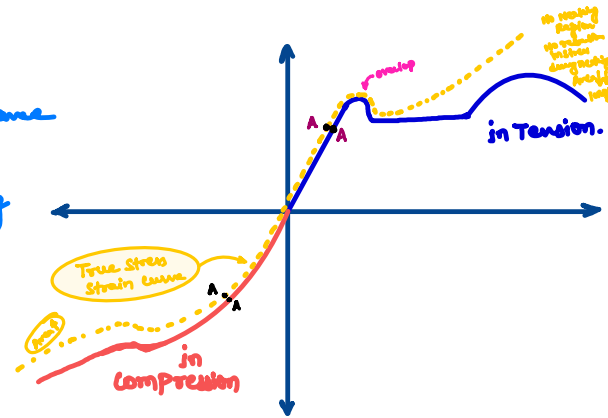
Hence, Brittle material subjected to Torsion fails at 45° .
Ductile material subjected to Torsion fails at 90° .

$$\theta \neq \phi$$



True Stress and True Strain curve (Mild Steel)

- As elongation or compression takes place during the performance of the test, C.S.A. and length of the specimen changes.
- If original area and length of the specimen is used for computing stress and strain, it is termed as "Engineering Stress" and "Engineering Strain".
- If actual area and length is being used for computing stress and strain, it is termed as "True Stress" and "True Strain".
- In strain softening region, σ_e even if load is reducing but True Stress is not decreasing instead it is increasing due more reduction in area than load. $\uparrow \sigma = \frac{P \downarrow}{A \downarrow}$
- The results obtained in tension and compression gives same plot of true stress and true strain but in case of Engineering stress and Engineering strain plot it is not same for large value of strain.
- upto proportional limit both true stress vs true strain curve and Engineering stress vs Engineering strain curve are same, however for all practical purposes we use Engineering stress vs Engineering strain.



Engineering Stress, $\sigma_e = \frac{P}{A_0}$

$A_0 \rightarrow$ original Area of Specimen.

Engineering Strain, $\epsilon_0 = \frac{\delta}{L_0}$

$L_0 \rightarrow$ original length of specimen (2")

Values are known to us because we made it

True Stress, $\sigma = \frac{P}{A}$

$A \rightarrow$ Actual Area of Specimen during straining

True Strain, $\epsilon = \frac{\delta}{L}$

$L \rightarrow$ Actual length of Specimen during straining.

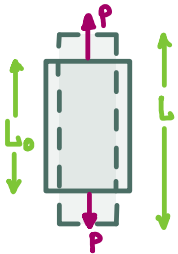
Note:-

True strain for the finite increment of loading such that length changes from L_0 to L is given by,

$$\epsilon = \int_{L_0}^L \frac{dL}{L} = \left| \ln L \right|_{L_0}^L$$

$$= \ln L - \ln L_0$$

$$\epsilon = \ln\left(\frac{L}{L_0}\right)$$



Note

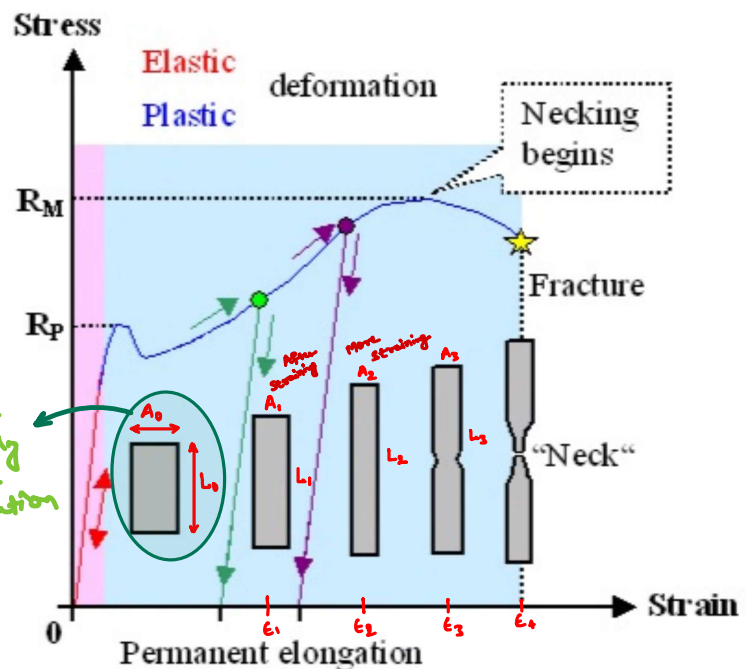
Assuming volume of specimen to remain constant during straining (However it is not constant in Elastic region.)

$$V = A_0 L_0 = A L$$

$$A = \frac{A_0 L_0}{L} = \frac{A_0}{\frac{L_0 + \delta L}{L_0}} = \frac{A_0}{1 \pm \epsilon_0}$$

$$\sigma = \frac{P}{A} = \frac{P}{A_0 / (1 \pm \epsilon_0)} = \frac{P}{A_0} (1 \pm \epsilon_0)$$

$$\sigma = \sigma_0 (1 \pm \epsilon_0)$$



Now,

$$\epsilon = \ln\left(\frac{L}{L_0}\right)$$

$$\epsilon = \ln\left(\frac{L_0 + \delta L}{L_0}\right) = \ln\left(1 \pm \frac{\delta L}{L_0}\right)$$

$$\epsilon = \ln(1 \pm \epsilon_0)$$

Q> The volume of a tensile specimen is essentially same while plastic deformation occurs in it. If the initial diameter of specimen is d_0 , show that when the dia is d , the true strain is $\epsilon = 2 \ln(d_0/d)$.

Soln

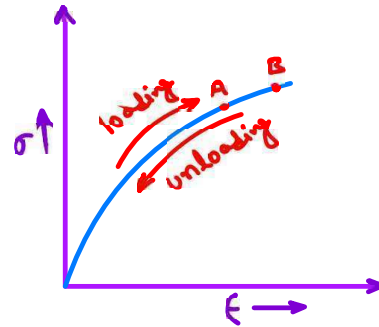
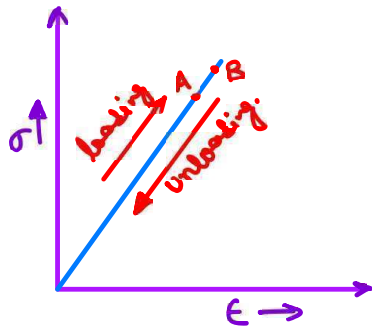
$$\frac{\pi d_0^2}{4} L_0 = \frac{\pi d^2}{4} \cdot L$$

$$\frac{L}{L_0} = \left(\frac{d_0}{d}\right)^2$$

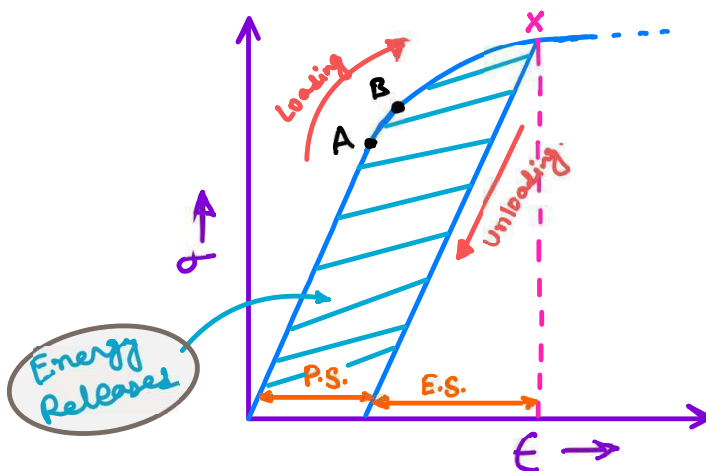
$$\epsilon = \ln\left(\frac{L}{L_0}\right) = \ln\left(\frac{d_0}{d}\right)^2 = 2 \ln\left(\frac{d_0}{d}\right).$$

Elasticity.

- It is the property of material by the virtue of which it is capable of attaining its original dimension after being unloaded and the material is termed as Elastic material.
- Graphically if the material is unloaded before elastic limit B, it retraces its original path.



- If however, the material is stressed beyond elastic limit and unloaded it does not retrace the original path but it is parallel to initial portion of Stress-Strain Curve.
- In this case it also has residual strain/permanent set/permanent deformation.

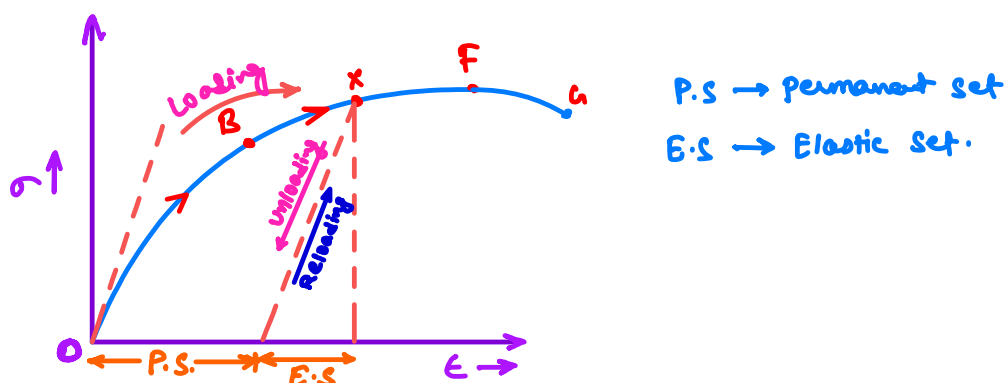


P.S = Permanent Set.

E.S = Elastic Set

Plasticity

- It is the property of material by virtue of which it undergoes inelastic strain or plastic strain before the strain at elastic limit.

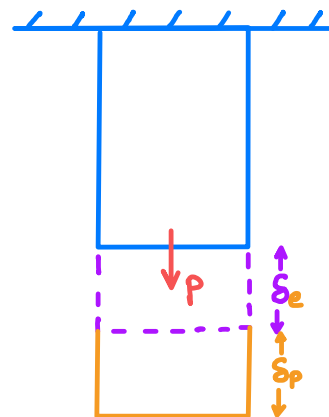


NOTE:—

- If the material is Reloaded after being unloaded beyond elastic limit (B), it follows same path as that of unloading.
- Hence, it behaves like linear elastic material.
- In this process new proportional limit is established (X), which is higher than the original one O.
- But it reduces the ductility of the material because yielding zone reduces ($XG < BG$).

Creep

- It is property of the material by the virtue of which material undergoes additional deformation which is over and above elastic deformation, with passage of time due to sustained loading (constant) within elastic limit.
- It depends upon following Parameters :-
 - a) Temperature level $\uparrow \Rightarrow$ creep $\uparrow$
 - b) Stress level $\uparrow \Rightarrow$ creep $\uparrow$
 - c) Time $\uparrow \Rightarrow$ creep $\uparrow$
 - d) Type of loading (Static or dynamic) $\Rightarrow$ for dynamic it is more.

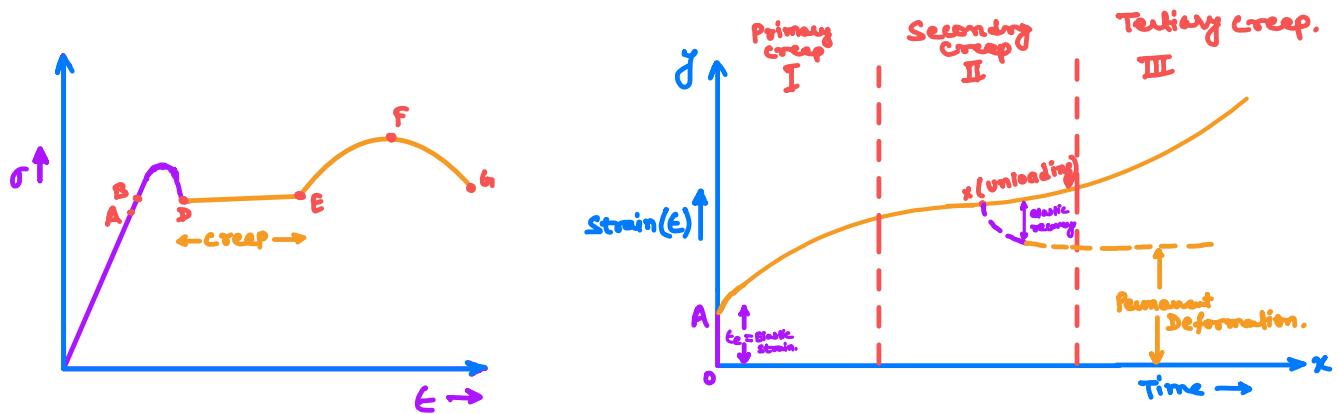


Note :-

Creep is more important at higher temperature level than stress level.

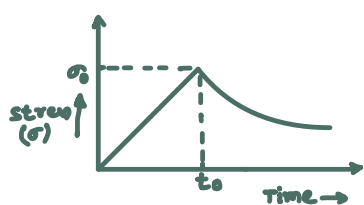
- Generally the effect of the creep becomes noticeable when the temperature is raised to approximately (30-50)% of Melting point (known as Homologous Temperature).

Eg:- pressure vessel at high temperature, gas turbine, rails, aircraft wings are all subjected to constant stress for long duration.



- In Phase I, Primary strain increases at ^{↑ axis.} faster rate because of slipping of molecules along oblique planes.
- In Phase II, secondary strain increases at very slow rate due to strain hardening.
- In Phase III, Tertiary strain again starts increasing at fast rate due to strain softening.

Note:-

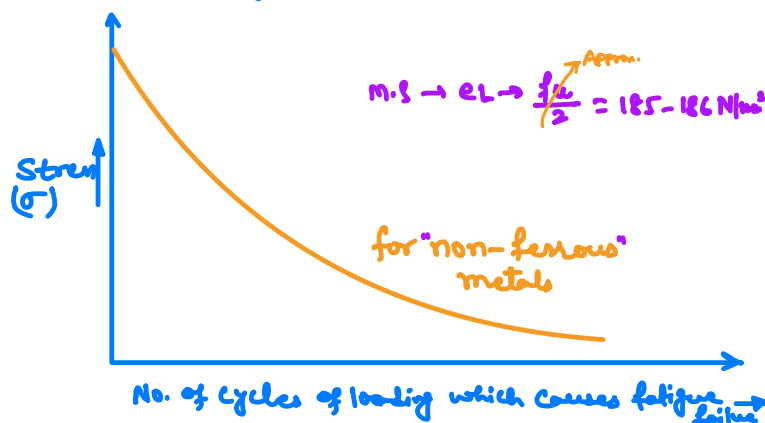
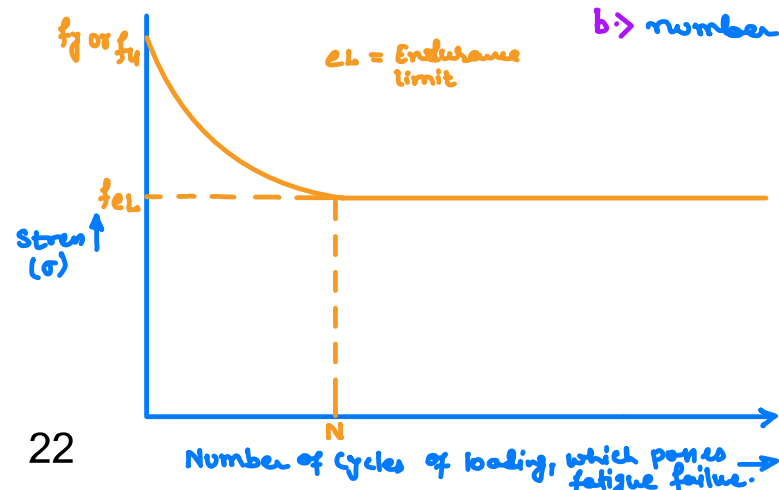


- If the material is stressed to σ_0 in time t_0 and left thereafter stress in the material goes on reducing due to development of creep in it under prolonged strain.
- This decrease in stress is termed as stress relaxation.

V. Fatigue

- Deterioration of the material under repeated cycles of stress or strain resulting in progressive cracking that finally/eventually produces fracture/failure of material is termed as fatigue.
- On microscopic level typically fatigue initiates the formation of the crack at the point of high stress which gradually increases due to repeated cycles of load and finally lead to the **Sudden** (brittle) failure.
- Magnitude of the load at which fatigue failure takes place is less than the load that can be resisted by the material in statically.
- Fatigue failure depends upon a) magnitude of the load and

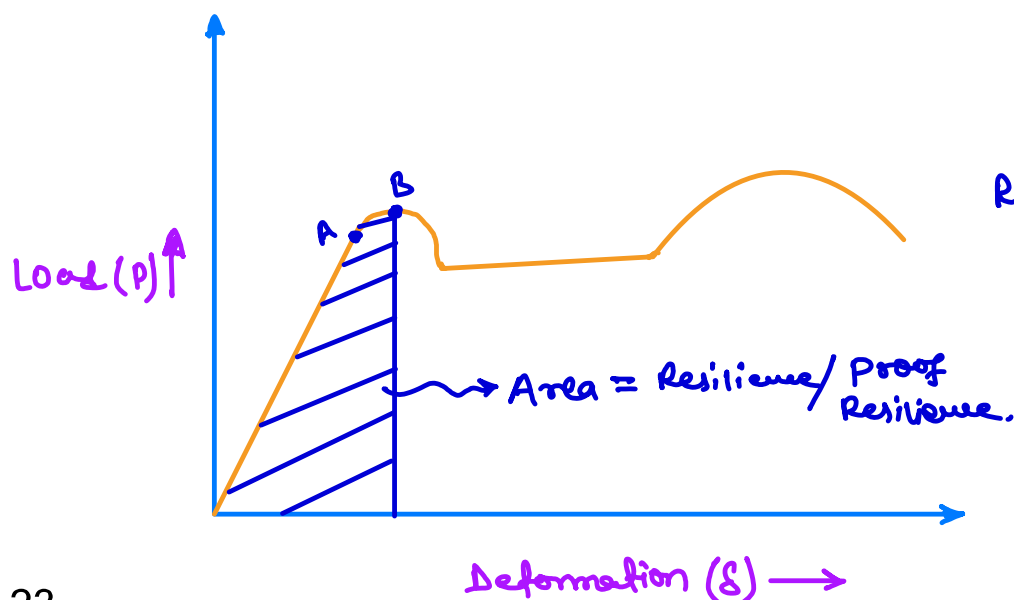
b) number of cycles of loading.



- on increasing the number of cycles of loading which causes fatigue failure stress at the failure goes on reducing
- **Endurance limit** is the stress level below which even the infinite number of cycles of loading can't produce the fatigue failure.
- For Non-ferrous metal, stress at the failure due to fatigue continuously decreases with increase in number of cycles of loading which causes fatigue failure.
- Hence in such case we define **fatigue limit** as the stress corresponding to particular number of cycles of loading which causes fatigue failure.
- for mild steel / structural steel
Endurance limit $\approx \frac{1}{2}$ ultimate strength.
for eg:- in case of Mild Steel, $f_y = 250 \text{ MPa}$
 $f_u = 410 \text{ MPa}$
 $f_e = 186 \text{ MPa}$
- Hence for static loading, material is designed for yield strength (for ductile material) or ultimate strength (for brittle material) and for dynamic loading material is designed for endurance limit.
- Endurance/fatigue limit is not the property of the material.

Vi. Resilience (strain energy density)

- It is the property of the material by the virtue of which it is capable of absorbing the energy when loaded up to elastic limit then upon unloading to have the energy recovered.
- The area under load-deformation curve within **elastic limit** is termed as Resilience / proof Resilience.
deformation is elastic



$$\text{Resilience} = \frac{1}{2} P \delta$$

Resilience - M.S (El. limit = proof defn)
P. Resilience - B point.

- Here strain energy stored per unit volume in linearly elastic material is termed as Modulus of Resilience.

$$\text{Resilience} = \frac{1}{2} P \delta \Rightarrow \frac{1}{2} P \delta \cdot \frac{\text{Volume}}{\text{Volume}} = \frac{1}{2} P \delta \frac{V}{V}$$

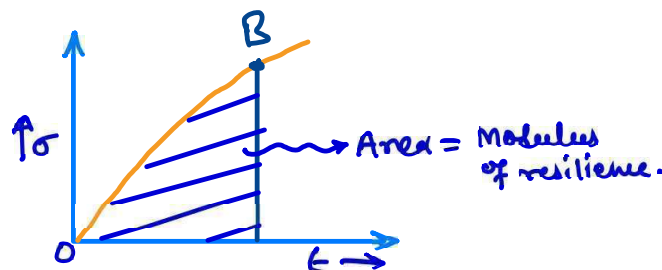
$$= \frac{1}{2} P \delta \frac{V}{A \cdot L} = \frac{1}{2} \left(\frac{P}{A} \right) \cdot \left(\frac{\delta}{L} \right) \cdot V$$

$$\text{Resilience} = \frac{1}{2} \sigma \cdot \epsilon \cdot \text{Volume}.$$

$$\frac{\text{Resilience}}{\text{Volume}} = \frac{1}{2} \sigma \cdot \epsilon$$

$$\text{Modulus of Resilience} = \frac{1}{2} \sigma \cdot \epsilon = \frac{1}{2} \sigma \cdot \frac{\sigma}{E} = \frac{\sigma^2}{2E}$$

- Hence from above analysis, Modulus of Resilience is equal to area under stress-strain curve upto elastic limit.



Not depends upon size → so it is property used where vibration occurs

- Since modulus of Resilience does not depend upon dimensions of specimen, it is property of material.
- This property is desirable for the materials to be used as springs, bearings, or which are being subjected to shock and vibrations.

Q.7 In a non-linear elastic material subjected to axial loading, strain is 0.15% (elastic strain) & stress-strain relation is given by $\sigma = C\epsilon^2$, where C is a constant given by $C = 10^5 \text{ N/mm}^2$, σ is stress & ϵ is strain, calculate the resilience of the material.

Soln

$$\sigma = C\epsilon^2$$

$$\text{Modulus of Resilience} = \text{Area} = A = \int_0^{\epsilon} dA = \int_0^{0.15\%} \sigma d\epsilon$$

$$\text{MOR} = \int_0^{0.15\%} C\epsilon^2 d\epsilon = C \left[\frac{\epsilon^3}{3} \right]_0^{0.15\%}$$

$$= 10^5 \times \left(\frac{0.15}{100} \right)^3 \cdot \frac{1}{3}$$

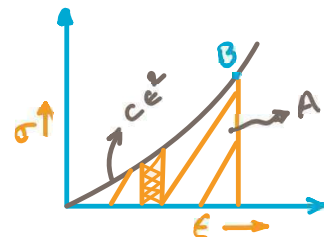
$$\text{MOR} = 1.125 \times 10^{-4} \text{ N/mm}^2$$

$$\text{Resilience} = \text{MOR} \cdot \text{Volume}$$

$$= 1.125 \times 10^{-4} \times 1$$

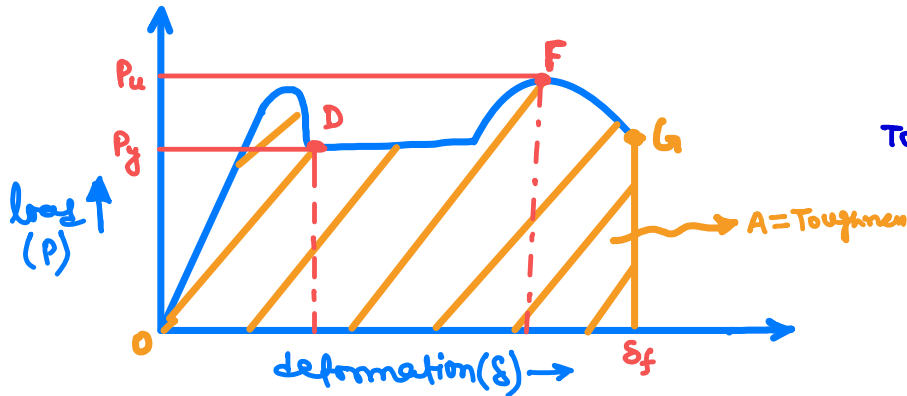
$$= 1.125 \times 10^{-4} \text{ N-mm}$$

[Assume vol. $V = 1 \text{ mm}^3$]



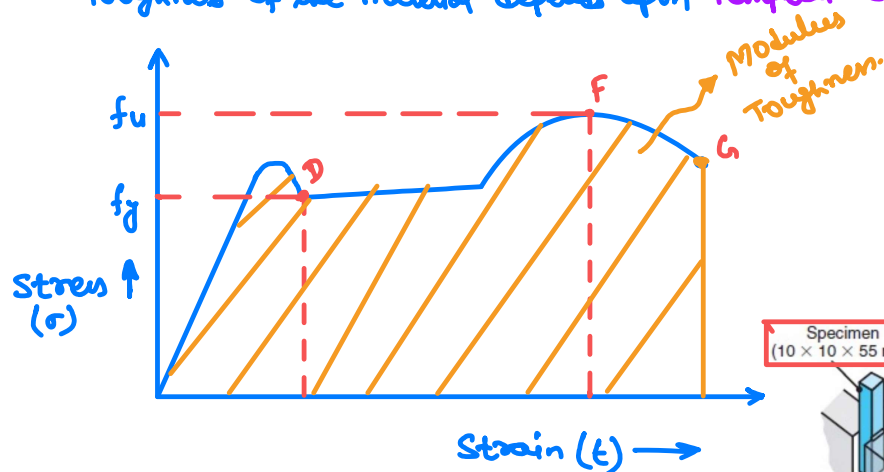
vii Toughness

- It is the ability of material to absorb the energy upto fracture (ϵ_f) hence in this case material is deforming **Plastically**.
- It may also be defined as Area Under load-deformation curve upto fracture limit.

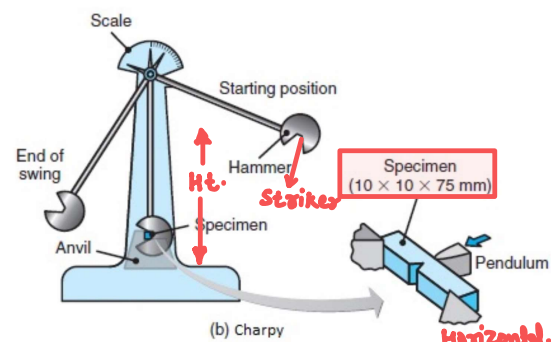
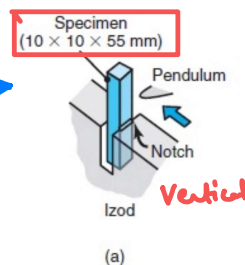


$$\text{Toughness} = \left(\frac{P_y + P_u}{2} \right) \cdot \delta_f$$

- Here, total amount of energy stored by the material upto fracture per unit volume is termed as **Modulus of Toughness** which is also equal to area under stress-strain curve.
- Since modulus of Toughness does not depend upon dimensions of material, so it is the characteristic property of material.
- Toughness is required in material where it is subjected to **impact loading**
eg:- Railway tracks,
- Toughness of the material depends upon Temperature directly. ($T \uparrow \Rightarrow T' \uparrow$)



$$\text{Modulus of Toughness} = \left(\frac{\sigma_u + \sigma_y}{2} \right) \epsilon_f$$



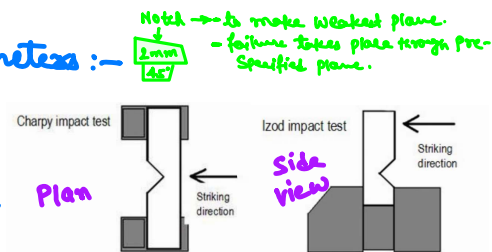
- Toughness of material is found by performing impact test which are of following types in which the material is being subjected to impact loading and load deformation is plotted to find the toughness.

a) Charpy test

b) Izod test

In both these tests, difference lies in following parameters:-

- i) position of specimen:- izod $\rightarrow$ vertically.
Charpy $\rightarrow$ Horizontal.
- ii) position of Notch :- izod $\rightarrow$ facing/ Towards the striker.
Charpy $\rightarrow$ Away from the striker.
- iii) Specimen dimension:- izod $\rightarrow$ (55 x 10 x 10 mm)
Charpy $\rightarrow$ (75 x 10 x 10 mm)
- iv) Type of Hammer : izod $\rightarrow$ Framing type Hammer
Charpy $\rightarrow$ Ball Pin type Hammer.



vii. Tenacity

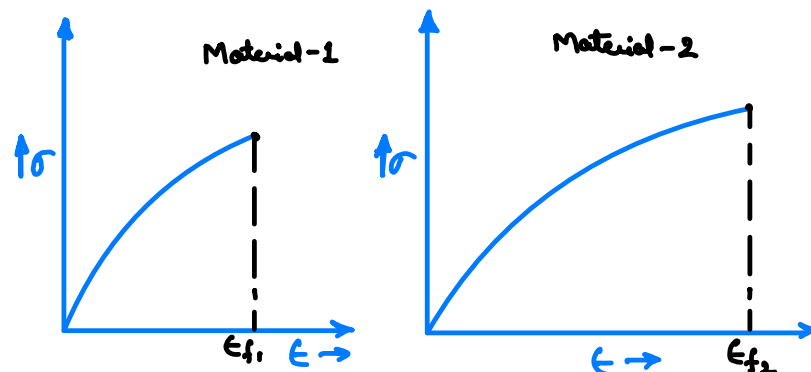
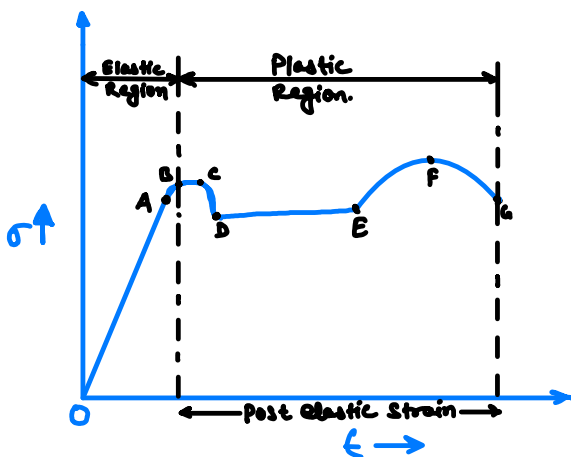
- The property of material by virtue of which it resist fracture when subjected to tensile loading is termed as Tenacity.

vii. Ductility

- It is the property of material which gives the measure of extent to which a material will deform before fracture.
- Ductility is the ability of material by which it can be drawn into thin wires.
- Hence, it is the ability of the material to deform under tensile loading.

Note:-

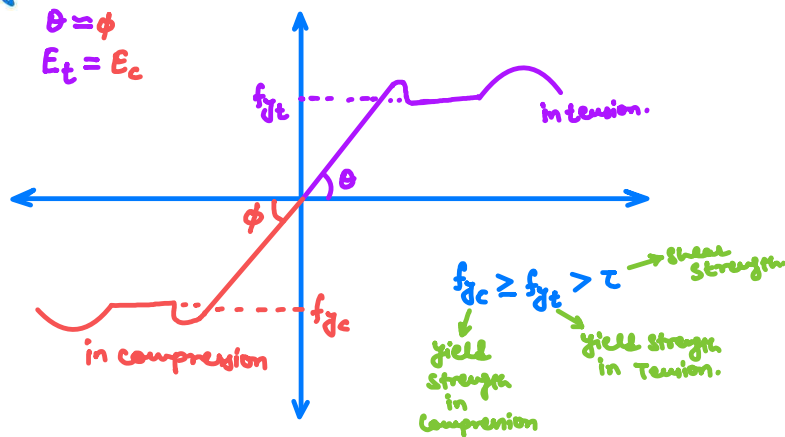
- However, the ability of the material to deform under compressive load is termed as Malleability.
- By virtue of malleability, material can be drawn into thin sheets.
- Ductile material is weakest in shear. Hence, it is designed for shear.
for eg:- Mild Steel, Aluminium, copper, lead, Silver, Gold, Platinum.
- A malleable material need not be ductile and vice-versa.
for eg:- Lead is highly malleable but possess low ductility.
- Platinum is the most ductile but Gold is the most malleable.
- A material is termed as Ductile if it has Post Elastic Strain (PES) more than 5%.



$$\epsilon_{f1} < \epsilon_{f2}$$

$$\text{Ductility}_1 < \text{Ductility}_2$$

- The designing of ductile material is being done on the basis of yield strength (f_y).

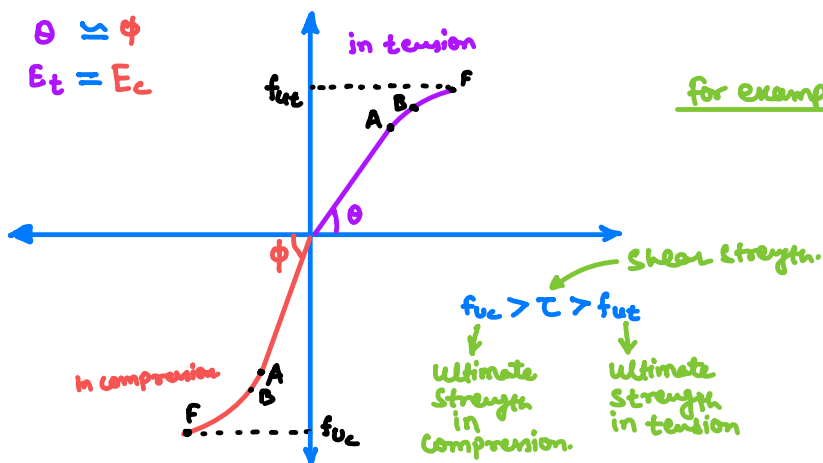


for eg:- in mild steel
 $f_{yt} = 250 \text{ MPa}$ → Limit stress
 $f_{yc} = 263 \text{ MPa}$.

- with increase in temperature, fracture strain increases that results in higher ductility
- Eg:- Tungsten is brittle at room temperature, but at elevated temperature it shows Ductile failure.
- Similarly by sufficient increase in pressure, brittle material can be transformed into ductile material.

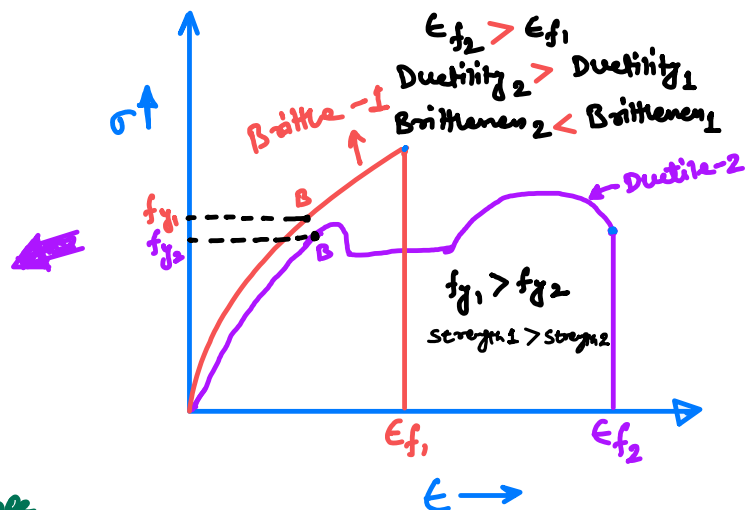
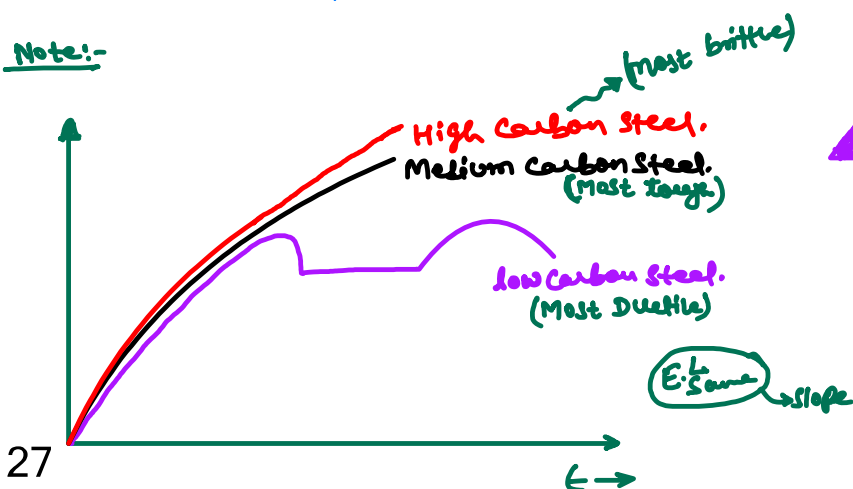
ix. Brittleness

- Lack of ductility is termed as Brittleness.
- It is not the absolute property of material.
- material is termed as brittle if PES is less than 5%, hence brittle material shows sudden failure.
- Brittle material is weakest in tension
- Designing of brittle material is done on the basis of ultimate strength.



for example :- Gray Cast Iron
 $f_{ut} = 160 \text{ MPa}$
 $f_{uc} = 550 \text{ MPa}$.

Note:-



- By increasing percentage Carbon, yield strength, ultimate strength, abrasion/wear resistance, stiffness, resilience, cost of material increases.

* Hardness :-

- It is the ability of material to resist wear and tear or Abrasion
- It is also related with yield strength directly.
- By decreasing Percentage Carbon, ductility, toughness, formability, flexibility, malleability increases.

Note :-

- Ductility is not the criteria for material to be tough.
- As tough material signifies both high strength & high ductility.
- But in general ductile material is also tough.

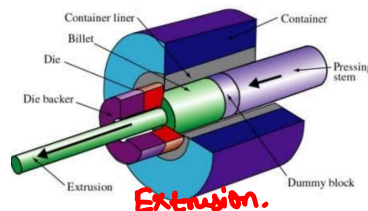
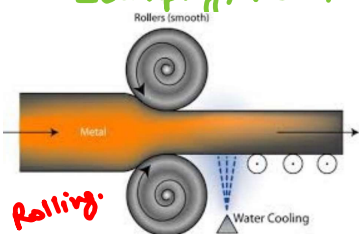
* for design purposes mild steel is preferred as :-

- its formability is comparatively higher than other metals.
- It is ductile in nature hence gives sufficient warning (yielding) prior to its failure (fracture).
- coefficient of thermal expansion (α) is approximately same as that of concrete, hence due to temperature variation no relative displacement of concrete and steel takes place hence, NO thermal stresses are developed.

$$\alpha_{\text{concrete}} \approx \alpha_{\text{mild steel}} = 12 \times 10^{-6}/^{\circ}\text{C}$$
- Its young's modulus of Elasticity is same in Tension and Compression (i.e. equally strong in tension and compression)
- It can be easily welded at joints.
- It is recyclable & offers higher scrap value.
- It is easily available & comparatively cheaper.

Note :-

- formability is the ability of material to undergo plastic deformation without being damaged.
- This property is required in processes like Rolling, Extrusion, forging, Hydroforming, Stamping, Roll forming.

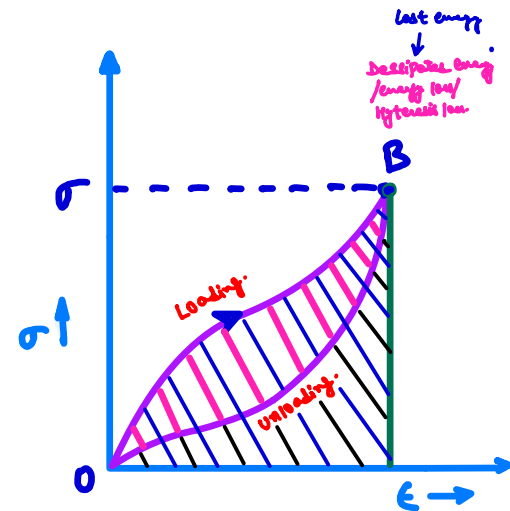


- while designing structural members FOS is to be applied to account for following:-
 - for poor workmanship
 - for uncertainty in loading.
 - for future Expansion
 - for Lack of quality control.

$\text{FOS} \uparrow \rightarrow \text{Cost} \uparrow \uparrow$
 $\text{FOS} \downarrow \rightarrow \text{Risk/failure} \uparrow \uparrow$

28. for accounting the effect of earthquake & wind

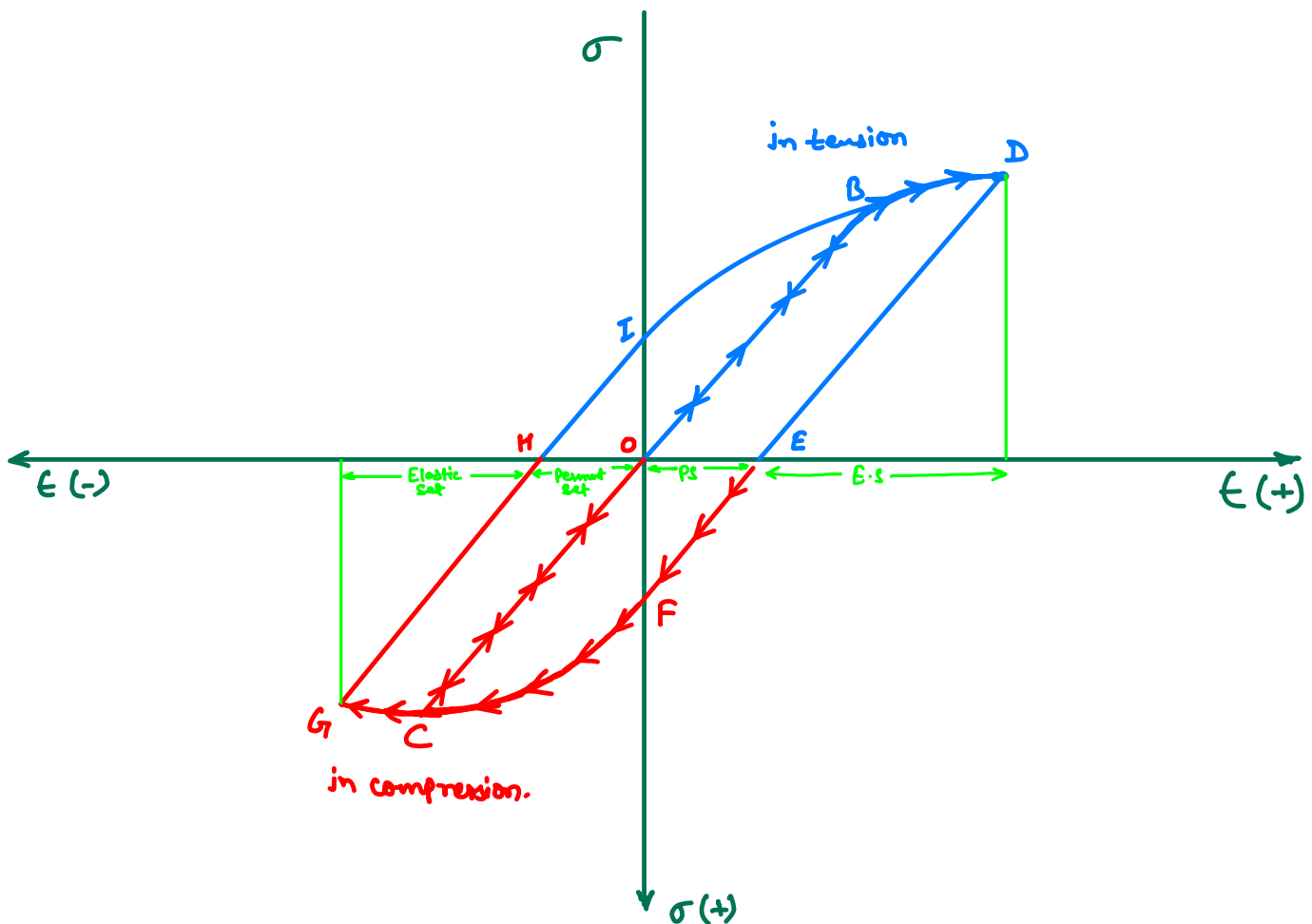
- It is the difference between the strain energy required to generate a given stress in the material and the materials elastic energy at that particular stress.
- This energy is lost due to internal friction in the form of heat.
- Due to this elastic hysteresis, loading & unloading stress-strain curve of the material is not observed to be same.
- It is the property of every material, for hard metals it is negligible & for elastomer it is very high.



Note :-

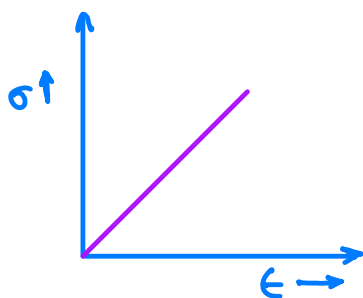
Bouschiger Effect:

- Bauschinger effect is also termed as Strain softening.
- Region OB: This region is elastic region in tension, if unloading of the material is carried out in this region, it will follow same path from B to O.

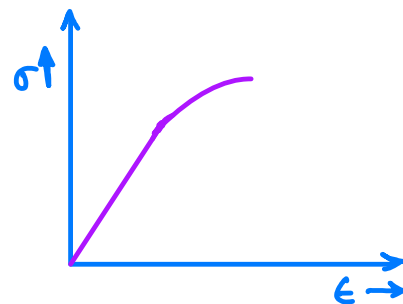


- Region OC :- This region is elastic region in compression, if unloading of the material is carried out in this region so, it will follow same path from C to O.
- Region BD :- Due to increase in tensile load, tensile stresses overcome the Bond Strengths resulting in yielding of material and dislocation starts moving towards the grain boundary.
 - Here, accumulation of dislocation at grain boundary generates back pressure which opposes tension.
- Region DE :- If unloading is carried out, material undergoes elastic recovery by following path DE which is parallel to initial tangent to the curve. and does not follow the original path.
 - Here OE signifies permanent set.
- Region EF :- Here compression of the material takes place from EF and dislocation which were opposing, support this compression.
- Region FC :- Here dislocation/back pressure which supports compression helps in attaining the original dimension at comparatively lower applied compression. (termed as Strain Softening).
- Region CG :- Due to further increase in compressive load material starts yielding in compression and back pressure due to accumulation of dislocation is generated which now opposes compression.
- Region GH :- It represents the path followed during unloading of compression. Here OH represents permanent set in compression.
- Region HI :- In this region material is again subjected to tensile loading which is supported by back pressure now.
- Region IB :- Due to the effect of back pressure tensile load required to regain the permanent set is lowered (termed as Strain Softening).

Approximate stress-strain curve for different materials

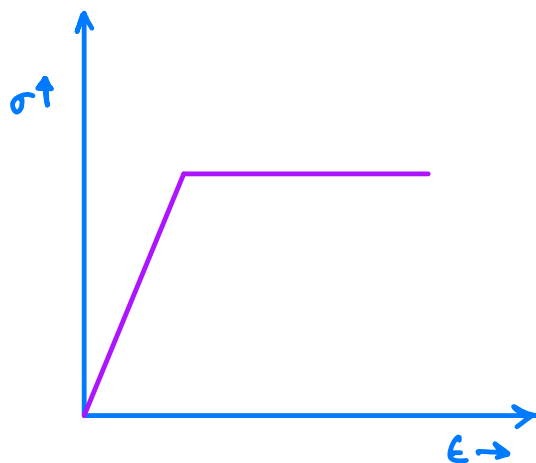


Linear Elastic. {eg:- Steel (for low stress)}

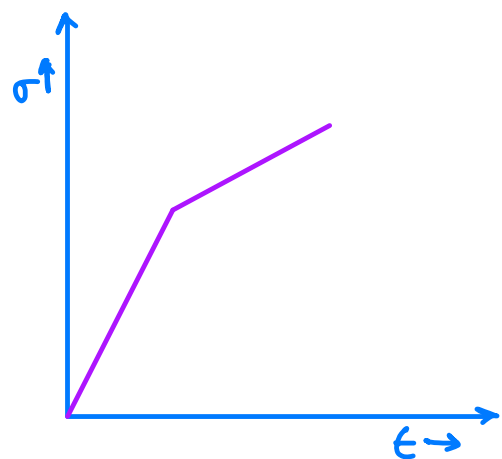


Non-linear Elastic.

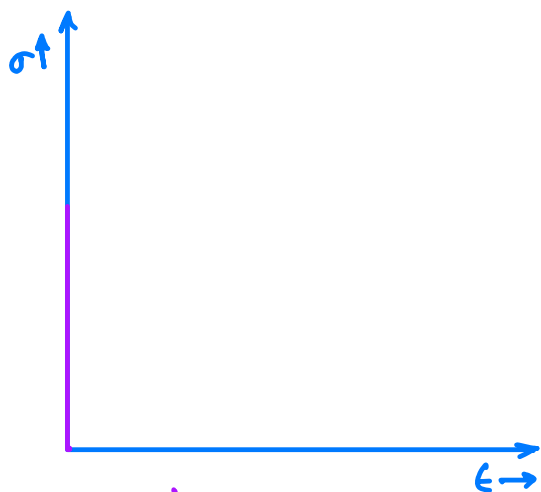
{eg:- Concrete, rubber}



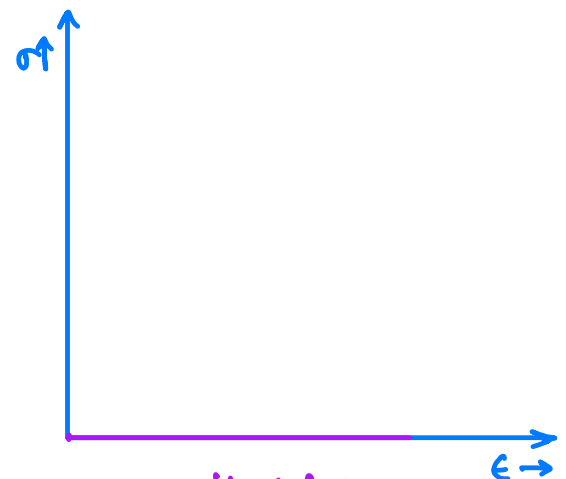
Elasto - plastic
{ eg:- Aluminium }



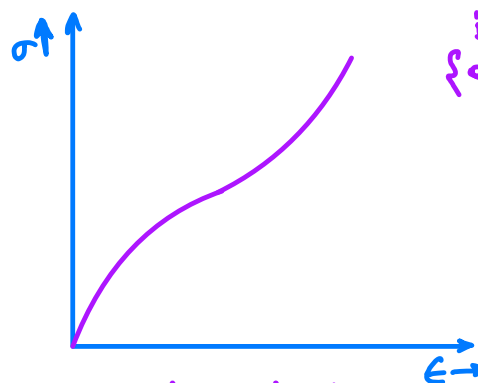
Elasto - plastic with
strain hardening.
{ eg:- Aluminium Alloy }



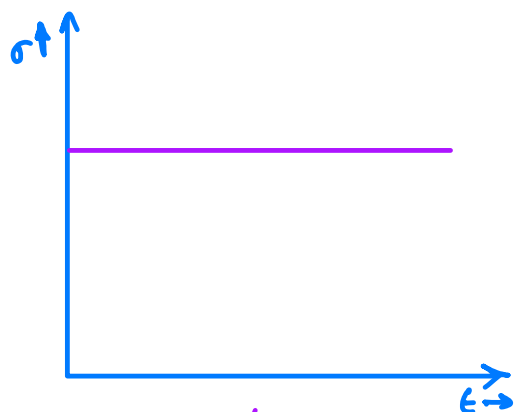
ideal rigid
{ eg:- Ball bearings made up
of hardened steel }



ideal fluid.
{ eg:- water }.



visco-elastic
{ eg:- Bitumen }



ideal/perfectly plastic
{ eg:- mud, putty, wax }

Note:-

— These are the materials which possess both viscosity and elasticity along with time dependent strain.

(Elasticity is due to stretching of Bonds b/n Particles whereas viscosity is due to Slipping of Particles over each other).

— These materials also possess

a.) Hysteresis

b.) Stress relaxation

c.) Creep

Hooke's Law

— According to this law for homogeneous isotropic and linear elastic material stress is directly proportional to strain.

$$\sigma \propto \epsilon$$

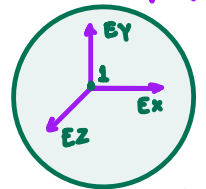
$$\sigma = E \epsilon$$

here E = modulus of Elasticity.

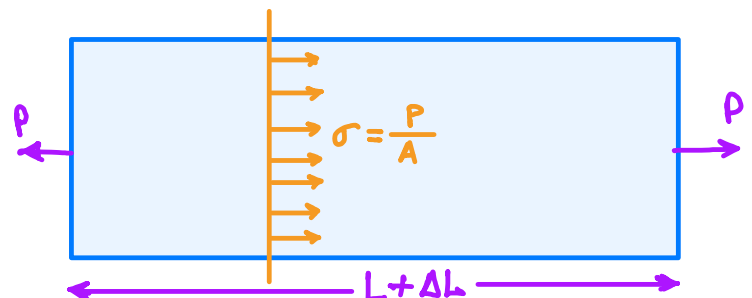
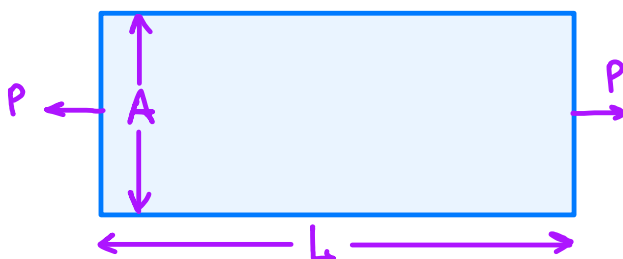
Note:-

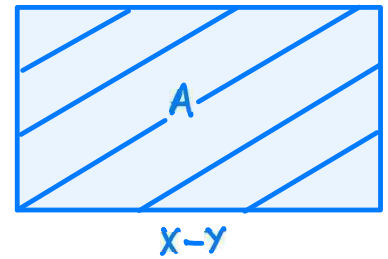
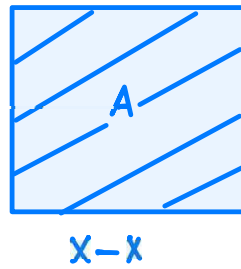
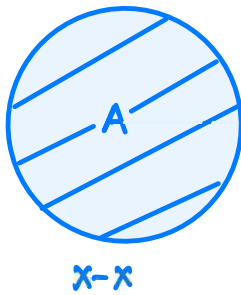
Along with isotropic, orthotropic & Anisotropic material also exist.

Orthotropic material :- It is the type of material which possess different elastic properties at any given point in perpendicular direction.

Anisotropic material

:- It is the type of material which have different elastic properties at any given point in all directions.
eg:- materials having cracks in it.

Deformation of member under Axial loadCase-1: Bar of uniform section (Prismatic bar)



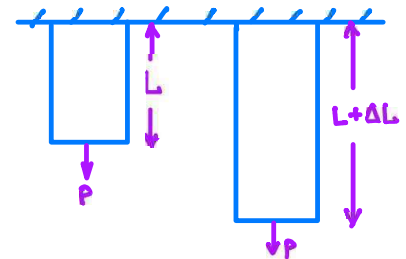
$$\sigma = \frac{P_m}{A}, \quad \epsilon = \frac{L_f - L_i}{L_i} = \frac{L + \Delta L - L}{L} = \frac{\Delta L}{L}$$

→ As per Hooke's Law,

$$\sigma = E \epsilon$$

$$\epsilon = \frac{\sigma}{E}$$

$$\frac{\Delta L}{L} = \frac{P}{A \cdot E}, \quad \boxed{\Delta L = \frac{PL}{AE}}$$



Note :-

- Stiffness is defined as property of material to resist deformation against the external loading.
- This is complementary of flexibility/pliability.
- Stiffness may also be termed as force required to produce unit deformation in material.

$$\text{Stiffness} = \frac{\text{force (P)}}{\text{deformation } (\Delta L)}$$

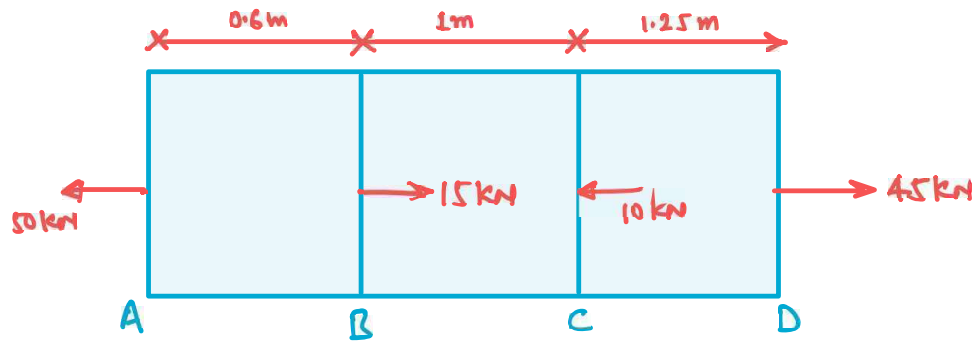
$$\text{Stiffness} = \frac{P}{\Delta L} = \frac{AE}{L}$$

- Flexibility may also be defined as deformation produced by unit loading.

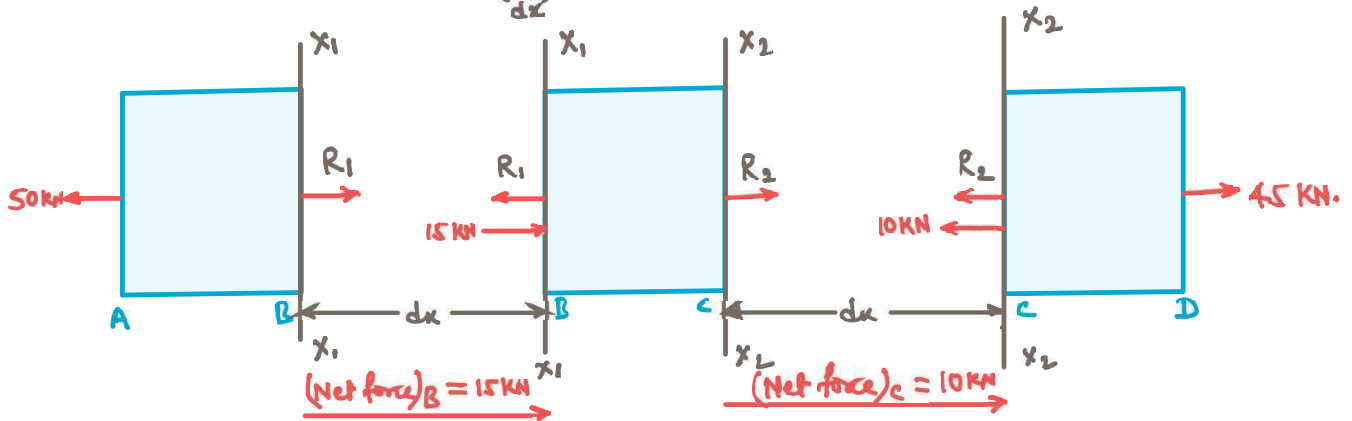
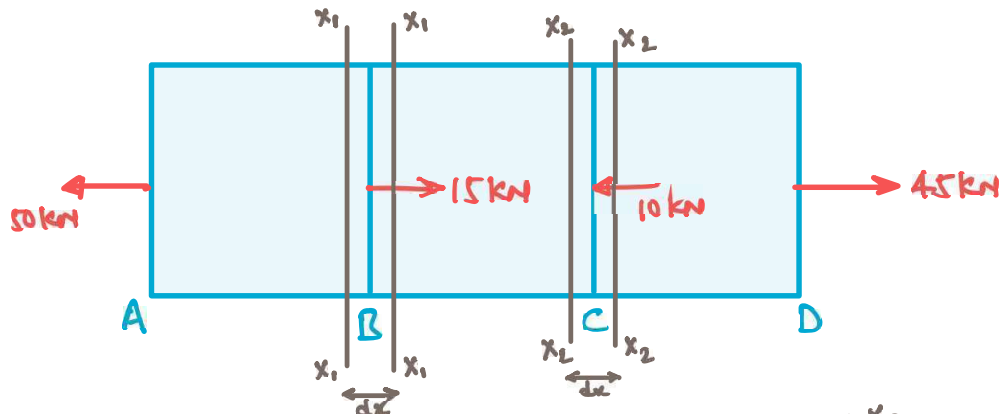
$$\text{flexibility} = \frac{\text{Deformation } (\Delta L)}{\text{force (P)}}$$

$$\text{flexibility} = \frac{\Delta L}{P} = \frac{L}{AE}$$

Q for the given bar compute the total deformation under the given condition of loading:-
 Area = 500 mm^2 , $E = 200 \text{ GPa}$.



Soln



for segment AB

$$\sum f_x = 0$$

$$50 - R_1 = 0$$

$$\boxed{R_1 = 50} \text{ kN}$$

$$\Delta_{AB} = \frac{R_1 L_{AB}}{AE}$$

for segment BC, $\sum f_x = 0$

$$R_1 - 15 - R_2 = 0$$

$$R_2 = R_1 - 15$$

$$\boxed{R_2 = 50 - 15 = 35} \text{ kN}$$

$$\Delta_{BC} = \frac{R_2 L_{BC}}{AE}$$

for segment CD

$$\Delta_{CD} = \frac{R_3 L_{cd}}{AE}, \quad R_3 = 45 \text{ kN.}$$

MPa - N/mm²
GPa - KN/mm²

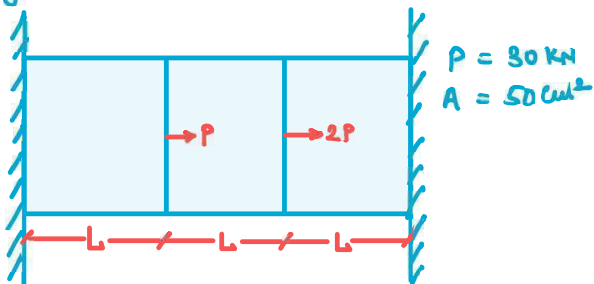
$$\Delta_{Total} = \Delta_{AB} + \Delta_{BC} + \Delta_{CD}$$

$$= \frac{1}{AE} [R_1 L_{AB} + R_2 L_{BC} + R_3 L_{CD}]$$

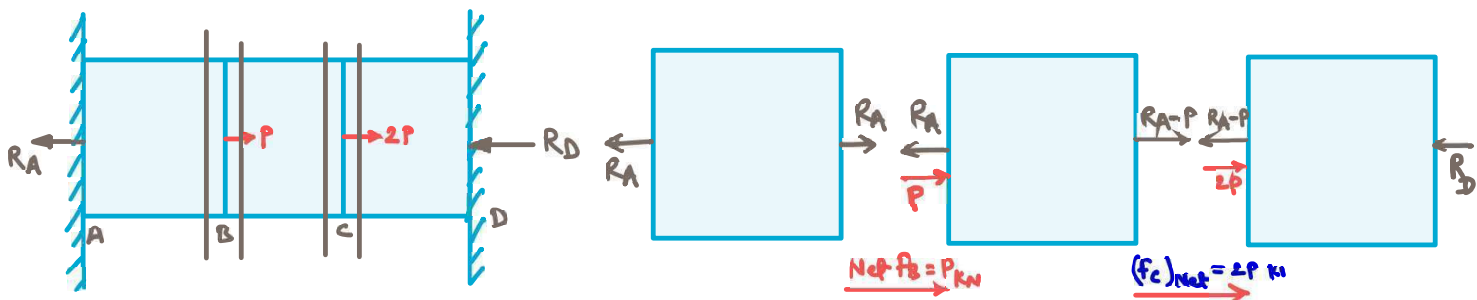
$$= \frac{1}{500 \times 200} [50 \times 0.6 \times 10^3 + 35 \times 1 \times 10^3 + 45 \times 1.25 \times 10^3]$$

$$\Delta_{Total} = 1.215 \text{ mm}$$

Q. For the given system draw the axial force diagram:-



Soln



$$R_A + R_D - P - 2P = 0 \quad \leftarrow (+) \quad \rightarrow (-)$$

$$R_A + R_D = 3P = 3 \times 30 = 90 \text{ kN.} \quad (i)$$

for segment AB:-

$$\Delta_{AB} = \frac{R_A L}{AE} \quad (\text{extension})$$

for segment BC:-

$$\Delta_{BC} = \frac{(R_A - P) \cdot L}{AE} \quad (\text{extension})$$

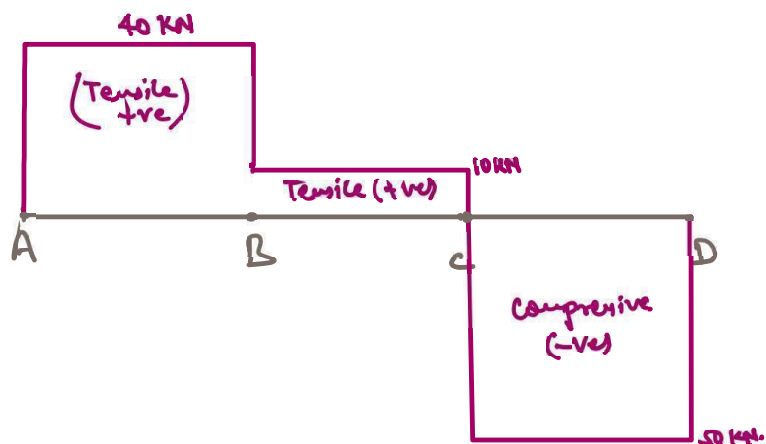
for segment CD:-

$$\sum F_x = 0$$

$$R_A - P + R_D - 2P = 0.$$

$$R_A + R_D = 3P \Rightarrow R_D = 3P - R_A$$

$$\Delta_{CD} = \frac{-R_D L}{AE} \quad (\text{Compression})$$



from eqn (i) & (ii)

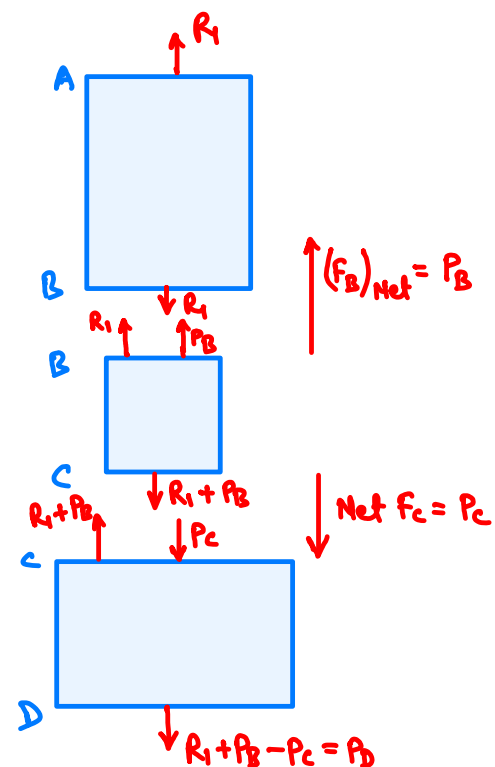
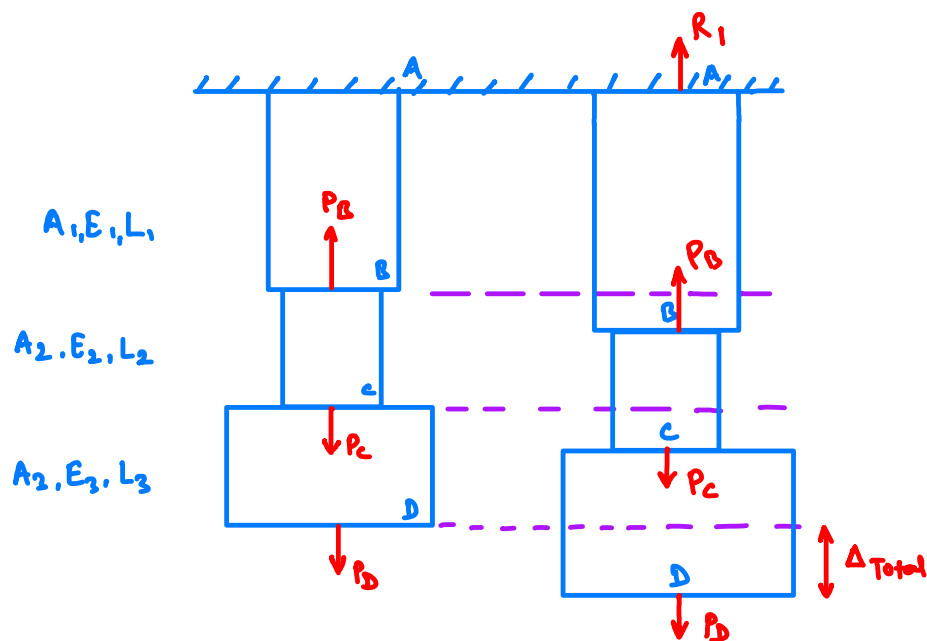
$$R_A = 40 \text{ kN}$$

$$R_D = 50 \text{ kN}$$

$$\Delta_{Total} = \Delta_{AB} + \Delta_{BC} + \Delta_{CD} = 0 \quad (\text{since both ends are fixed})$$

$$\frac{L}{AE} [R_A + R_A - P - R_D] = 0 \Rightarrow 2R_A - R_D = P = 30 \text{ kN} \quad (ii)$$

Case-2: Stepped Bar



For segment CD, $\sum F_y = 0$.

$$R_1 + P_B - P_C - P_D = 0$$

$$R_1 = P_D + P_C - P_B \quad \dots (i)$$

$$\Delta_{CD} = \frac{P_D L_3}{A_2 E_2}$$

for segment BC, $\Delta_{BC} = \frac{(R_1 + P_B) L_2}{A_2 E_2}$

$$\Delta_{BC} = \frac{(P_D + P_C) L_2}{A_2 E_2}$$

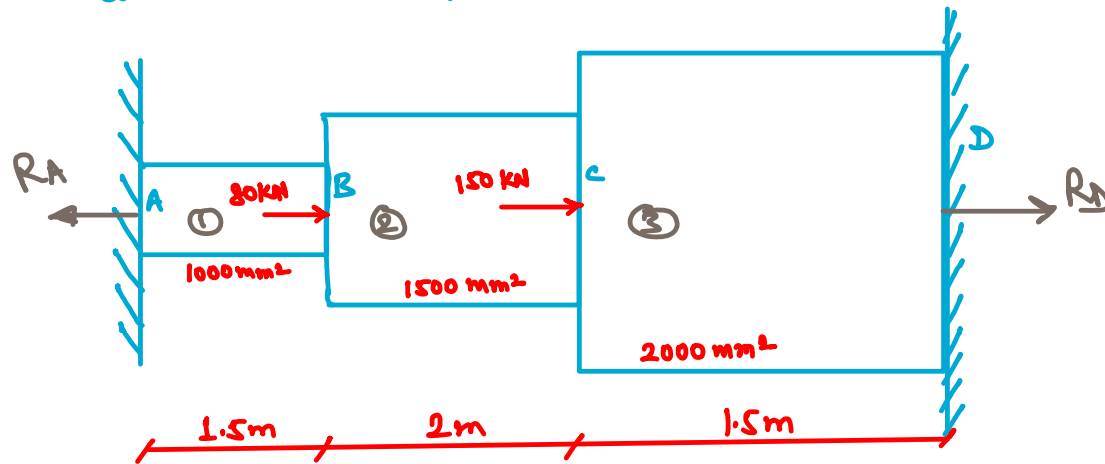
for segment AB, $\Delta_{AB} = \frac{R_1 L_1}{A_1 E_1}$

$$\Delta_{AB} = \frac{(P_D + P_C - P_B) \cdot L_1}{A_1 E_1}$$

$$\Delta_{Total} = \Delta_{AB} + \Delta_{BC} + \Delta_{CD}$$

Q.7 A circular bar ABCD is rigidly fixed at A & D and is subjected to axial forces as shown:-
Determine the reactions, the force in each portion of the bar and the displacement of point B & C.

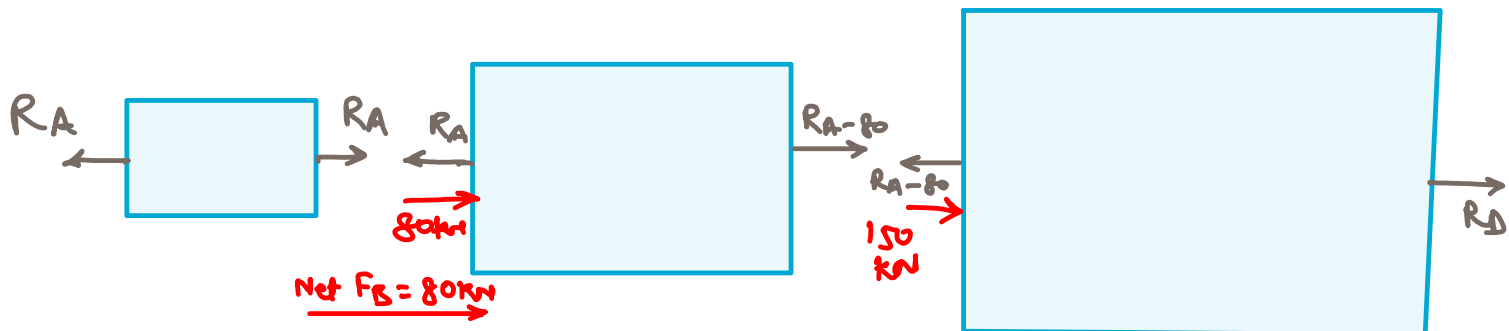
Consider the value of $E = 200 \text{ kN/mm}^2$



for equilibrium of the bar, $\sum F_x = 0$. $\rightarrow (+)$
 $\leftarrow (-)$

$$80 + 150 - R_A + R_D = 0$$

$$R_A - R_D = 230 \dots\dots (1)$$



for Segment AB, $\Delta_{AB} = \frac{R_A L_1}{A_1 E}$

for segment BC, $\Delta_{BC} = \frac{(R_A - 80) L_2}{A_2 E}$

for Segment CD, $\sum F_x = 0$
 $150 + R_D - (R_A - 80) = 0$
 $R_A - R_D = 230 \dots\dots (i)$

$$\Delta_{CD} = \frac{(R_A - 230) L_3}{A_3 E}$$

$$\Delta_{Total} = \Delta_{AB} + \Delta_{BC} + \Delta_{CD} = 0 \quad \left\{ \begin{array}{l} \text{both ends} \\ \text{A \& D are fixed} \end{array} \right\}$$

$$\frac{R_A L_1}{A_1 E} + \frac{(R_A - 80) L_2}{A_2 E} + \frac{(R_A - 230) L_3}{A_3 E} = 0$$

$$R_A = 77.91$$

$$R_D = R_A - 230 = 77.91 - 230 = -152.09 \text{ kN}$$

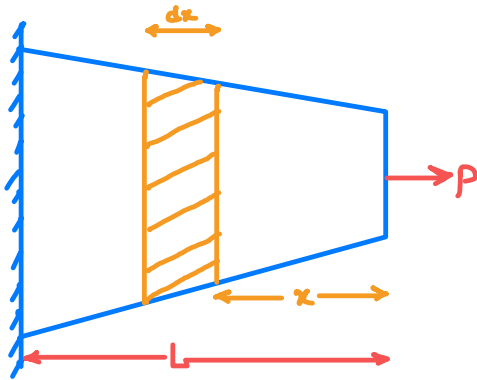
ii) displacement of joint B = $\Delta_{AB} = \frac{R_A L_1}{A_1 E}$
 $= 0.5832 \text{ mm} \rightarrow$

iii) Displacement of joint C = $\Delta_{CD} = \frac{R_D L_3}{A_3 E}$

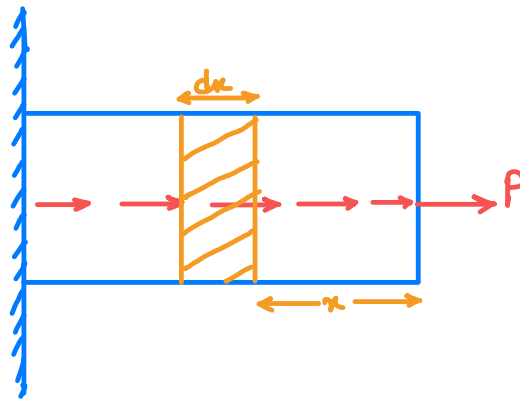
$= -0.57 \text{ mm} \leftarrow$
 $= 0.57 \text{ mm} \rightarrow$

Displacement of C = $\Delta_{AB} + \Delta_{BC}$
 $= \frac{R_A L_1}{A_1 E} + \frac{(R_A - 80) L_2}{A_2 E}$
 $= 0.57 \text{ mm} \rightarrow$

Case-3 Bars of varying cross-section / Bar with varying force



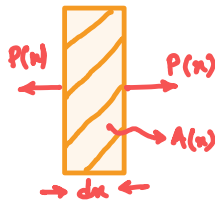
a) Area Varying



b) force varying.

$$d\delta_{\text{Strip/Element}} = \frac{P(x) dx}{A(x) E}$$

$$\Delta_{\text{Total}} = \int_0^L d\delta_{\text{Strip}} = \int_0^L \frac{P(x) dx}{A(x) E}$$



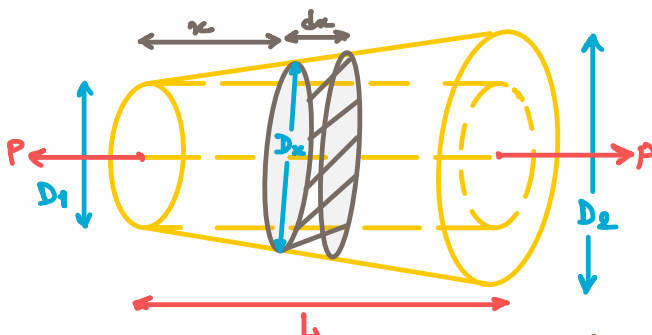
★ If Area is variable & force is constant (a)

$$\Rightarrow \Delta = \int_0^L \frac{P dx}{A(x) E}$$

★ If Area is constant & force is variable (b)

$$\Rightarrow \Delta = \int_0^L \frac{P(x) dx}{A E}$$

Eg:- find $\Delta = ?$



over the length of Lm, change in dia = $D_2 - D_1$

" " 1m " " = $\frac{D_2 - D_1}{L}$

xm " " = $\frac{D_2 - D_1}{L} x$

$$D_x = D_1 + (D_2 - D_1) \frac{x}{L} = D_1 + cx, \text{ Here } c = \frac{D_2 - D_1}{L}$$

$$\delta_{\text{Strip}} = \frac{P dx}{A_x E} = \frac{P dx}{\frac{\pi d^2}{4} E} \Rightarrow \frac{4P dx}{\pi (D_1 + cx)^2 E}$$

$$\Delta = \int_0^L \delta_{\text{Strip}} = \int_0^L \frac{4P dx}{\pi E (D_1 + cx)^2} \Rightarrow \frac{4P}{\pi E} \left[\frac{-1}{(D_1 + cx)c} \right]_0^L \Rightarrow \frac{4P}{\pi E} \left[\frac{1}{(D_1 + cx)c} \right]_L^0$$

$$\Delta = \frac{4P}{c \pi E} \left[\frac{1}{D_1} - \frac{1}{(D_1 + cL)} \right] \Rightarrow \frac{4P}{(D_2 - D_1) \pi E} \left[\frac{1}{D_1} - \frac{1}{D_1 + \frac{(D_2 - D_1)L}{L}} \right]$$

$$\Delta = \frac{4PL}{(D_2 - D_1) \pi E} \left[\frac{1}{D_1} - \frac{1}{D_2} \right] \Rightarrow \frac{4PL (D_2 - D_1)}{(D_2 - D_1) \pi E D_1 D_2} \Rightarrow \Delta = \frac{4PL}{\pi E D_1 D_2}$$

Eg:- 2 → Rectangular section.

find Δ .

$$\delta_{\text{strip}} = \frac{P dx}{A(x) E}$$

$$= \frac{P dx}{(b_x t) E}$$

$$\text{Here, } b_x = b_2 + (b_1 - b_2) \frac{x}{L}$$

$$= b_2 + cx$$

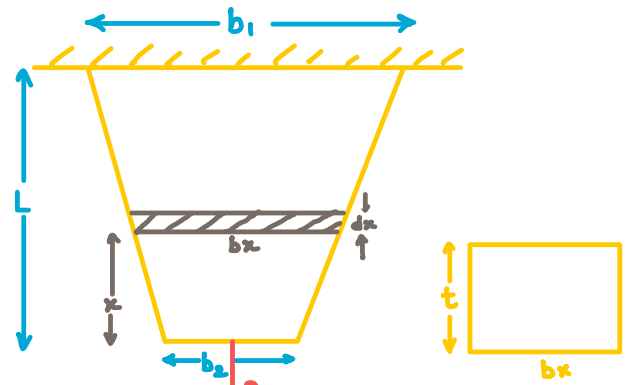
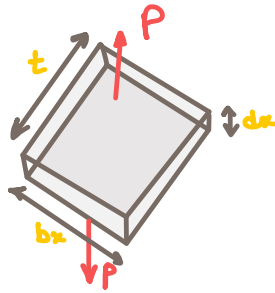
$$\text{Here } c = \frac{b_1 - b_2}{L}$$

$$\delta_{\text{strip}} = \frac{P dx}{(b_2 + cx) t E}$$

$$\Delta = \int_0^L \delta_{\text{strip}} = \int_0^L \frac{P dx}{(b_2 + cx) t E} \Rightarrow \frac{P}{t E} \left| \frac{\ln(b_2 + cx)}{c} \right|_0^L$$

$$= \frac{P}{t E c} [\ln(b_2 + cL) - \ln b_2] \Rightarrow \frac{P}{t E \frac{(b_1 - b_2)}{L}} \left[\frac{\ln \left\{ b_2 + \frac{(b_1 - b_2) \cdot L}{L} \right\}}{b_2} \right]$$

$$\Delta = \frac{PL}{t E (b_1 - b_2)} \ln(b_1/b_2)$$



Case - 4 Deformation due to Self weight

a) Rectangular Bar

$$\delta_{\text{strip}} = \frac{P(x) dx}{A E}$$

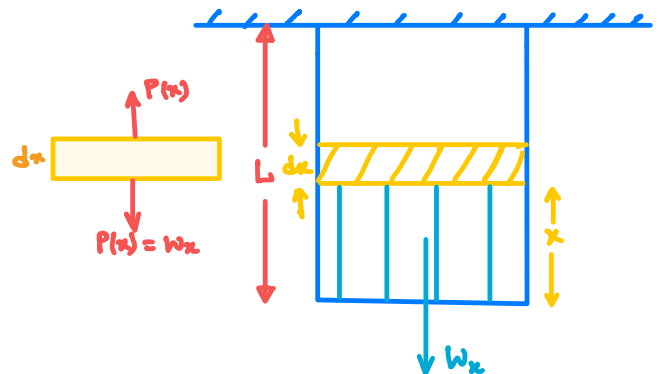
$$P(x) = W_x = V_x \gamma \quad \gamma \rightarrow \text{unit wt. of material}$$

$$P(x) = A \cdot x \cdot \gamma$$

$$\delta_{\text{strip}} = \frac{A \cdot x \cdot \gamma dx}{A E} = \frac{x \gamma dx}{E}$$

$$\Delta = \int_0^L \delta_{\text{strip}} = \int_0^L \frac{x \gamma dx}{E} = \frac{\gamma}{E} \left| \frac{x^2}{2} \right|_0^L$$

$$\Delta = \frac{\gamma L^2}{2 E}$$



Note :-

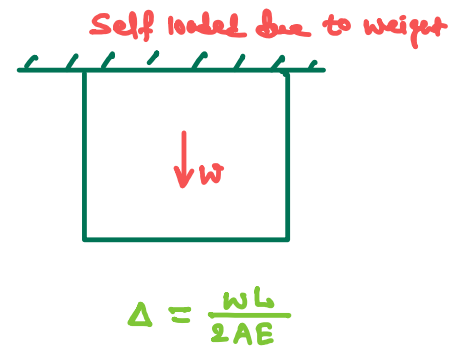
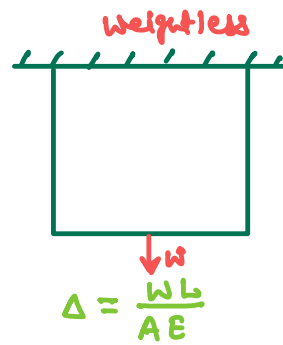
- If along the length of bar, cross-sectional Area is constant [i.e. bar is prismatic] deformation due to Self weight would be $\frac{\gamma L^2}{2 E}$ irrespective of geometry of bar [i.e circular, square, rectangular or any other geometry]

$$\text{Here } \gamma = \frac{W}{V} = \frac{W}{A L}$$

Hence, $\Delta = \frac{W}{AL} \cdot \frac{L^2}{2E}$

$$\Delta = \frac{WL}{2AE}$$

- Deformation in the bar due to self weight is half the deformation due to applied load equal self-weight.



b. > Conical bar

$$\delta_{\text{strip}} = \frac{P(x) dx}{A(x) E}$$

$$P_x = W_x = V_x \gamma = \frac{1}{3} A_x \cdot x \cdot \gamma$$

$$\delta_{\text{strip}} = \frac{\frac{1}{3} A_x \cdot x \cdot \gamma \cdot dx}{A_x \cdot E}$$

$$= \frac{\gamma x dx}{3E}$$

$$\Delta = \int_0^L \delta_{\text{strip}}$$

$$= \int_0^L \frac{\gamma x dx}{3E}$$

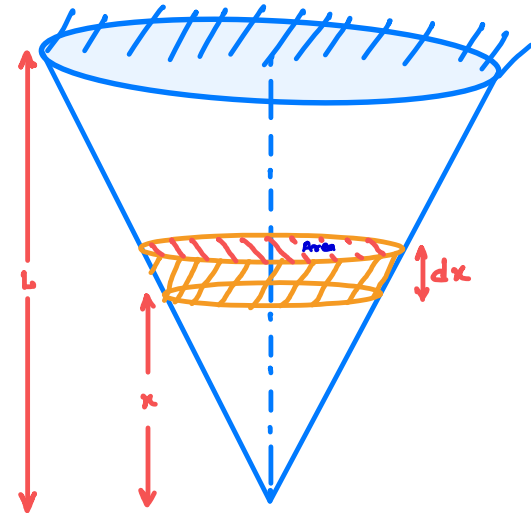
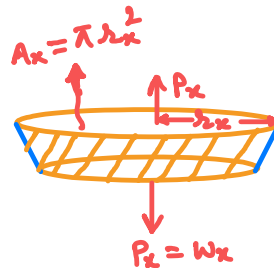
$$\Rightarrow \frac{\gamma}{3E} \left| \frac{x^2}{2} \right|_0^L$$

$$\Delta = \frac{\gamma L^2}{6E}$$

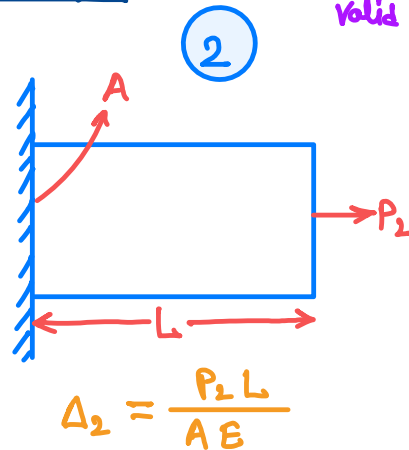
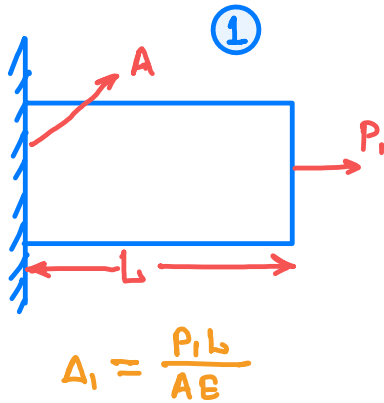
$$\text{or, } \gamma = \frac{W}{V} = \frac{W}{\frac{1}{3} AL} = \frac{3W}{AL}$$

$$\Delta = \frac{3WL^2}{AL6E}$$

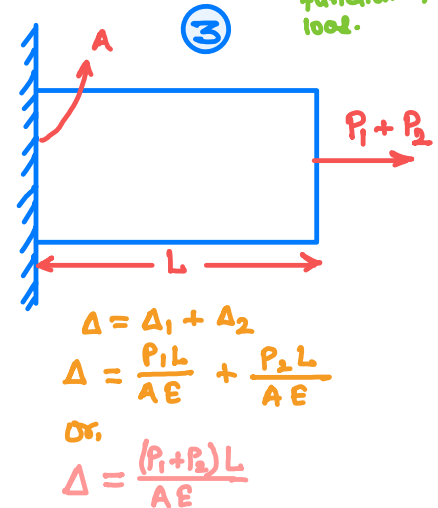
$$\Delta = \frac{WL}{2AE}$$



#

Superposition Theorem

Valid only for linear function → Here, deformation is our linear function of load.



Here, $[\Delta \propto P]$

→ This principle states that for all the linear systems, net response caused by two or more factors is the sum of responses of these factors taken individually

★ Strain Energy

$$u_1 = \frac{1}{2} P_1 \Delta_1, \quad u_2 = \frac{1}{2} P_2 \Delta_2, \quad u_3 = \frac{1}{2} (P_1 + P_2) \Delta$$

$$= \frac{P_1^2 L}{2AE}, \quad = \frac{P_2^2 L}{2AE}, \quad = \frac{(P_1 + P_2)^2 L}{2AE}$$

Here, $u_1 + u_2 \neq u_3$ [$u_3 > u_1 + u_2$]
as strain energy is not the linear function of load. ($u \propto P^2$)

Case - 5 Deformation in composite section

$$P_1 + P_2 = P \quad \text{--- (i)}$$

Total load applied by us is resisted by both materials.

for composite section,

$$\Delta_1 = \Delta_2 \quad \text{--- (ii)}$$

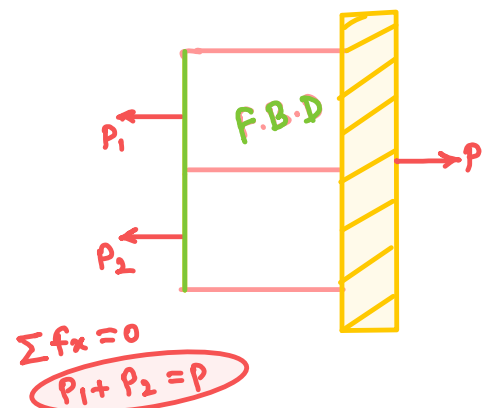
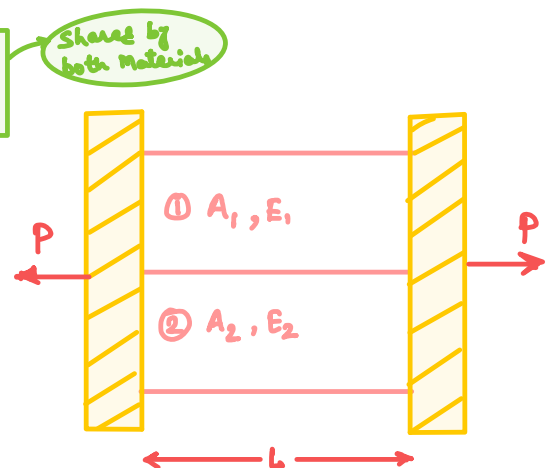
$$\frac{P_1 L}{A_1 E_1} = \frac{P_2 L}{A_2 E_2}$$

$$P_2 = \frac{P_1 A_2 E_2}{A_1 E_1}$$

Substituting in equation (i)

$$P_1 + \frac{P_1 A_2 E_2}{A_1 E_1} = P$$

$$P_1 \left[\frac{A_1 E_1 + A_2 E_2}{A_1 E_1} \right] = P$$



$$P_1 = \frac{P A_1 E_1}{A_1 E_1 + A_2 E_2}$$

$$P_2 = \frac{P A_2 E_2}{A_1 E_1 + A_2 E_2}$$

$$\Delta_1 = \Delta_2 = \Delta = \frac{P_1 L}{A_1 E_1} = \frac{P_2 L}{A_2 E_2}$$

$$\Delta = \frac{PL}{A_1 E_1 + A_2 E_2}$$

Note:-

If this Composite section is replaced by another section of same length subjected to same load and area equal to area of these two section for deformation to be same what must be equivalent modulus of elasticity of this section.

$$\Delta' = \frac{PL}{(A_1 + A_2) E_{eq}}$$

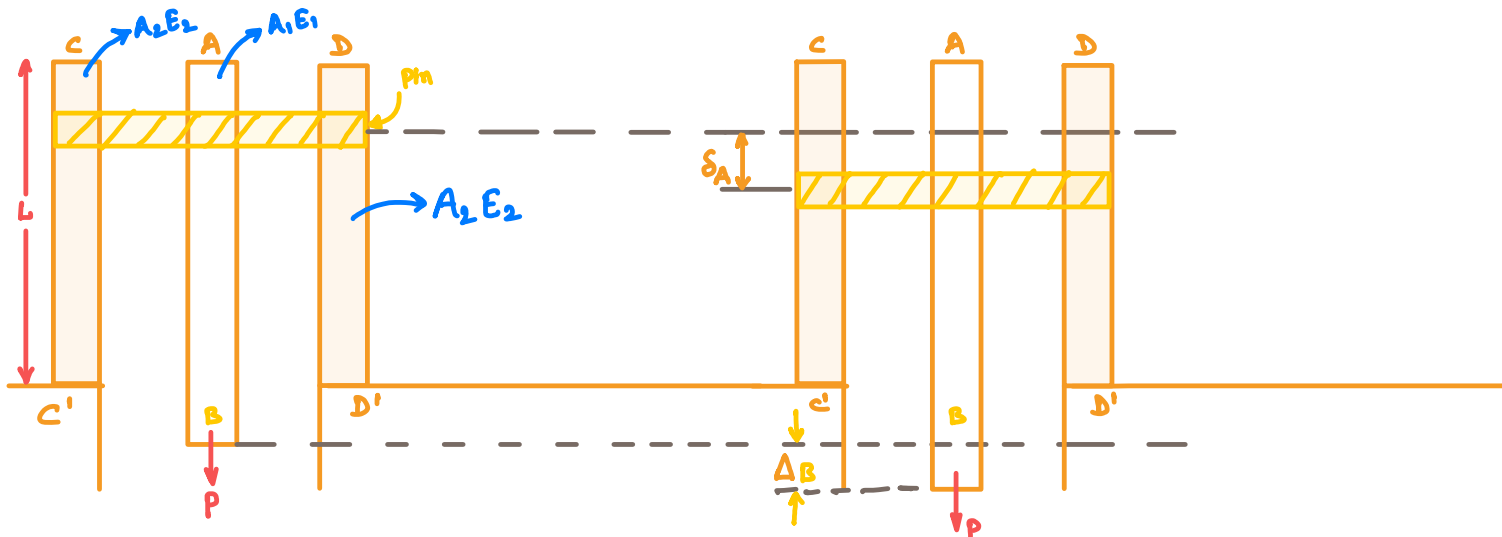
$$\Delta' = \Delta$$

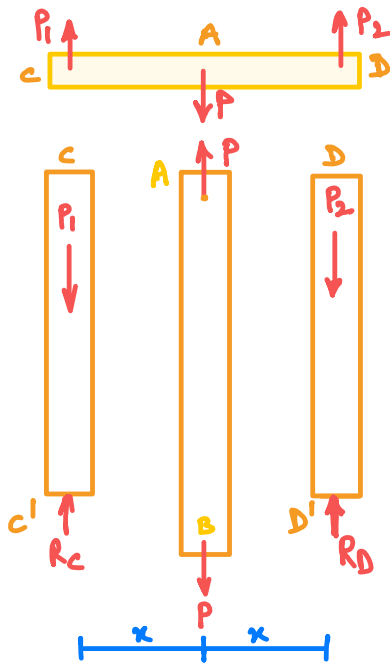
$$\frac{PL}{(A_1 + A_2) E_{eq}} = \frac{PL}{A_1 E_1 + A_2 E_2}$$

$$E_{eq} = \frac{A_1 E_1 + A_2 E_2}{A_1 + A_2} \quad *$$



Case - 6:- Member subjected to Rigid body Movement





for pin ACD,

$$\sum f_y = 0 \quad \uparrow (+), \downarrow (-)$$

$$P_1 + P_2 - P = 0$$

$$P_1 + P_2 = P \quad \dots (i)$$

$$\sum M_A = 0$$

$$P_1 x - P_2 x = 0$$

$$P_1 = P_2 \quad \dots (ii)$$

$$P_1 = P_2 = \frac{P}{2}$$

for member CC', $\sum f_y = 0$.

$$R_C - P_1 = 0$$

$$R_C = P_1 = \frac{P}{2}$$

for member DD', $\sum f_y = 0$

$$R_D - P_2 = 0$$

$$R_D = P_2 = \frac{P}{2}$$

Compression of C and D, $\delta_A = \frac{(P/2)L}{A_2 E_2}$

Elongation of AB, $\delta_{AB} = \frac{PL}{A_1 E_1}$

Since here point A undergoes rigid body movement of δ_A , Total displacement of point B

$$\Delta B = \delta_A + \delta_{AB}$$

- from here it can be stated that rigid body movement/displacement does not have any effect over the elongation of the bar.

Q1)

A long Rectangular Copper bar under a tensile load "P" hangs from a pin that is supported by two steel posts. The copper bar has a length of 2m. Cross-section area of 4800 mm^2 . Modulus of Elasticity $E = 120 \text{ GPa}$, for each steel post height is 0.5 m and C.S.A 4500 mm^2
 $E = 200 \text{ GPa}$

Determine,

- Downward displacement of lower end of Copper bar due to loading of 180 kN ?
- Maximum permissible load if the displacement is limited to 1 mm ?

Soln

$$\Delta_B = \delta_A + \delta_{AB}$$

δ_A = rigid body displacement of point A.

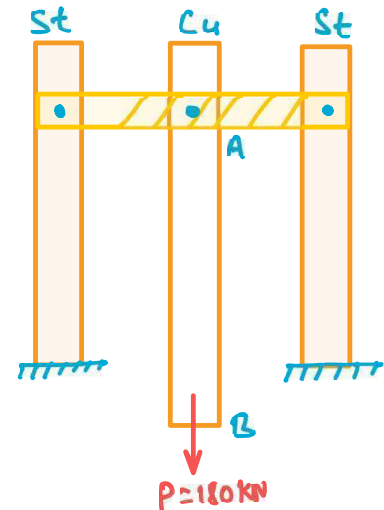
= Compression of Steel post

$$\delta_A = \frac{(P/2) L_{st}}{A_{st} E_{st}} = \text{---} = 0.05 \text{ mm.}$$

δ_{AB} = Elongation of bar AB

$$\delta_{AB} = \frac{P L_{cu}}{A_{cu} E_{cu}} = \text{---} = 0.625 \text{ mm.}$$

$$\Delta_B = 0.05 + 0.625 = \underline{0.675 \text{ mm}}$$



$$\text{ii)} \quad \Delta_B \propto P$$

$$\frac{\Delta_{B_{max}}}{\Delta_B} = \frac{P_{max}}{P}$$

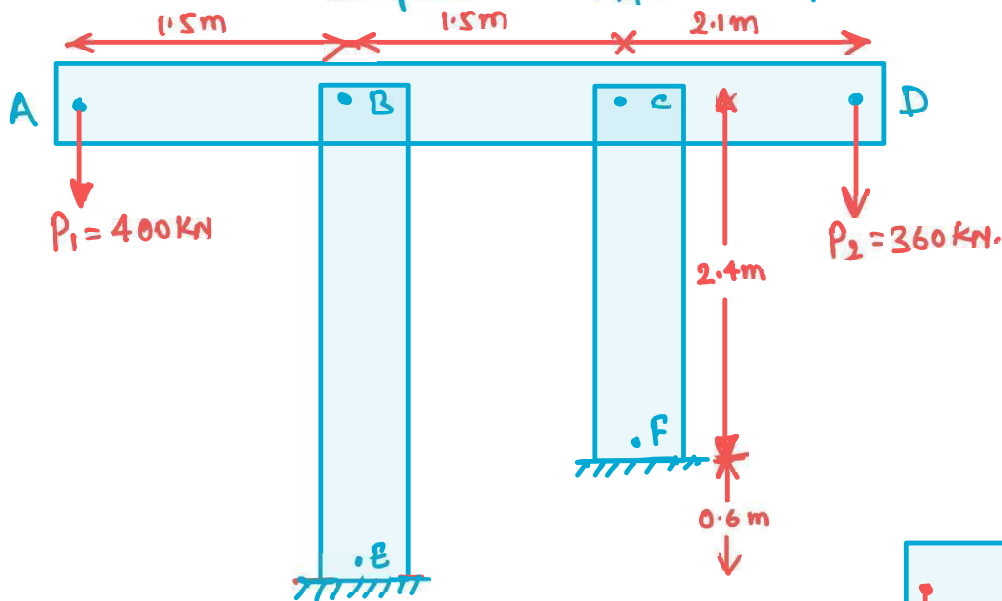
$$P_{max} = \frac{\Delta_{B_{max}} \cdot P}{\Delta_B}$$

$$= \frac{1 \times 180}{0.675}$$

$$\underline{P_{max} = 266.67 \text{ kN}}$$



The horizontal rigid beam ABCD is supported by vertical bars BE and CF and is loaded by a vertical forces at the ends of magnitude 400 kN and 360 kN acting at point A & D respectively. Bar BE & CF are made up of steel $E = 200 \text{ GPa}$ and C.S.A of BE is 11000 mm^2 & CF is 9300 mm^2 & bars are arranged as shown. Compute the displacement of A & D.



Soln

for rigid beam ABCD

$$\sum M_C = 0 \quad \curvearrowright$$

$$P_{BE} \times 1.5 + P_2 \times 2.1 - P_1 \times (1.5 + 1.5) = 0.$$

$$P_{BE} = 296 \text{ kN}$$

$$\sum M_B = 0.$$

$$P_2 \times (1.5 + 2.1) - P_{CF} \times 1.5 - P_1 \times 1.5 = 0.$$

$$P_{CF} = 464 \text{ kN}.$$

$$\textcircled{6} \sum f_y = 0 \Rightarrow P_{BE} + P_{CF} - P_1 - P_2 = 0. \quad \uparrow + \downarrow -$$

$$P_{CF} = 464 \text{ kN}$$

for steel post BE, $\sum f_y = 0.$

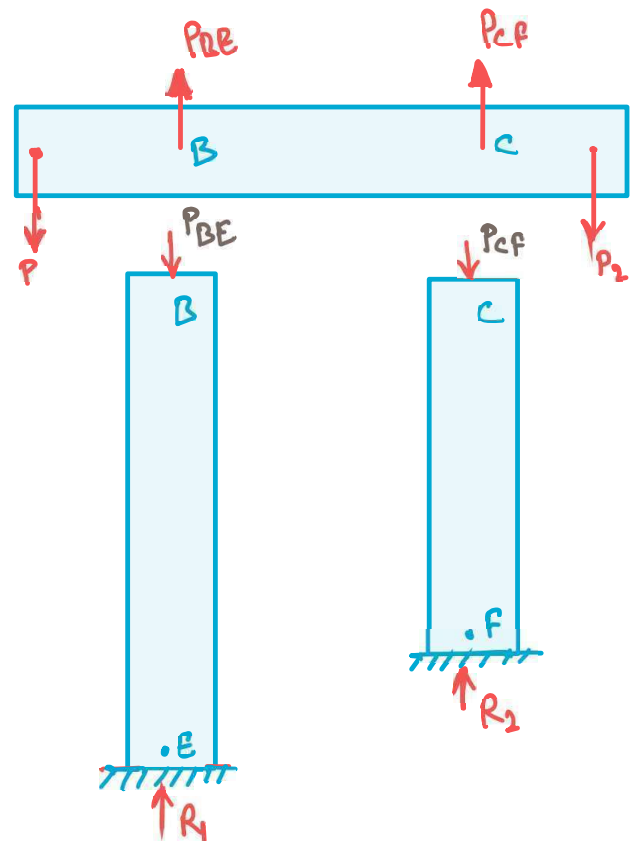
$$R_1 - P_{BE} = 0 \Rightarrow R_1 = P_{BE} = 296 \text{ kN}.$$

$$\Delta_{BE} = \frac{P_{BE} L_{BE}}{A_{BE} E} = \frac{296 \times 2.4}{11000 \times 200 \times 10^9} = 0.4 \text{ m (comp.)}$$

for steel post CF, $\sum f_y = 0.$

$$R_2 - P_{CF} = 0 \Rightarrow R_2 = P_{CF} = 464 \text{ kN}.$$

$$\Delta_{CF} = \frac{P_{CF} L_{CF}}{A_{CF} E} = \frac{464 \times 0.6}{9300 \times 200 \times 10^9} = 0.67 \text{ (comp.)}$$



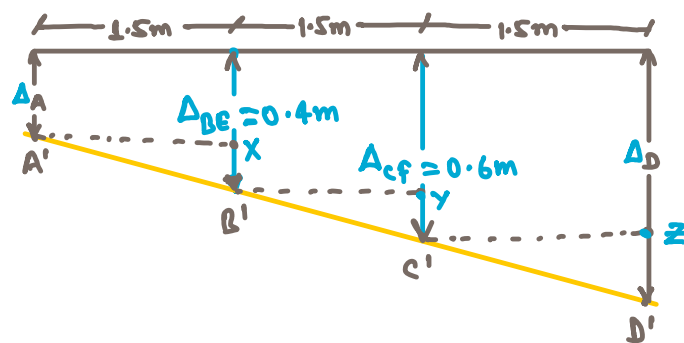
Triangle $\Delta A'B'C' \sim \Delta B'C'D'$

$$\frac{EB'}{f_{C'}} = \frac{A'E}{B'F} \Rightarrow \cancel{XB'} = \cancel{YC'} = \Delta_{CF} - BE$$

$$\cancel{XB'} = 0.2 \text{ mm}$$

$$\Delta_A = \Delta_{BE} - \cancel{XB'}$$

$$\Delta_A = 0.4 - 0.2 = 0.2 \text{ mm}$$



$\Delta B'C'D' \sim \Delta C'D'ZD'$

$$\frac{ZD'}{YC'} = \frac{ZC'}{YB'} \Rightarrow ZD' = \frac{ZC' \cdot YC'}{YB'} = \frac{2.1 \times 0.2}{1.5}$$

$$ZD' = 0.28 \text{ mm}$$

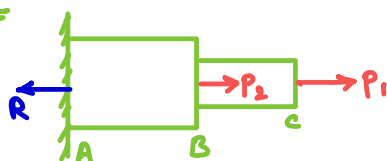
$$\Delta_D = \Delta_{CF} + ZD' = 0.6 + 0.28$$

$$\Delta_D = 0.88 \text{ mm}$$

Case-7 Statically indeterminate case

- If the forces in different components of the member in a structure is computed by the equation of statics ($\sum F = 0, \sum M = 0$). It is termed as **Statically determinate structures**.

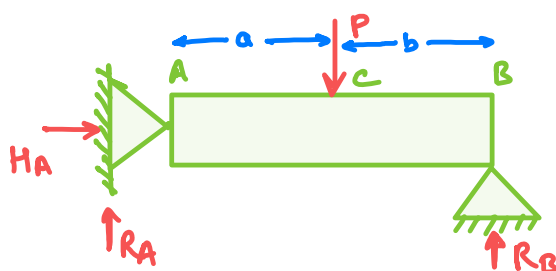
Eg:-



$$\sum F_x = 0 \Rightarrow R = P_1 + P_2$$

$$F_{AB} = R = P_1 + P_2$$

$$F_{BC} = P_1$$



$$\sum F_x = 0$$

$$H_A = 0$$

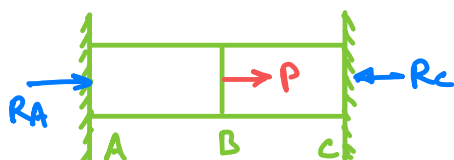
$$\sum F_y = 0 \Rightarrow R_A + R_B = P$$

$$\sum M_A = 0 \Rightarrow R_B = \frac{Pa}{a+b}$$

- If however the forces can't be determined by equation of statics it is termed as **indeterminate structures**.

- In such cases, to find the additional unknown forces **compatibility equation** is used.

Eg:-



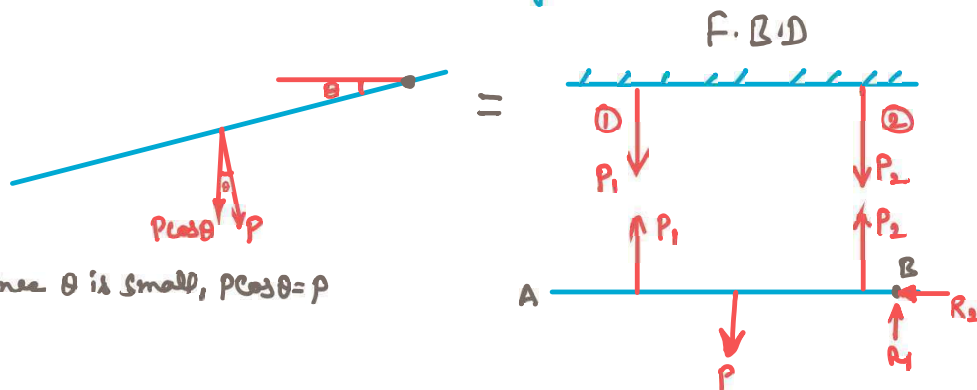
$$F_x = 0$$

$$R_A + P - R_C = 0$$

Since in this case two forces are unknown R_A & R_C and one equation of static is available ($F_x = 0$), it is indeterminate by 1 degree. $(2 - 1 = 1)$

Hence, here compatibility eqⁿ $\Delta_{Total} = 0$ is also to be used.

Q. A bar is hanging from a ceiling from two wires. The bar is provided with a hinge at a distance of "a" from the right wire "b" from the right wire. A load P is applied at the bar at a distance of "L" from the hinge. Find the force developed in the both wires.



Since θ is small, $P \cos \theta = P$

for bar AB:

$$\sum F_x = 0 \Rightarrow R_2 = 0$$

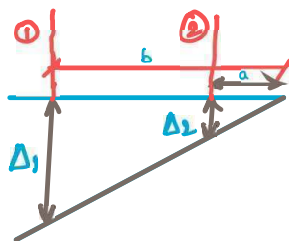
$$\sum F_y = 0 \Rightarrow P_1 + P_2 + R_1 = P$$

$$\sum M_B = 0 \Rightarrow P_1 b + P_2 a = PL \dots (i)$$

$$\tan \theta = \frac{\Delta_1}{b} = \frac{\Delta_2}{a}$$

$$\frac{P_1 L w}{A_w E_w b} = \frac{P_2 L w}{A_w E_w a}$$

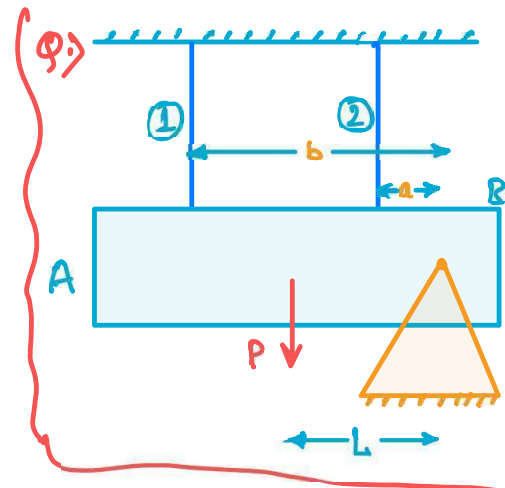
$$P_1 a - P_2 b = 0 \dots (ii)$$



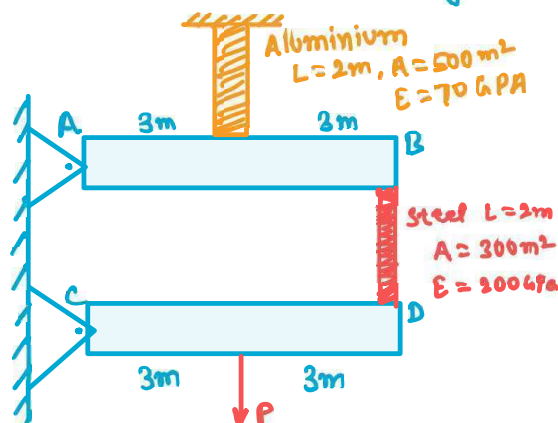
from (i) & (ii),

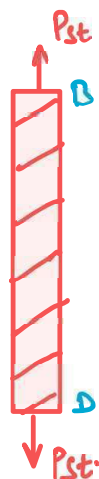
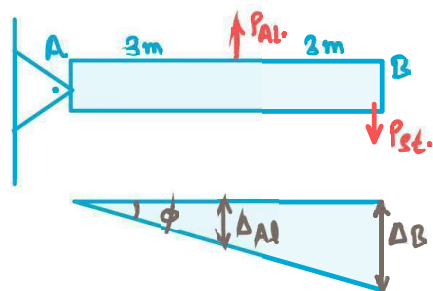
$$P_1 = \frac{P b b}{a^2 + b^2}$$

$$P_2 = \frac{P b a}{a^2 + b^2}$$



Q. A rigid bar AB and CD as shown are supported by pins at A & C and two rods. Determine the maximum force 'P' that can be applied as given if its vertical movement is limited to 5 mm. Neglect the weight of all the members.





$$\sum M_A = 0 \quad [\text{for Beam AB}]$$

$$P_{AL} \times 3 = P_{St} \times 6$$

$$P_{AL} = 2 P_{St} \quad \dots (i)$$

$$\tan \phi = \frac{\Delta A}{3} = \frac{\Delta B}{6}$$

AL. $\rightarrow$ Aluminium.

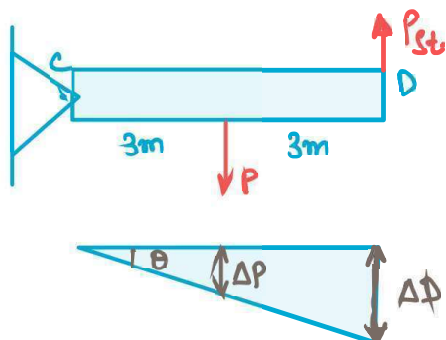
$$\Delta B = 2 \Delta A \quad \dots (ii)$$

$$\Delta B = 2 \left[\frac{P_{AL} \cdot L_{AL}}{A_{AL} \cdot E_{AL}} \right]$$

$$\Delta B = \frac{4 P_{AL}}{35} = \frac{8 P_{St}}{35}$$

$$\Delta D = \Delta B + \Delta BD$$

$$= \frac{8}{35} P_{St} + \frac{P_{St} \cdot L_{St}}{A_{St} \cdot E_{St}} = \frac{8}{35} P_{St} + \frac{1}{30} P_{St} = \frac{11}{42} P_{St}$$



$$\sum M_C = 0 \quad [\text{for beam CD}]$$

$$P \times 3 = P_{St} \times 6$$

$$P = 2 P_{St} \quad \dots (iii)$$

$$\tan \theta = \frac{\Delta P}{3} = \frac{\Delta D}{6}$$

$$\Delta P = \frac{\Delta D}{2} = \frac{11}{42} P_{St} \cdot \frac{1}{2} = \frac{11}{84} P_{St} = \frac{P}{2} \cdot \frac{11}{84} = \frac{11}{168} P$$

$$\Delta P = 5 \text{ mm}$$

$$5 = \frac{11}{168} P \quad \Rightarrow \quad P = \frac{168 \times 5}{11} = 76.36 \text{ KN.}$$

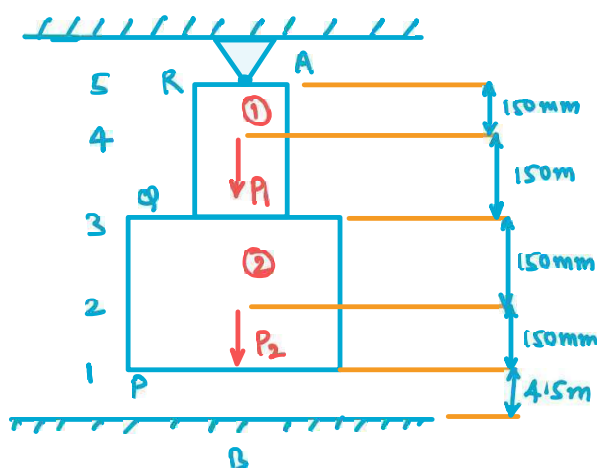
Q.) Determine the reactions at A & B for the Steel bar and loading as given assuming that 4.5 mm clearance exist b/w the bar and the ground before the loads are applied. Assume $E = 200 \text{ GPa}$.

$$P_1 = 300 \text{ KN}$$

$$P_2 = 600 \text{ KN}$$

$$A_1 = 250 \text{ mm}^2$$

$$A_2 = 400 \text{ mm}^2$$



Soln

Elongation of bar PQR under external load.

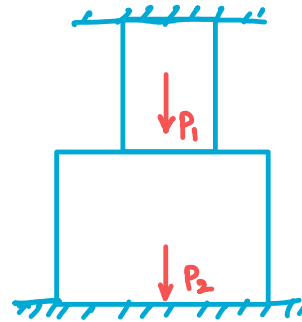
$$\Delta = \left[0 + \frac{600 \times 10^3}{400 \times 10^{-6}} + \frac{600 \times 10^3}{250 \times 10^{-6}} + \frac{900 \times 10^3}{250 \times 10^{-6}} \right] \frac{0.15}{200 \times 10^3}$$

$$\Delta = 5.625 \text{ mm}$$

Compression due to reaction at point 'B'

$$\Delta = - \left[\frac{R_B}{400 \times 10^{-6}} + \frac{R_B}{250 \times 10^{-6}} \right] \frac{0.3}{2 \times 10^5}$$

$$\Delta = \frac{-39}{4000} R_B$$

Approach

a) Extension prevented by $R_B = 5.625 - 4.5 = 1.125 \text{ mm}$
 Compression by R_B

$$\frac{39}{4000} R_B = 1.125$$

$$R_B = 115.4 \text{ kN.} \quad \star$$

b) $\Delta_L - \Delta_{R_B} = 4.5 \text{ m.}$

$$5.625 - \frac{39}{4000} R_B = 4.5$$

$$R_B = 115.4 \text{ kN} \quad \star$$

$$\sum f_f = 0$$

$$R_A + P_1 + P_2 = 0.$$

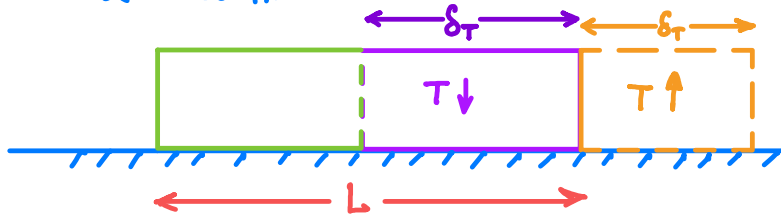
$$R_A = 784.6 \text{ kN}$$

Case-8 Members Subjected to Temperature change

$$\delta_T = L \alpha \Delta T$$

ΔT = change in temperature of bar.

α = Coefficient of thermal Expansion.



If a body is free and is subjected to change in temperature (either increasing/decreasing) the body undergoes deformation either expansion or contraction (δ_T).

$$\text{Hence, Strain} = \frac{\Delta L}{L} = \frac{\delta_T}{L} = \frac{L \alpha \Delta T}{L}$$

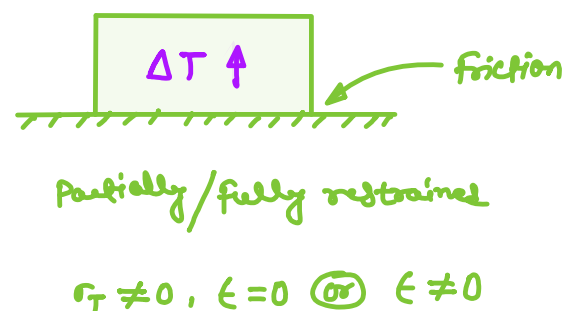
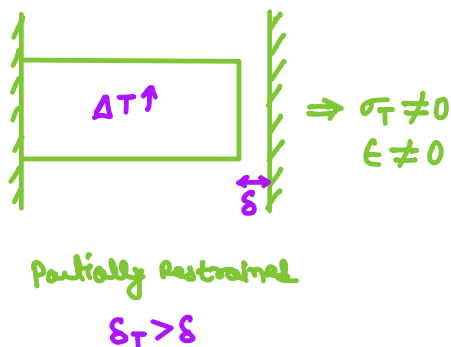
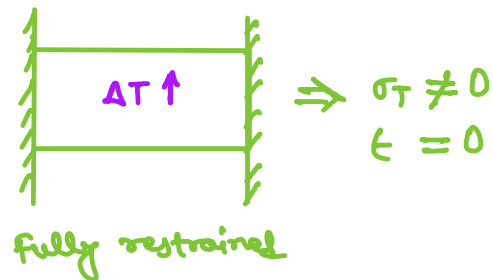
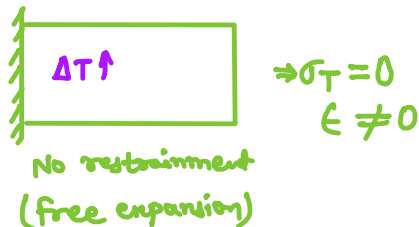
$$\text{Strain} = \alpha \Delta T$$

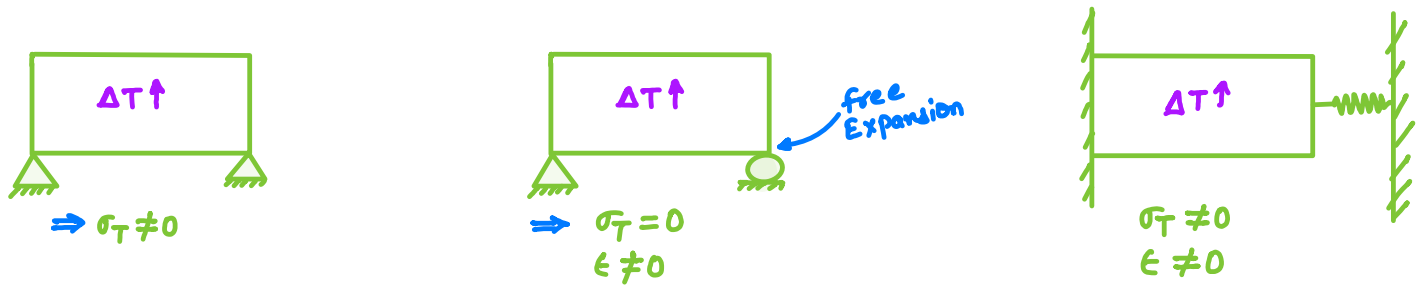
and the Stress generated is Zero (Since body is free).

- If however the body is restrained either fully or partially and temperature is changed then stress will occur termed as "Thermal Stress" / "Temperature Stress" and the strain would depend upon extent of restraintment.

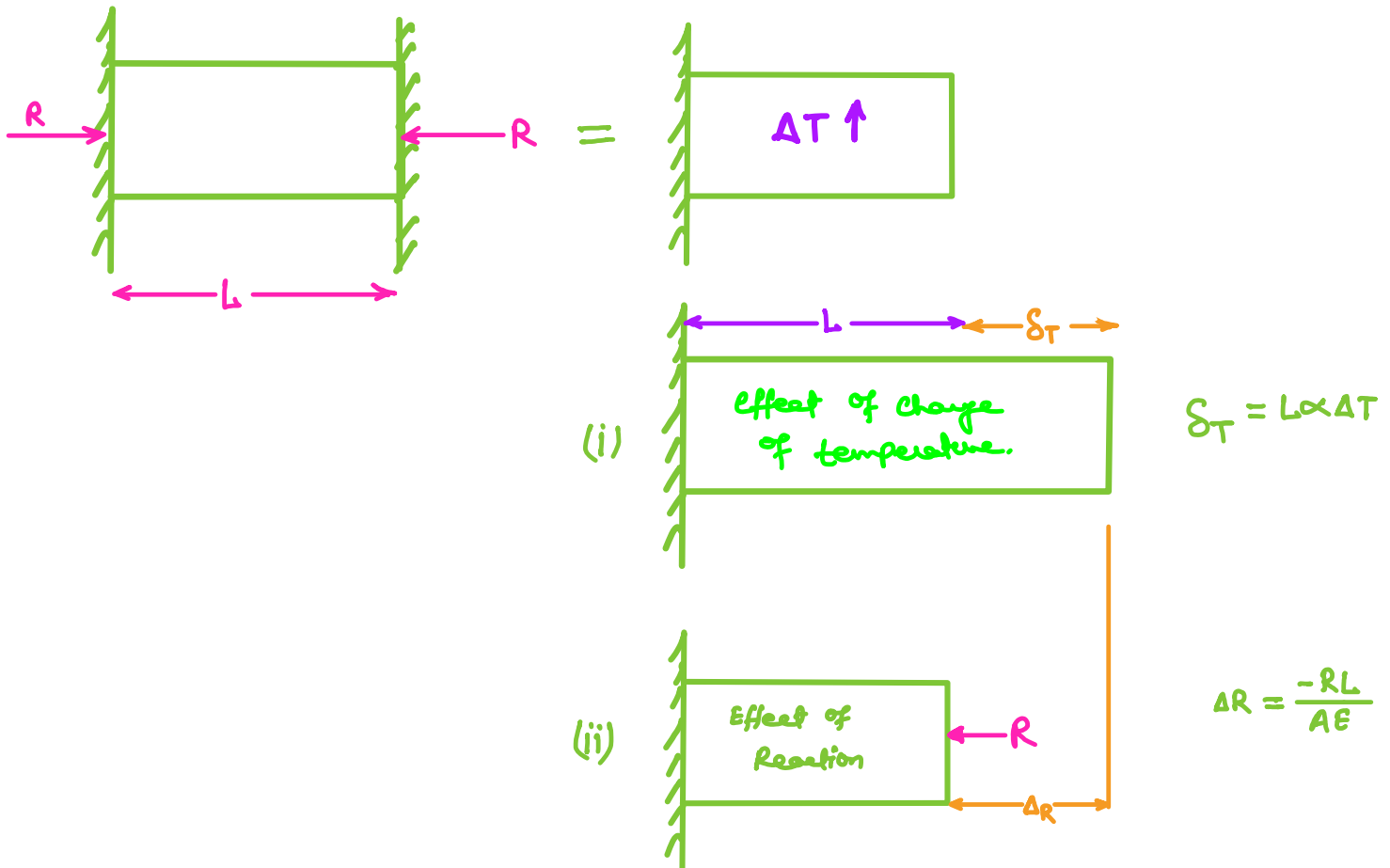
- To find out this thermal stress principle of Superposition can be applied. (Applicable upto Proportional limit).

for eg:-





Analysis



- from compatibility equation,

$$\Delta T = 0$$

$$\delta_T + \Delta R = 0$$

$$\delta_T = -\Delta R$$

$$L\alpha\Delta T = \frac{+RL}{AE}$$

$$\frac{R}{A} = \sigma_T = E\alpha\Delta T$$

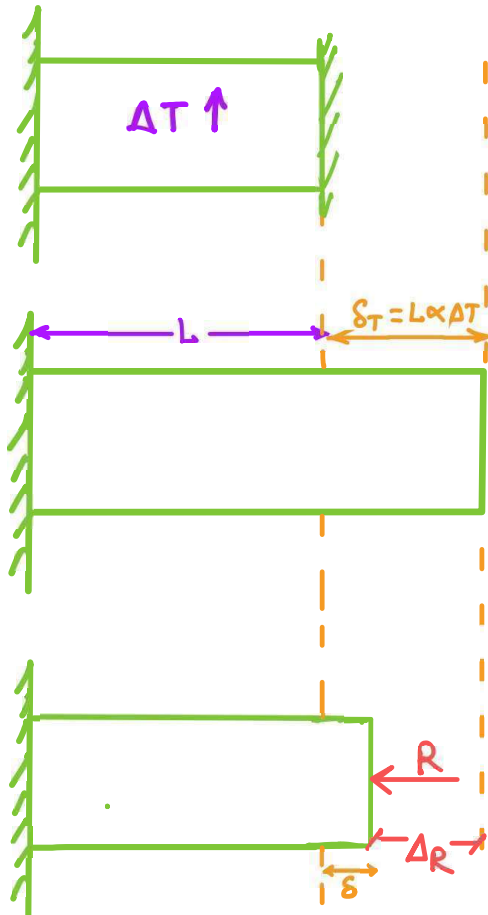
$$\sigma_T = E(\epsilon_T)_p$$

$$(\epsilon_T)_p = \text{Thermal strain prevented} = \frac{\Delta L_{\text{prevented}}}{L}$$

- Here Nature of thermal stress depends upon Nature of change in temperature.
- If temperature is increased, thermal stress causes compression in bar, hence it is compressive in nature.
- If temperature is decreased, thermal stress causes expansion in bar hence, it is tensile in nature.

Note:-

- If temperature change is restrained, but supports yield by amount " δ ", in this case also the thermal stress would be developed corresponding to strain prevented.



from compatibility equation

$$\Delta R = \delta_T - \delta \rightarrow \text{Strain prevented.}$$

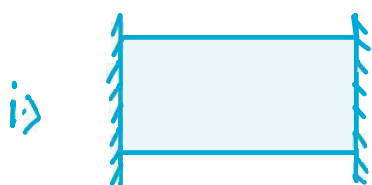
$$\frac{RL}{AE} = L\alpha\Delta T - \delta$$

$$\sigma_T = \frac{E[L\alpha\Delta T - \delta]}{L}$$

$$\sigma_T = \frac{E \Delta L_{\text{prevented}}}{L}$$

$$\sigma_T = E(\epsilon_T)_{\text{prevented}}$$

 Find out the thermal stress developed due to increase in temperature of 15°C . $E = 200 \text{ GPa}$, $\alpha = 10 \times 10^{-6}/^\circ\text{C}$

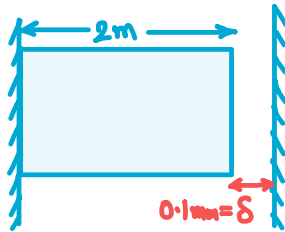


Soln

$$\sigma_T = E(\epsilon_T)_p = E\alpha\Delta T = 200 \times 10^3 \times 10^{-5} \times 15$$

$$\sigma_T = 200 \times 10^3 (1.5 \times 10^{-4}) = 30 \text{ MPa}$$

ii.)



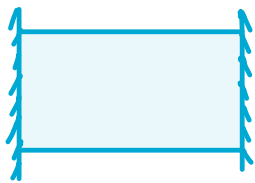
$$\Delta L = L \alpha \Delta T = 2 \times 10 \times 10^{-6} \times 15 \times 10^3 = 0.3 \text{ mm}$$

$$\Delta L_{\text{prevented}} = 0.3 - 0.1 = 0.2 \text{ mm.}$$

$$\sigma_T = E (\epsilon_T)_{\text{prevented}} = 200 \times 10^3 \times \left(\frac{0.2}{2 \times 10^3} \right)$$

$$\sigma_T = 20 \text{ MPa}$$

iii.)



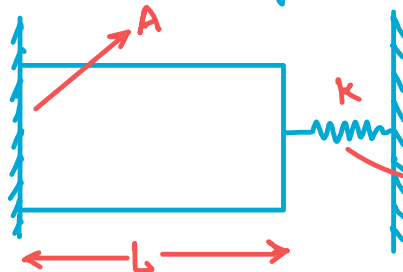
$$\Delta L = 0.3 \text{ mm}$$

$$\Delta L_p = 0.3 - 0.2 = 0.1 \text{ mm.}$$

$$\sigma_T = E (\epsilon_T)_p = 200 \times 10^3 \left(\frac{0.1}{2 \times 10^3} \right) = 10 \text{ MPa.}$$

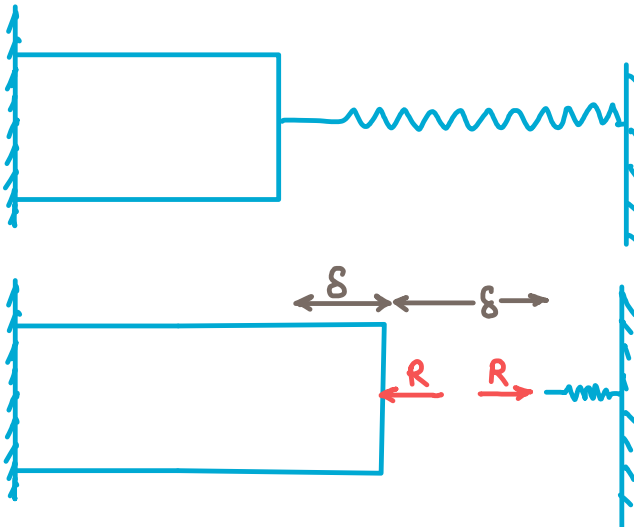
Q.)

for the member shown in figure compute the thermal stress developed



Maximum deformation is " δ " $\ll \delta_T$

Soln



$$\sigma_T = E (\epsilon_T)_p$$

$$= E \frac{\Delta L_p}{L}$$

$$\sigma_T = \frac{E [L \alpha \Delta T - \delta]}{L}$$

$$K_{\text{spring}} = \frac{R}{\delta} \dots (i) \text{ for spring}$$

$$R = \delta K_{\text{sp.}} \Rightarrow \frac{R}{A} A = \delta K_{\text{sp.}}$$

$$\Rightarrow \sigma_T A = \delta K_{\text{sp}} \Rightarrow \delta = \frac{\sigma_T A}{K_{\text{sp}}}$$

$$\sigma_T = \frac{E \left[L \alpha \Delta T - \frac{\sigma_T A}{K_{\text{sp}}} \right]}{L} = E \alpha \Delta T - \frac{E \sigma_T A}{K_{\text{sp}} L}$$

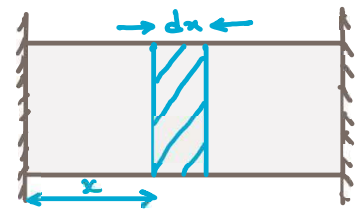
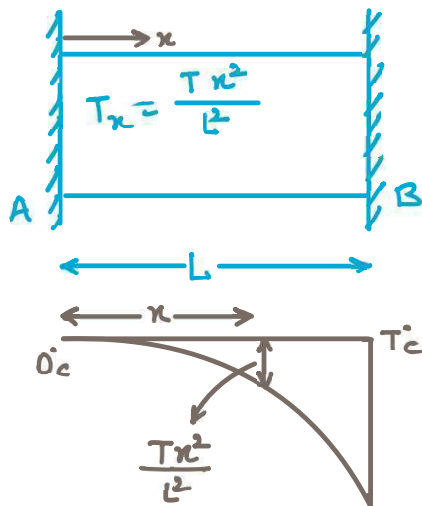
$$\sigma_T = E \alpha \Delta T - \frac{K_{\text{bar}} \sigma_T}{K_{\text{sp}}} \left[\text{here } \frac{AE}{L} = K_{\text{bar}} \right]$$

$$\sigma_T \left(1 + \frac{K_{\text{bar}}}{K_{\text{sp}}} \right) = E \alpha \Delta T$$

$$\sigma_T = \frac{E \alpha \Delta T}{K_{\text{sp}}}$$

ESE-Main

Q.7 for the given system system, find thermal stresses:-



Solution:-

$$\begin{aligned}\Delta L_{\text{Strip}} &= dx \propto \Delta T \\ &= dx \propto \frac{T x^2}{L^2}\end{aligned}$$

$$\Delta L_{\text{Total}} = \int_0^L \propto \frac{T x^2}{L^2} dx$$

$$\Rightarrow \frac{T \propto}{L^2} \left[\frac{x^3}{3} \right]_0^L$$

$$\Rightarrow \frac{T \propto}{L^2} \cdot \frac{L^3}{3} \Rightarrow \underline{\underline{\frac{\propto T L}{3}}}$$

$$\sigma_T = E (\epsilon_T)_p$$

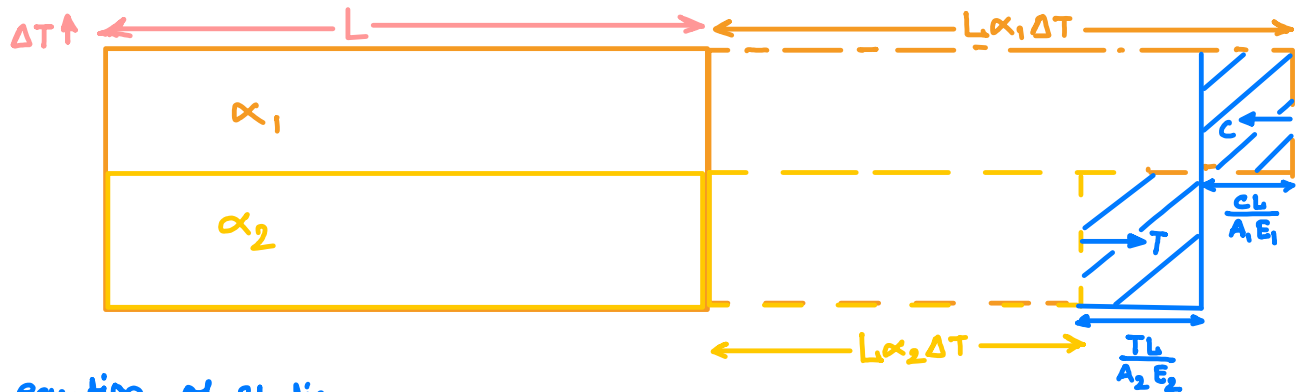
$$= E \left(\frac{\Delta L_{\text{Total}}}{L} \right)_p$$

$$= E \left(\frac{\propto T L}{3L} \right)$$

$$\boxed{\sigma_T = \frac{E \propto T}{3}}$$

Composite bar subjected to Temperature change :-

i> Temperature rise case. ($\alpha_1 > \alpha_2$)



from equation of statics

$$\sum F_x = 0$$

$$T - C = 0$$

$$T = C$$

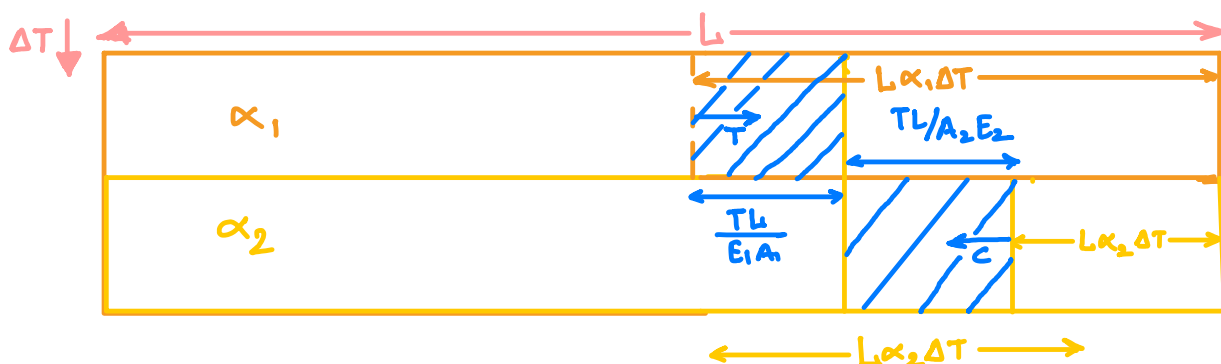
$$\sigma_2 A_2 = \sigma_1 A_1 \quad \text{--- (i)}$$

from compatibility equation

$$L\alpha_2\Delta T + \frac{TL}{A_2E_2} = L\alpha_1\Delta T - \frac{CL}{A_1E_1}$$

$$\boxed{\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2} = (\alpha_1 - \alpha_2)\Delta T} \quad \text{----- (ii)}$$

ii> Temperature fall case. ($\alpha_1 > \alpha_2$)



$$T = C$$

$$\sigma_1 A_1 = \sigma_2 A_2 \quad \text{--- (i)}$$

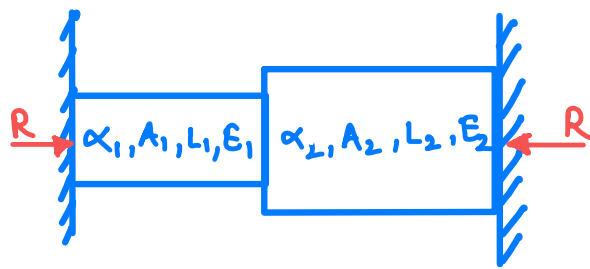
$$\alpha_1 (-\Delta T) + \frac{\sigma_1 L_1}{E_1} = L_2 \alpha_2 (-\Delta T) - \frac{\sigma_2 L_2}{E_2}$$

$$\boxed{\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2} = (\alpha_1 - \alpha_2) \Delta T}$$

Note:-

- in both the cases either temperature is increased or decreased, relationship comes out to be same.
- If Temperature $\uparrow$: Material for which α is more is subjected to compression α is less is subjected to tension.
- If Temperature $\downarrow$: The material for which α is more is subjected to Tension

★ Series connection with end supports



In this case if temperature is increased or decreased, both the members are subjected to compression or tension respectively.

- If temperature is increased free expansion $= L_1 \alpha_1 \Delta T + L_2 \alpha_2 \Delta T$
- Since these bars are restrained against expansion compression will be induced/developed in them $= \frac{\sigma_1 L_1}{E_1} + \frac{\sigma_2 L_2}{E_2}$

from compatibility equation,

$$L_1 \alpha_1 \Delta T + L_2 \alpha_2 \Delta T = \frac{\sigma_1 L_1}{E_1} + \frac{\sigma_2 L_2}{E_2} \dots (i)$$

$$\sigma_1 A_1 = \sigma_2 A_2 \dots (ii)$$

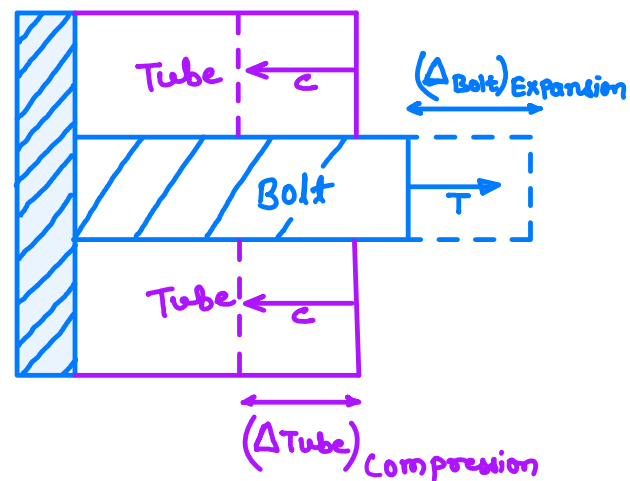
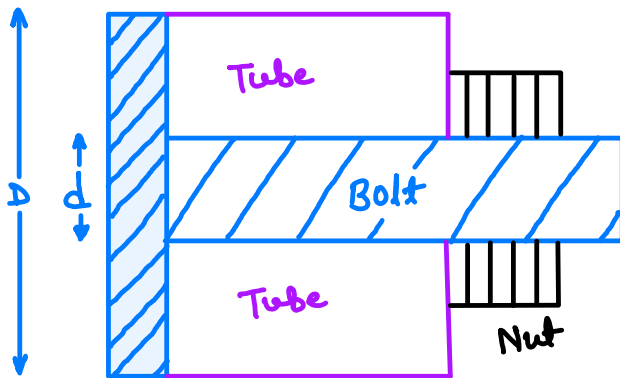
- If the supports yield by an amount δ , then in such case

$$\frac{\sigma_1 L_1}{E_1} + \frac{\sigma_2 L_2}{E_2} = \underbrace{L_1 \alpha_1 \Delta T + L_2 \alpha_2 \Delta T}_{\text{Expansion prevented}} - \delta \dots (i)$$

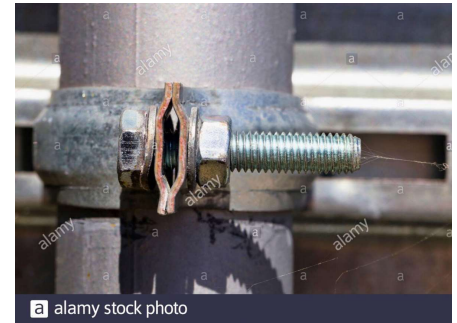
$$\sigma_1 A_1 = \sigma_2 A_2 \dots (ii)$$

Expansion prevented.

Case-9 Nut and Bolt System



- In Nut and bolt assembly a tight fitted assembly is made and if further rotation to the Nut is provided it will lead to the development of compression in tube and tension in bolt.



- Now if Nut and bolt assembly is subjected to change in temperature thermal stresses are also introduced in tube and Bolt.
- Total stresses in this case is computed by principle of Superposition.

for Analysis : from compatibility Equation :-

|Compression of tube| + |Extension of bolt| = Movement of Nut

- If 'n' be the number of rotation of Nut & 'p' be its pitch.

$$\left| \frac{\sigma_{\text{Tube}} L_{\text{Tube}}}{E_{\text{Tube}}} \right| + \left| \frac{\sigma_{\text{Bolt}} L_{\text{Bolt}}}{E_{\text{Bolt}}} \right| = np \quad \text{--- (i)}$$

from statics equation :-

$$C_{\text{Tube}} = T_{\text{Bolt}}$$

$$\sigma_{\text{Tube}} \cdot A_{\text{Tube}} = \sigma_{\text{bolt}} \cdot A_{\text{Bolt}} \dots\dots (ii)$$

from eqn (i) & (ii)

Q> Following system shows Copper in Steel bar fixed at the their ends using plates. If Temperature of the assembly is raised by 30°C . find the stresses in the ROD.

$$E_s = 2 \times 10^5 \text{ N/mm}^2, E_c = 10^5 \text{ N/mm}^2, \alpha_s = 1.2 \times 10^{-5} / ^\circ\text{C}$$

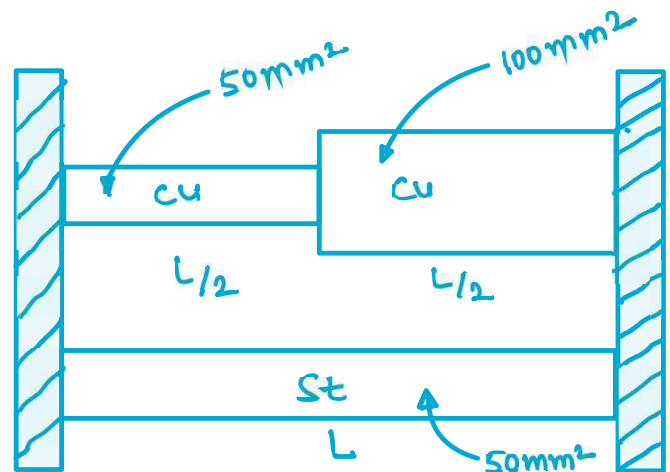
$$\alpha_c = 1.5 \times 10^{-5} / ^\circ\text{C}$$

Solution

free Expansion of

$$\text{Copper} = \frac{L}{2} \alpha_{cu} \Delta T + \frac{L}{2} \alpha_{cu} \Delta T$$

$$= L \alpha_{cu} \Delta T$$



$$\text{free Expansion of Steel} = L \alpha_{st} \Delta T$$

from compatibility equation:-

$$L \alpha_{cu} \Delta T - \left(\overset{\text{Compression.}}{\frac{\sigma_{cu_1} L}{2 E_{cu}}} + \frac{\sigma_{cu_2} L}{2 E_{cu}} \right) = L \alpha_{st} \Delta T + \frac{\sigma_{st} L}{E_s} \dots\dots (i)$$

from Static Equation :-

$$T_{st} = C_{cu_1} = C_{cu_2}$$

$$\sigma_{st} A = \sigma_{cu_1} A = \sigma_{cu_2} (2A) \dots\dots (ii)$$

$$\sigma_{st} = 7.2 \text{ N/mm}^2$$

$$\sigma_{cu_1} = 7.2 \text{ N/mm}^2$$

$$\sigma_{cu_2} = 3.6 \text{ N/mm}^2$$

Q.7 At a temperature of 80°C , a Steel tyre of 12 mm thickness, and 90 mm wide that is to be shrunk on to a locomotive wheel of 2 m diameter. Just fits over the wheel is at a temperature of 25°C . Determine the Contact pressure between the tyre and the wheel after the assembly cools to 25°C . Neglect the deformation of the wheel caused by the pressure of tyre, $\alpha = 11.7 \mu\text{m/m}^\circ\text{C}$
 $E = 200 \text{ GPa}$

Solution.

$$\Delta = (-\delta_T) + \Delta_p = 0$$

$$\Rightarrow \delta_T = \Delta_p$$

$$\alpha L \Delta T = \frac{TL}{AE}$$

$$T = AE \alpha \Delta T$$

$$= (90 \times 12) \times 200 \times 10^3 \times 11.7 \times 10^{-6} \times (80 - 25)$$

$$= 138.99 \text{ KN.}$$

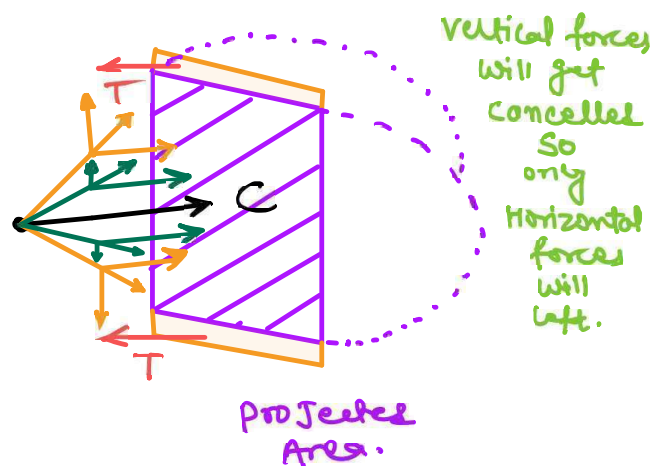
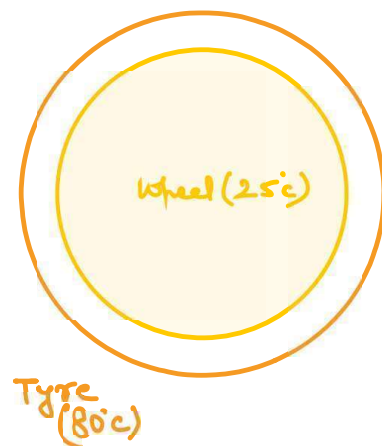
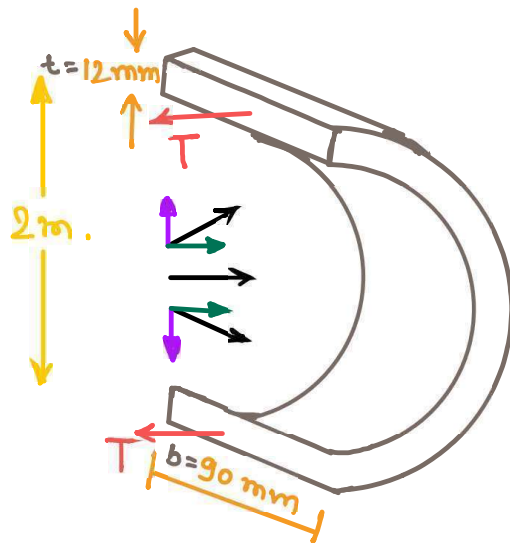
from Static equation:-

$$2T = C$$

$$2T = \sigma_{\text{Contact}} D.b$$

$$\sigma_{\text{Contact}} = \frac{2 \times 138.99 \times 10^3}{2 \times 90 \times 10^3}$$

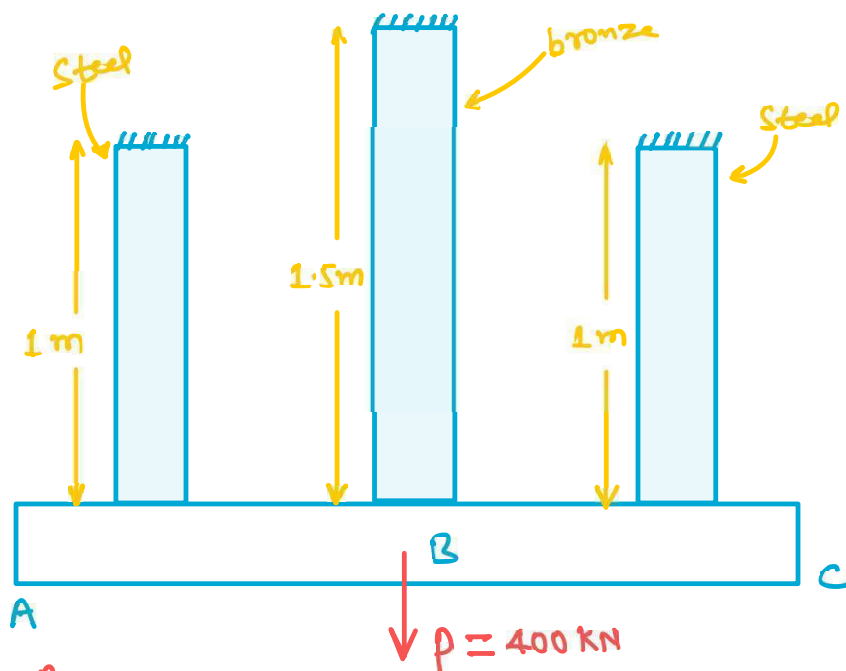
$$\sigma_{\text{Contact}} = 1.544 \text{ MPa.}$$



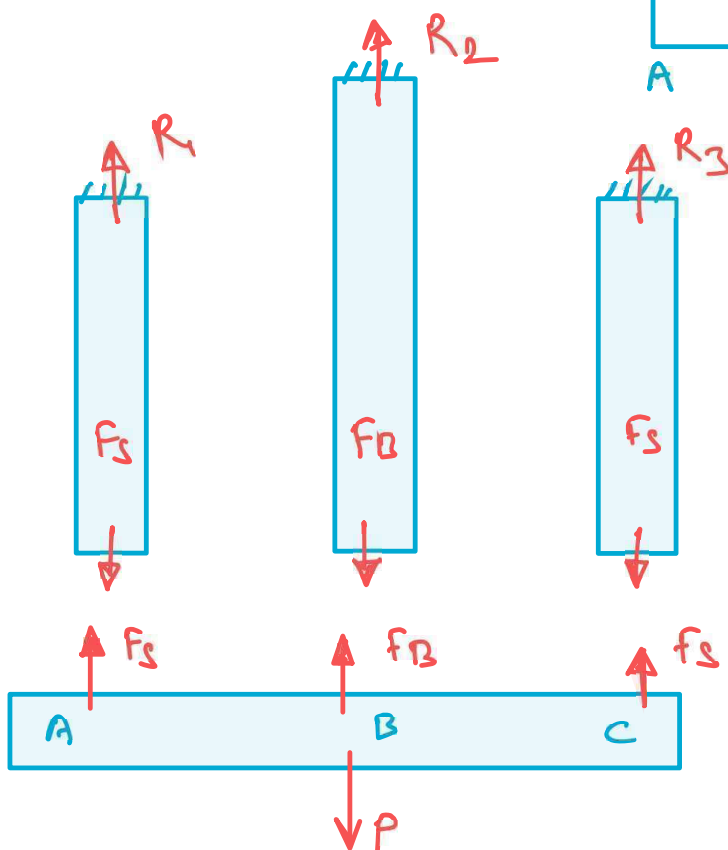
Q.7

A rigid block ABC weighing 400 kN is supported by 3 rods symmetrically placed as shown in given figure. The lower end of the rods are at the same level before the block is attached and temperature is changed. Assume block to remain horizontal. Determine the stress in each rod after a temperature increase of 20°C .

$A_s = 900\text{ mm}^2$, $L_{st} = 1\text{ m}$, $E_s = 205\text{ kN/mm}^2$, $\alpha = 1.2 \times 10^{-6}/^\circ\text{C}$
 $A_b = 1500\text{ mm}^2$, $L_b = 1.5\text{ m}$, $E_b = 80\text{ kN/mm}^2$, $\alpha = 20 \times 10^{-6}/^\circ\text{C}$.



Solution



from equation of Statics
for Block ABC,

$$\sum F_y = 0$$

$$2F_s + F_b - P = 0.$$

$$2F_s + F_b = 400 \text{ --- (i)}$$

from compatibility equation,

$$\Delta_S = \Delta_B$$

$$L_S \alpha_S \Delta T + \frac{F_S L_S}{A_S E_S} = L_B \alpha_B \Delta T + \frac{F_B L_B}{A_B E_B} \quad \dots (ii)$$

$$F_S = 94.01 \text{ kN}$$

$$F_B = 211.98 \text{ kN}$$

$$\begin{aligned} \sigma_S &= \frac{F_S}{A_S} \\ &= \frac{94.01 \times 10^3}{900} \end{aligned}$$

$$= 104.45 \text{ N/mm}^2$$

$$\sigma_B = \frac{F_B}{A_B}$$

$$\sigma_B = \frac{211.98 \times 10^3}{1500}$$

$$= 141.32 \text{ N/mm}^2$$

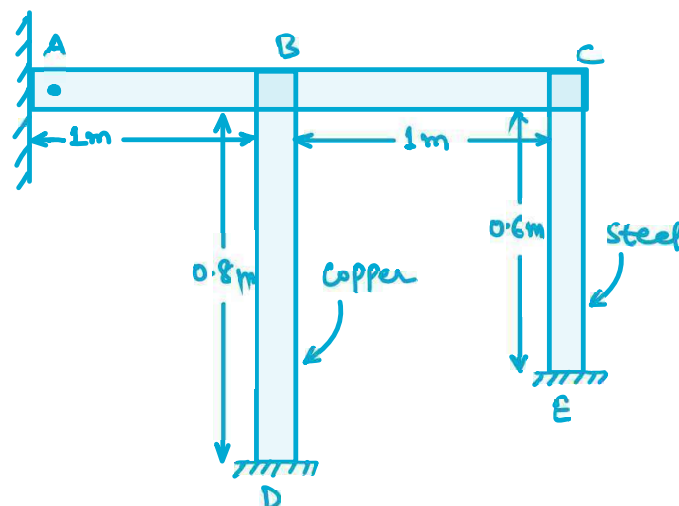
Q.2 A rigid bar ABC is pinned at 'A' and is connected by a steel bar 'CE' and a copper bar 'BD' as shown:-

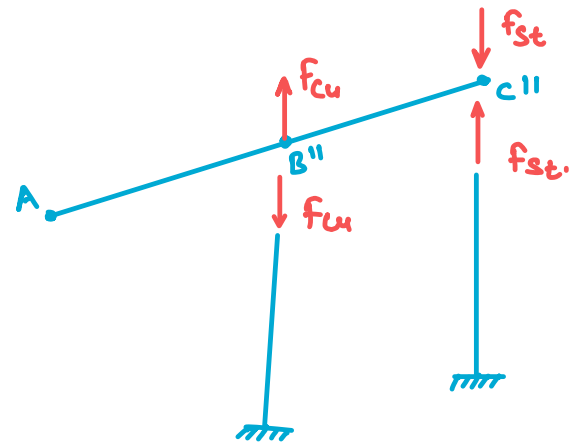
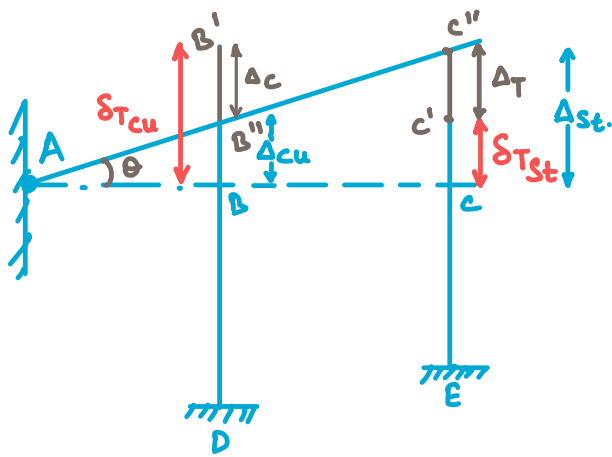
a) If the temperature of the whole assembly is raised by 40°C , find the stresses induced in steel.

b) Copper rods

$$A_S = 400 \text{ mm}^2, E_S = 2 \times 10^5 \text{ N/mm}^2, \alpha_S = 12 \times 10^{-6}/^\circ\text{C}$$

$$A_{Cu} = 600 \text{ mm}^2, E_{Cu} = 1 \times 10^5 \text{ N/mm}^2, \alpha_C = 18 \times 10^{-6}/^\circ\text{C}$$





from equation of statics

$$\sum M_A = 0$$

$$f_{cu} \times 1 - f_s \times 2 = 0$$

$$F_{cu} = 2 F_s \text{ ---- (i)}$$

from compatibility equation

$$\tan \theta = \frac{\Delta_{cu}}{1} = \frac{\Delta_s}{2}$$

$$L_{cu} \alpha_{cu} \Delta T - \frac{F_{cu} L_{cu}}{A_{cu} E_{cu}} = \frac{L_s \alpha_s \Delta T + \frac{f_s L_s}{A_s E_s}}{2} \text{ ---- (ii)}$$

$$f_s = 14.2 \text{ kN}$$

$$F_{cu} = 28.4 \text{ kN}$$

$$\sigma_s = \frac{f_s}{A_s}$$

$$\sigma_{cu} = \frac{F_{cu}}{A_{cu}}$$

$$= \frac{14.2 \times 10^3}{400}$$

$$= \frac{28.4 \times 10^3}{600}$$

$$= 35.5 \text{ N/mm}^2$$

$$= 47.342 \text{ N/mm}^2$$

Q.) A Steel Rod of 20mm dia passes centrally through a tight fitting copper tube of 40mm externally diameter. The Tube is closed with the help of a rigid washer of negligible thickness and Nuts are threaded on the rod. The Nuts are tightened till the Compressive load on the tube is 50 kN. Determine the stresses in the rod and tube when the temperature of the assembly falls by 50°C.

consider, $E_s = 200 \times 10^3 \text{ N/mm}^2$, $E_{cu} = 100 \times 10^3 \text{ N/mm}^2$

$\alpha_s = 12 \times 10^{-6} / ^\circ\text{C}$, $\alpha_{cu} = 18 \times 10^{-6} / ^\circ\text{C}$

Solution:-

i) when Nut is Rotated

$$\sigma_{cu} = \frac{F_{cu}}{A_c} = \frac{-50 \times 10^3}{\frac{\pi}{4} (40^2 - 20^2)}$$

$$= -53.05 \text{ N/mm}^2 \text{ (Compression)}$$

$$\sigma_s = \frac{F_s}{A_s} = \frac{50 \times 10^3}{\pi \times \frac{20^2}{4}} = 159.15 \text{ N/mm}^2$$

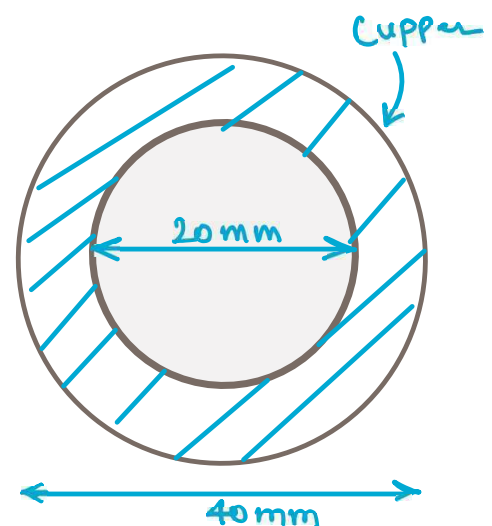
ii) when temperature is reduced from compatibility equation:-

$$\Delta_{cu} = \Delta_{st.}$$

$$- L_{cu} \alpha_{cu} \Delta T + \frac{\sigma_{cu} L_{cu}}{E_{cu}} = - L_s \alpha_s \Delta T - \frac{\sigma_s L_s}{E_s} \dots (i)$$

↓
Free
Contraction

↓
Compression
due to Cu.



from statics equation :-

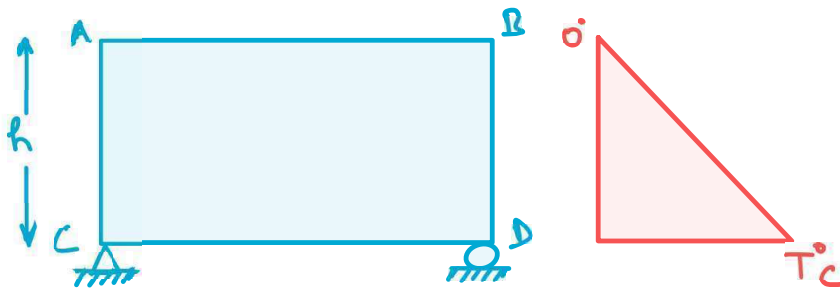
$$(f_u)_T = (f_s)_c$$

$$\sigma_{cu} A_{cu} = \sigma_s A_s \dots (ii)$$

$$\left. \begin{array}{l} \sigma_{cu} = 12 \text{ N/mm}^2 \\ \sigma_{cu} = -36 \text{ N/mm}^2 \end{array} \right\} \text{Due to temp change}$$

Total Stress, a) in Copper = $-52.05 + 12 = -41.05 \text{ N/mm}^2$ (C)

b) in Steel = $159.15 - 36 = 123.15 \text{ N/mm}^2$ (T)

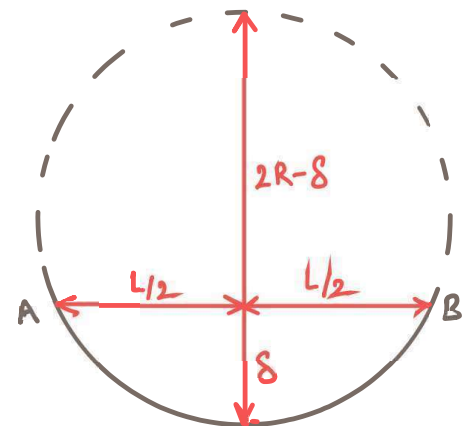


Compute downward deflection.

Solution

Since δ is small

Length of arc AB = AB = L



$$(2R - \delta)\delta = \frac{L}{2} \cdot \frac{L}{2}$$

$$2R\delta - \delta^2 = \frac{L^2}{4}$$

$$\delta = \frac{L^2}{8R}$$

$$\theta = \frac{L}{R} = \frac{L + \delta_T}{R + h}$$

$$\frac{R+h}{R} = \frac{L+\delta_T}{L}$$

$$\frac{h}{R} = \frac{\delta_T}{L}$$

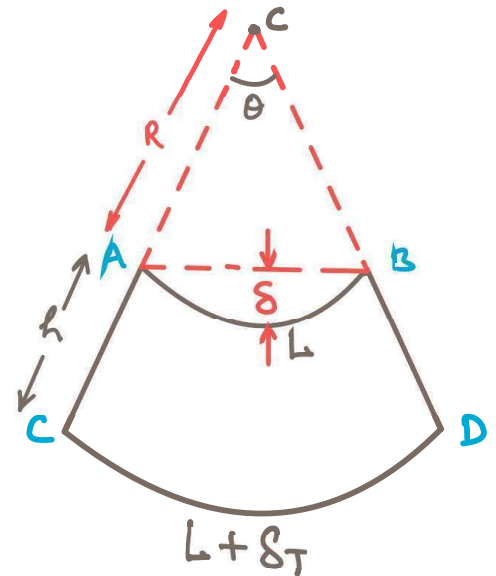
$$\Rightarrow R = \frac{hL}{\delta_T} \dots (ii)$$

from (i) & (ii)

$$\delta = \frac{L^2 \delta_T}{8hL}$$

$$\delta = \frac{L^2 \alpha \Delta T}{8hL}$$

$$\delta = \frac{L^2 \alpha \Delta T}{8h} \quad * \rightarrow (\text{objective})$$



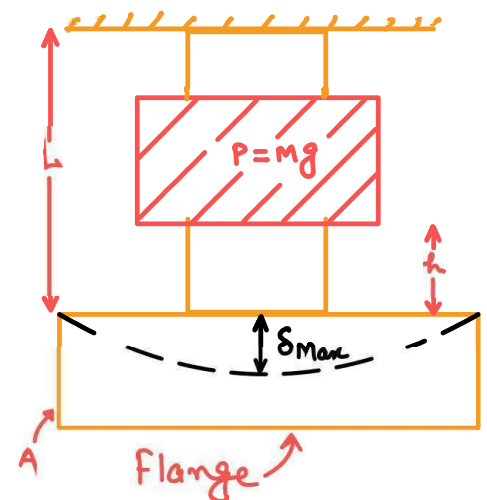
Case-10 Impact load.

In order to find deformation and Stress

$\Rightarrow$ Using Energy conservation

Loss of potential = Gain of strain
Energy Energy in given
component

$$mg(h + \delta_{max}) = \frac{\sigma_{max}^2}{2E} (AL)$$



$$Mg \left(h + \frac{\sigma_{\max} \cdot L}{E} \right) = \frac{\sigma_{\max}^2}{2E} AL$$

$$\left(\frac{AL}{2E} \right) \sigma_{\max}^2 + \left(-\frac{MgL}{E} \right) \sigma_{\max} - Mgh = 0$$

$$\sigma_{\max}^2 + \left(-\frac{2P}{A} \right) \sigma_{\max} + \left(-\frac{2PhE}{AL} \right) = 0$$

$$\sigma_{\max} = \frac{\frac{2P}{A} \pm \sqrt{\left(-\frac{2P}{A} \right)^2 - 4 \times 1 \times \left(-\frac{2PhE}{AL} \right)}}{2 \times 1}$$

$$\sigma_{\max} = \frac{P}{A} \left[1 \pm \sqrt{1 + \frac{2hAE}{PL}} \right]$$

$$\sigma_{\max} = \sigma_{\text{Static}} \left[1 + \sqrt{1 + \frac{2h}{\delta_{\text{Static}}}} \right]$$

→ If $h \rightarrow 0$

⇒ (i.e. in case of Suddenly applied load)

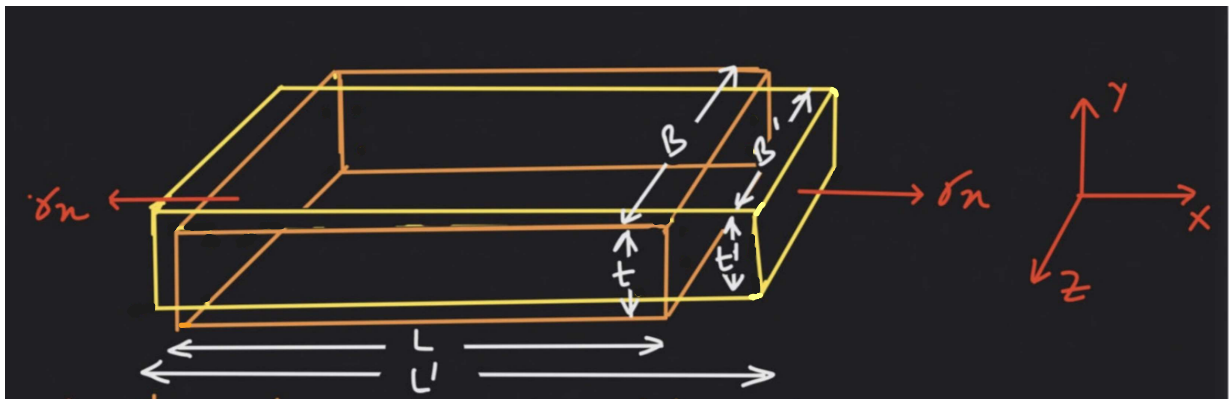
$$\sigma_{\max} \geq 2 \sigma_{\text{Static}}$$

$$\text{as } \delta = \frac{\sigma L}{E}$$

$$\delta_{\max} \geq 2 \delta_{\text{Static}}$$

POISSON'S RATIO

- For Homogeneous and isotropic material, Elongation or Contraction produced by any axial force in the direction of force is accompanied by Contraction or Elongation in all the transverse direction and all such contraction or elongations are same.



If σ_x is the stress in x -direction, then

ϵ_x = longitudinal strain (Normal strain)

$\epsilon_y = \epsilon_z$ = Lateral / Transverse strain (Normal strain)

$$L' > L$$

$$B' < B$$

$$t' < t$$

Case-(i) - Uniaxial

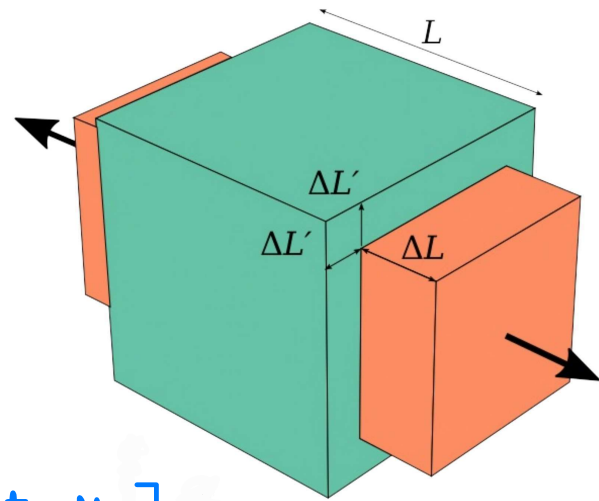
- This lateral strain is expressed in terms of parameter termed as poisson's ratio (μ/ν), which is defined as ratio of Lateral strain by longitudinal strain.

$$\mu = - \frac{\text{Lateral Strain}}{\text{Longitudinal Strain}}$$

$$\mu = - \frac{\epsilon_y}{\epsilon_x} = - \frac{\epsilon_z}{\epsilon_x}$$

$$\epsilon_x = \frac{\sigma_x}{E} \quad [\text{Here (+) = Elongation}]$$

$$\epsilon_y = \epsilon_z = -\mu \frac{\sigma_x}{E} \quad [\text{Here (-) = Contraction}]$$



Note:-

- volume of bar does not remain unchanged due to combined effect of elongation and contraction.

- for Engineering materials $\Rightarrow 0 \leq \mu \leq 0.5$

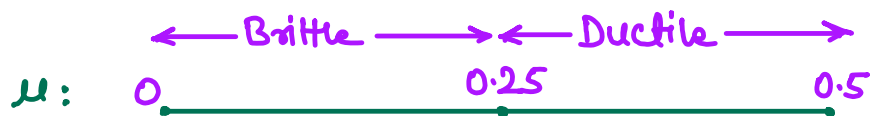
Materials	μ
Cork, Glass	0
Perfectly Elastic Rubber	0.5
Concrete	0.1 - 0.2
Metals	0.25 - 0.4
Aluminium	0.33
Cast iron	0.2 - 0.3
Steel	0.27 - 0.3
Gold	0.44
Bronze	0.35
Stainless Steel	0.35
Copper	0.355
Rubber	0.45 - 0.5

$\therefore$ In general $\Rightarrow -1.5 \leq \mu \leq 0.5$

This value is Negative for Rheotropic material.

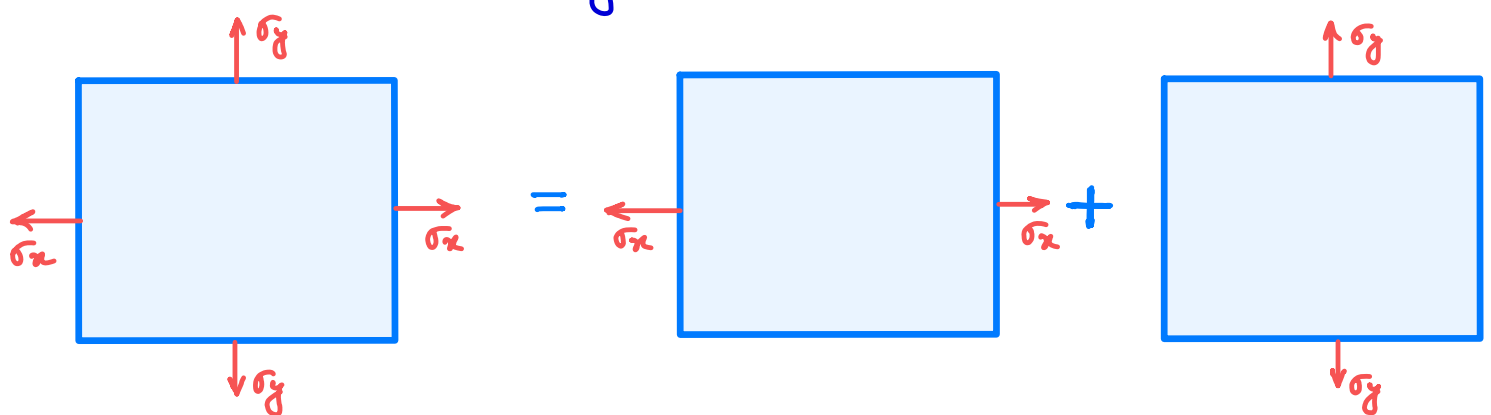
Eg:- Human Tissues or Polymer foams.

- Here Negative value of Poisson's ratio signifies that these materials expand laterally when stretched.
- This Poisson's ratio can Roughly be used to gauge the ductility of material.



- However for perfectly elastic Rubber, $\mu = 0.5$ but it is a brittle material.
- This Ratio of Lateral to Longitudinal strain upto elastic limit is termed as "Poisson's ratio", but beyond elastic limit it is termed as "Contraction Ratio".

Case-(ii) Bi-Axial loading



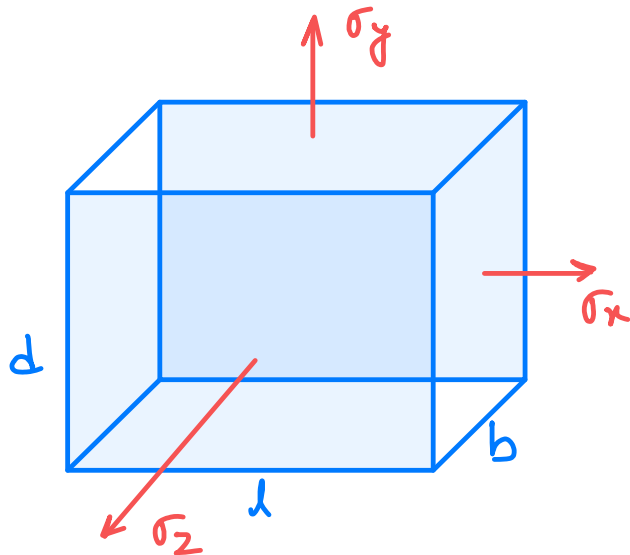
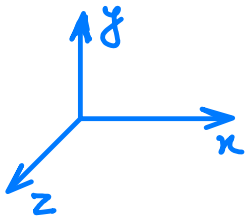
$$\epsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$$

$$\epsilon_y = -\mu \frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$\epsilon_z = -\mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$$

$$\left[E = \frac{\sigma_x}{\epsilon_x} = \frac{\sigma_y}{\epsilon_y} \right]$$

Case-(iii) Multi-Axial loading



- for homogeneous isotropic material value of strain is

$$\epsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} - \mu \frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$$

This above relationship is termed as generalised Hooke's law.

(+) Stress :- Tension

(-) Stress :- Compression

(+) Strain :- Elongation

(-) Strain :- Contraction

Note:-

- The above Result has been obtained by considering the effect of each stress component independently and combining their effect [termed as Principle of Superposition].
- If material is also being subjected to shear stress along with normal stress there would be no effect on normal strain if material is homogeneous and isotropic but if material is anisotropic or orthotropic, application of shear stress also affects normal strain.

★ Volumetric Strain : Dilation

Before stress : $V = lbd$

After stress : $V' = (l + \Delta l)(b + \Delta b)(d + \Delta d)$

$$V' = lbd \left(1 + \frac{\Delta l}{l}\right) \left(1 + \frac{\Delta b}{b}\right) \left(1 + \frac{\Delta d}{d}\right)$$

$$V' = V(1 + \epsilon_x)(1 + \epsilon_z)(1 + \epsilon_y)$$

$$\epsilon_v = \frac{\Delta V}{V} = \frac{V' - V}{V} = \frac{V'}{V} - 1$$

$$\epsilon_v = (1 + \epsilon_x)(1 + \epsilon_z)(1 + \epsilon_y)$$

$$\epsilon_v = \cancel{1} + \epsilon_x + \epsilon_y + \epsilon_z + \epsilon_x \epsilon_y + \epsilon_x \epsilon_z + \epsilon_y \epsilon_z + \epsilon_x \epsilon_y \epsilon_z - \cancel{1}$$

Since $\epsilon_x \epsilon_y \epsilon_z$ are very small

$$\epsilon_x \epsilon_y = \epsilon_y \epsilon_z \approx \epsilon_y \epsilon_z \approx 0$$

$$\boxed{\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z}$$

$$\epsilon_v = \left(\frac{\sigma_x}{E} - \frac{\mu \sigma_y}{E} - \frac{\mu \sigma_z}{E}\right) + \left(\frac{\sigma_y}{E} - \frac{\mu \sigma_x}{E} - \frac{\mu \sigma_z}{E}\right) + \left(\frac{\sigma_z}{E} - \frac{\mu \sigma_x}{E} - \frac{\mu \sigma_y}{E}\right)$$

$$\boxed{\epsilon_v = \frac{\Delta V}{V} = \frac{\sigma_x + \sigma_y + \sigma_z}{E}(1 - 2\mu)}$$

— This equation signifies that if Engineering material is stretched its volume increases.

Note:-

This equation is also applicable for those material which do not follow Hooke's law [as Non-linearity is Neglected in computation of Volumetric Strain].

- Here $\epsilon_v = 0$

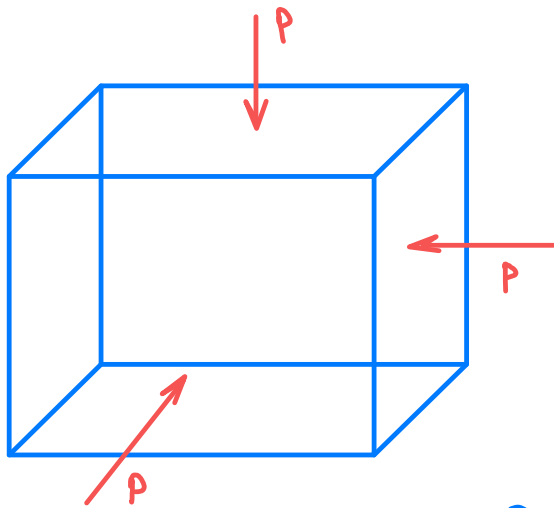
a.) $\frac{\sigma_x + \sigma_y + \sigma_z}{E} = 0$

(or)

b.) $\mu = 0.5$

- For linear elastic Rubber $\mu = 0.5$ hence it shows no volume change on being stressed
- For incompressible fluid, volume change is zero when stressed hence it's $\mu = 0.5$

★ Hydrostatic compression.



Here $(P = \gamma H)$

for Hydrostatic compression

$$\epsilon_v = \frac{(-P) + (-P) + (-P)}{E} (1 - 2\mu)$$

$$\epsilon_v = \frac{-3P}{E} (1 - 2\mu) \dots \dots (i)$$

$$K = \frac{P}{\epsilon_v} \Rightarrow \epsilon_v = \frac{-P}{K} \dots \dots (ii)$$

from (i) & (ii)

$$-\frac{3P}{E} (1 - 2\mu) = \frac{-P}{K}$$

$$K = \text{Bulk modulus of Elasticity} = \frac{E}{3(1 - 2\mu)}$$

- Also, in case of hydrostatic Condition Volume will decrease:

$$\epsilon_v < 0$$

$$-\frac{3P}{E} (1 - 2\mu) < 0$$

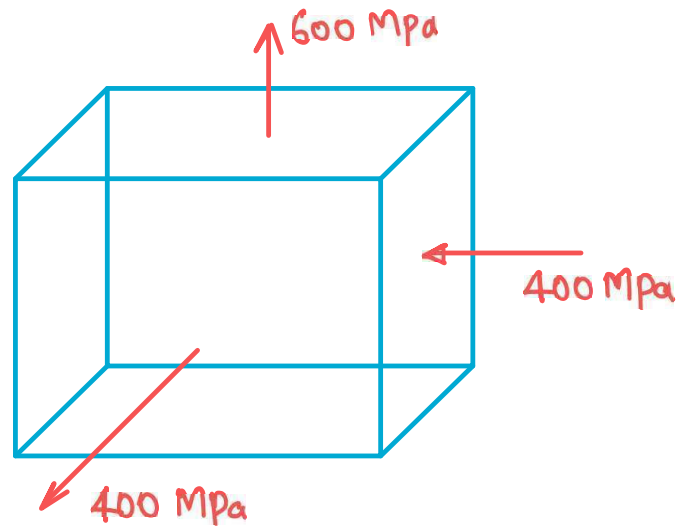
$$1 - 2\mu > 0$$

$$\mu < 0.5 \rightarrow \text{for Engineering Materials}$$

- Q.7 A rectangular block as shown in figure is stressed
 Determine (a.) Strain in x, y, z direction
 (b.) Total change in volume
 $\mu = 0.3$, $E = 2.1 \times 10^5 \text{ MPa}$.

Solution :-

$$\begin{aligned}\epsilon_x &= \frac{\sigma_x}{E} - \frac{\mu \sigma_y}{E} - \frac{\mu \sigma_z}{E} \\ &= \frac{(-400)}{2 \times 10^5} - \frac{0.3 \times 600}{2 \times 10^5} - \frac{0.3 \times 400}{2 \times 10^5} \\ &= -0.0033\end{aligned}$$



$$\begin{aligned}\epsilon_y &= \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_z}{E} \\ &= \frac{600}{2 \times 10^5} - \frac{0.3 \times (-400)}{2 \times 10^5} - \frac{0.3 \times 400}{2 \times 10^5} \\ &= 0.00285\end{aligned}$$

$$\begin{aligned}\epsilon_z &= \frac{\sigma_z}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} \\ &= \frac{400}{2 \times 10^5} - \frac{0.3 \times (-400)}{2 \times 10^5} - \frac{0.3 \times 600}{2 \times 10^5} \\ &= 0.0016\end{aligned}$$

$$(b.) \epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z = 0.001143$$

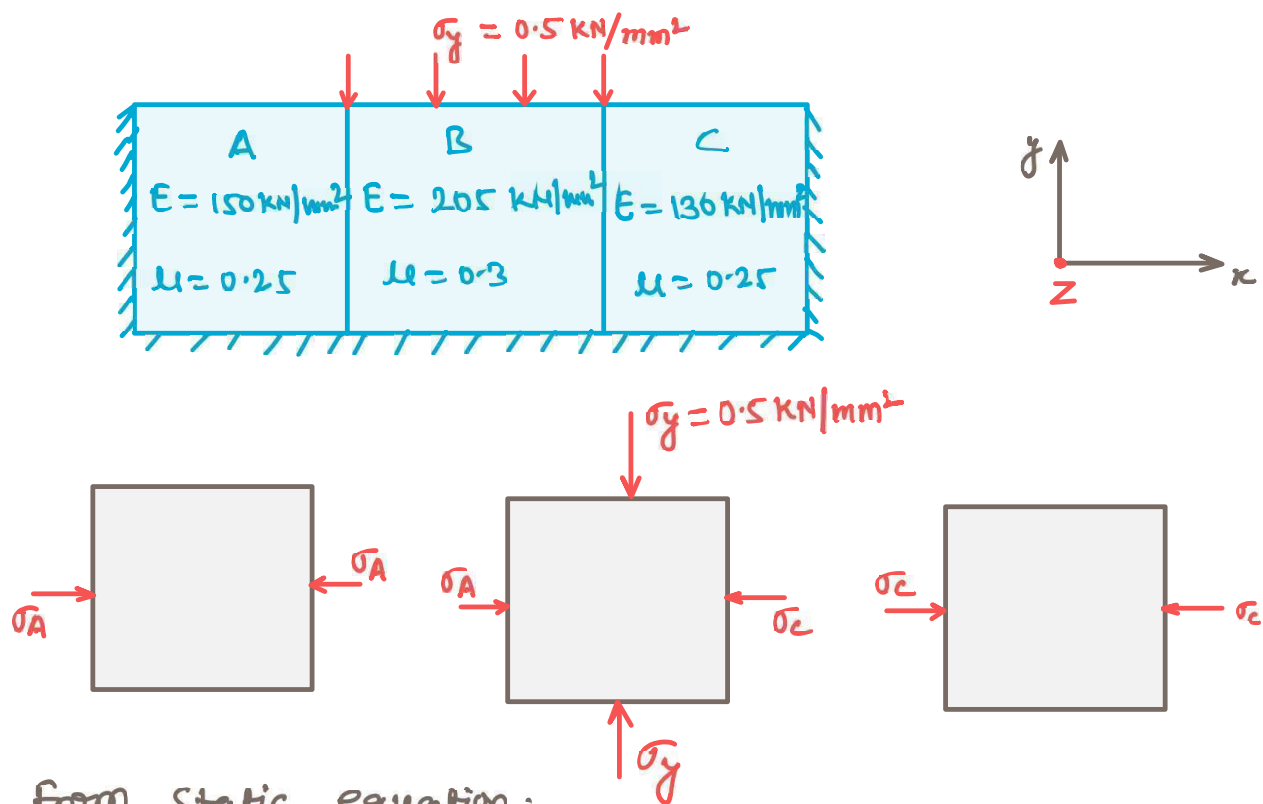
$$\epsilon_v = \frac{\Delta v}{v} \Rightarrow \Delta v = \epsilon_v \cdot v$$

$$\Delta v = 0.001143 v.$$

Φ

The given figure shows the three metal tubes A, B, C of side 100 mm in direct contact resting on a rigid base & confined in x-direction between two rigid end plates. If the upper face of the central cube "B" is subjected to Uniform compressive stress of 0.5 KN/mm^2 . Compute

- direct stress in x-direction
- direct strain in 3 co-ordinate directions. (x, y, z)
- volumetric strain.



from static equation:-

$$\sigma_A = \sigma_C = \sigma \quad \dots (i)$$

from compatibility equation:-

$$\Delta_T = \Delta_A + \Delta_B + \Delta_C = 0$$

$$\frac{-\sigma l_A}{E_A} + \left\{ \frac{-\sigma l_B}{E_B} - \frac{\mu (-0.5) l_B}{E_B} \right\} + \left(\frac{-\sigma l_C}{E_C} \right) = 0 \quad \dots (ii)$$

$$\sigma = 0.041 \text{ KN/mm}^2$$

$$ii) \quad \epsilon_x = \frac{-\sigma}{E} - \frac{\mu(-\sigma_y)}{E_B} + 0 = 5.3 \times 10^{-4}$$

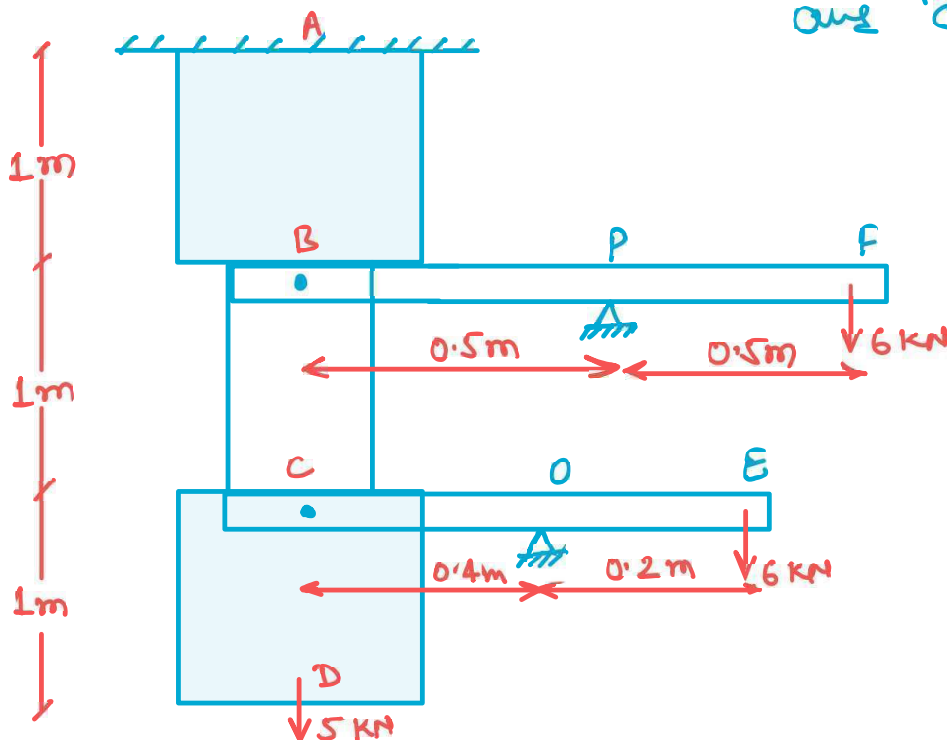
$$\epsilon_y = \frac{-\sigma_y}{E_B} - \frac{\mu(-\sigma)}{E_B} + 0 = -2.38 \times 10^{-3}$$

$$\epsilon_z = 0 - \frac{\mu(-\sigma)}{E_B} - \frac{\mu(-\sigma_y)}{E_B} = 7.92 \times 10^{-4}$$

$$iii) \quad \epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z = 5.3 \times 10^{-4} + (-2.38) \times 10^{-3} + 7.92 \times 10^{-4} \\ = \underline{\underline{-1.058 \times 10^{-3}}}$$

Q.7 A Stepped Vertical Steel bar ABCD is fixed at top end A. Each segment of the bar AB, BC & CD are 1m long & has cross-section of 20mm x 20mm, 10mm x 10mm and 20mm x 20mm respectively. A 5 kN load is applied directly at D & 6 kN loads are applied on the lever attached to the Stepped bar at B and C as given. Find the vertical displacement of D and the change in the volume of the bar.

$E = 2 \times 10^5 \text{ MPa}$, $\mu = 0.25$, connections b/w lever & Bar 'B' and 'C' are hinged.



Solution.

for lever CE

$$F_C \times 0.4 = 6 \times 0.2$$

$$\underline{F_C = 3 \text{ kN}}$$

for lever BF

$$F_B \times 0.5 = 6 \times 0.5$$

$$\underline{F_B = 6 \text{ kN}}$$

Hence for stepped bar

$$\Sigma F_y = 0$$

$$5 - 3 - 6 + R = 0$$

$$\underline{R = 4 \text{ kN}}$$

for segment AB

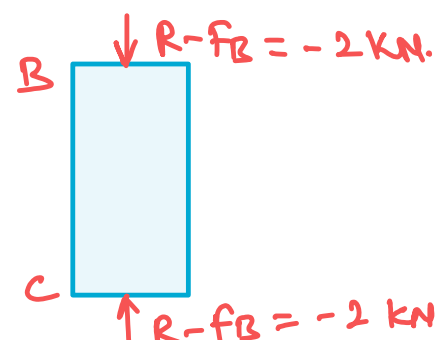
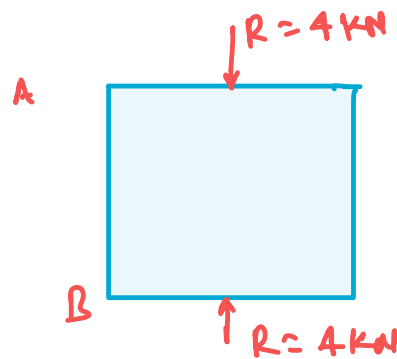
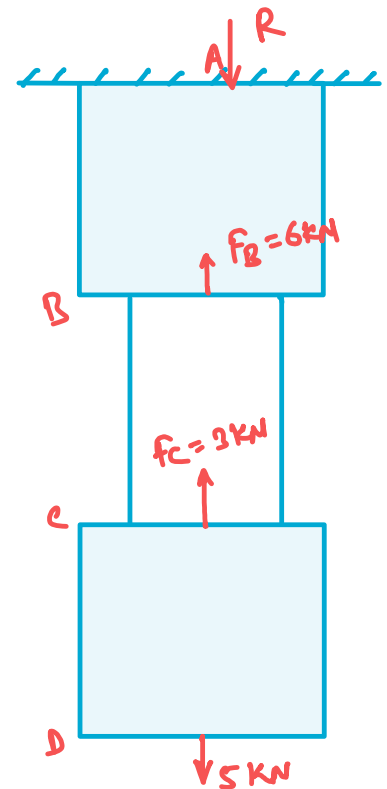
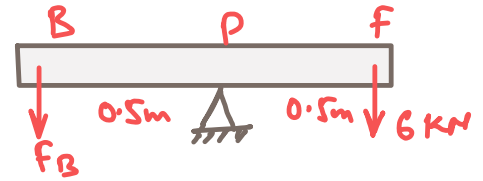
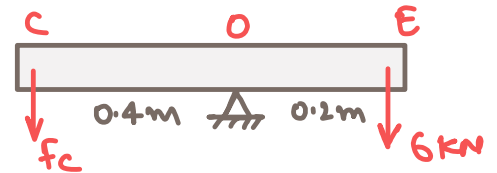
$$\Delta_{AB} = \frac{-R L_{AB}}{A_{AB} E}$$

$$= -0.05 \text{ mm.}$$

for segment of bar BC.

$$\Delta_B = \frac{-(R - F_B) L_{BC}}{A_{BC} \cdot E}$$

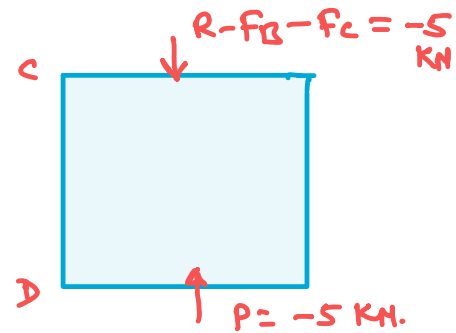
$$= 0.1 \text{ mm.}$$



for segment of CD

$$\Delta_{CD} = \frac{-(R - F_B - F_C) L_{CD}}{A_{CD} E}$$

$$= 0.0625 \text{ mm.}$$



Total deflection of point D = $\Delta_{AB} + \Delta_{BC} + \Delta_{CD}$

$$= \underline{0.1125 \text{ mm.}}$$

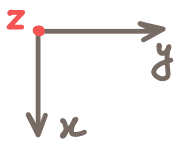
change in volume of bar

$$\Delta V = \Delta V_{AB} + \Delta V_{BC} + \Delta V_{CD}$$

$$(\epsilon_v)_{AB} = \frac{\Delta V_{AB}}{V_{AB}} = \epsilon_x + \epsilon_y + \epsilon_z$$

$$= \left(-\frac{\sigma_{AB}}{E} + 0 + 0 \right) + \left(0 - \frac{\mu(-\sigma_{AB}) + 0}{E} \right) + \left(0 + 0 - \frac{\mu(-\sigma_{AB})}{E} \right)$$

$$= \frac{\Delta V_{AB}}{V_{AB}} = -2.5 \times 10^{-5}$$



or, $\epsilon_{V_{AB}} = \frac{\Delta V_{AB}}{V_{AB}} = \frac{(\sigma_x + \sigma_y + \sigma_z)_{AB} (1 - 2\mu)}{E} = -2.5 \times 10^{-5}$

$$\Delta V_{AB} = -2.5 \times 10^{-5} \cdot V_{AB} = -10 \text{ mm}^3$$

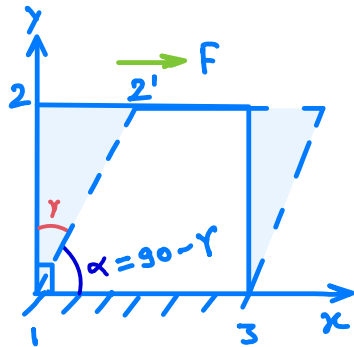
$$(\Delta V)_{BC} = \frac{(\sigma_x + \sigma_y + \sigma_z)_{BC} (1 - 2\mu)}{E} V_{BC} = 5 \text{ mm}^3$$

Similarly,

$$\Delta V_{CD} = \frac{(\sigma_x + \sigma_y + \sigma_z)_{CD} (1 - 2\mu)}{E} V_{CD} = 7.5 \text{ mm}^3$$

Shear Strain

- The change in angle that occurs between two line segments that are originally perpendicular to each other is termed as Shear Strain.



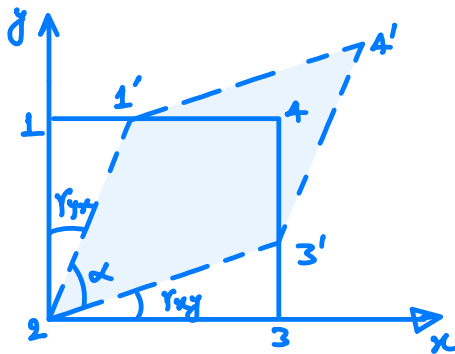
$$\begin{aligned} \text{Change in angle between lines } 12 \text{ \& } 13 &= 90^\circ - \alpha \\ &= 90^\circ - (90^\circ - \alpha) \end{aligned}$$

$$\text{Shear strain} = \phi_{xy}$$

If $\alpha < 90^\circ$, Shear strain positive.

If $\alpha > 90^\circ$, Shear strain Negative.

for eg:-



$$\text{Shear strain } \phi_{xy} = 90^\circ - \alpha$$

$$\text{Here, } \gamma_{yx} + \alpha + \gamma_{xy} = 90^\circ$$

$$90^\circ - \alpha = \gamma_{xy} + \gamma_{yx}$$

$$\phi_{xy} = \gamma_{xy} + \gamma_{yx}$$

$$\text{Since, } \alpha > 90^\circ, \phi_{xy} = (+ve)$$

Here,

$$\tau_{xy} = \tau_{yx}$$

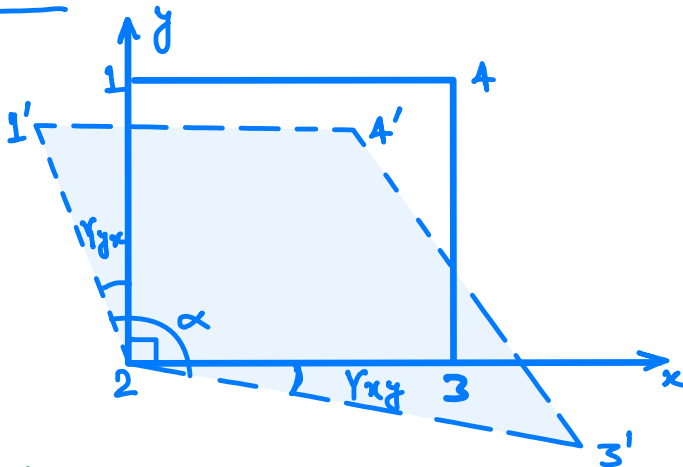
$$\Rightarrow \gamma_{xy} = \gamma_{yx} = \frac{\phi_{xy}}{2}$$

Here, $\phi_{xy} = \gamma_{xy} + \gamma_{yx}$: Shear strain in x - y plane

$\phi_{xz} = \gamma_{xz} + \gamma_{zx}$: Shear strain in x - z plane.

$\phi_{yz} = \gamma_{yz} + \gamma_{zy}$: Shear strain in y - z plane.

eg:- 2:->



Shear strain $\phi_{xy} = 90^\circ - \alpha$

Here $\alpha = 90^\circ + \gamma_{xy} + \gamma_{yx}$

$$\phi_{xy} = -(\gamma_{xy} + \gamma_{yx})$$

Since $\alpha > 90^\circ$

$$\phi_{xy} = (-) \text{ve.}$$

Note :-

If u, v, w are displacements in x, y, z directions respectively then differential form of Normal & Shear Strain is as follows:-

Normal strain

$$\epsilon_x = \frac{du}{dx}$$

$$\epsilon_y = \frac{dv}{dy}$$

$$\epsilon_z = \frac{dw}{dz}$$

Shear strain

$$\phi_{xy} = \frac{du}{dy} + \frac{dv}{dx}$$

$$\phi_{xz} = \frac{du}{dz} + \frac{dw}{dx}$$

$$\phi_{yz} = \frac{dv}{dz} + \frac{dw}{dy}$$

- while Normal strain results in change in volume of material (Dilation), Shear strain results in change in shape of material (Distortion).

So Strain tensor matrix is as follows:-

$$\epsilon_{ij} = \begin{bmatrix} \epsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \epsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \epsilon_z \end{bmatrix} = \begin{bmatrix} \epsilon_x & \frac{\phi_{xy}}{2} & \frac{\phi_{xz}}{2} \\ \frac{\phi_{yx}}{2} & \epsilon_y & \frac{\phi_{yz}}{2} \\ \frac{\phi_{zx}}{2} & \frac{\phi_{zy}}{2} & \epsilon_z \end{bmatrix}$$

$i = \text{face}$

$j = \text{direction}$

- unlike Normal Stress and Normal Strain, Shear Stress & Shear Strain is found by performing TORSION Test.
- The curve obtained between Shear Stress & Shear Strain is similar to that of Normal Stress and Normal Strain curve that yield strength and ultimate strength is approximately half as that of tension test.
- Initial portion of Shear Stress & Shear Strain is also linear.
- Hence, for homogeneous and isotropic material, the Shear Stress within proportional limit is proportional to Shear Strain.

$$\tau_{xy} \propto \phi_{xy}$$

$$\tau_{xy} = G \phi_{xy}$$

G = Shear modulus
or
Modulus of Rigidity

Here $G = \frac{E}{2(1+\mu)}$ *

for engineering material $\rightarrow 0 \leq \mu \leq 0.5$

if $\mu = 0$ $G = \frac{E}{2}$

$\mu = 0.5$ $G = \frac{E}{3}$

$$\Rightarrow \frac{E}{3} < G < \frac{E}{2}$$

- Thus for homogeneous & isotropic material, generalised Hooke's law is given by,

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\mu \sigma_y}{E} - \frac{\mu \sigma_z}{E}$$

$$\phi_{xy} = \frac{\tau_{xy}}{G}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\mu \sigma_x}{E} - \frac{\mu \sigma_z}{E}$$

$$\phi_{yz} = \frac{\tau_{yz}}{G}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\mu \sigma_x}{E} - \frac{\mu \sigma_y}{E}$$

$$\phi_{zx} = \frac{\tau_{zx}}{G}$$

- The above Hooke's law can also be written in the form of matrix as follows:-

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \phi_{xy} \\ \phi_{yz} \\ \phi_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\mu}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ -\frac{\mu}{E} & \frac{1}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ -\frac{\mu}{E} & -\frac{\mu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

- Hence for homogeneous and isotropic material only two independent and distinct elastic constants (E, μ) or (E, G) or (G, μ) are required to define the state of stress at a point in material.
- for orthotropic material like wood, also normal strain is not effected by shear stress but poisson's ratio, modulus of elasticity & modulus of rigidity are different in different perpendicular direction hence, in this case 9 independent and distinct elastic constants are required to define/describe the state of stress at a point in the material.

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \phi_{xy} \\ \phi_{yz} \\ \phi_{zx} \end{bmatrix} = \begin{bmatrix} \boxed{\frac{1}{E_x}} & \boxed{-\frac{\mu_{yx}}{E_y}} & \boxed{-\frac{\mu_{zx}}{E_z}} & 0 & 0 & 0 \\ -\frac{\mu_{xy}}{E_x} & \boxed{\frac{1}{E_y}} & \boxed{-\frac{\mu_{zy}}{E_z}} & 0 & 0 & 0 \\ -\frac{\mu_{xz}}{E_x} & \boxed{-\frac{\mu_{yz}}{E_y}} & \boxed{\frac{1}{E_z}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{\frac{1}{G_{xy}}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{\frac{1}{G_{yz}}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{\frac{1}{G_{zx}}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

- For Anisotropic material, Normal Strain also depends upon Shear Stress, thus number of independent and distinct elastic constants required in this case to define the state of stress at a point in the material are 21.

Note:-

- In this case all the zero terms, above diagonal elements will not be zero in strain matrix for orthotropic material
- Hence, there are 81 independent elastic constants for general Anisotropic material and 2 for isotropic material as follows :-

- For "general Anisotropic material", "81" independent Constants.
- With additional Stress Symmetry the number of constant reduces to "54".
- With additional Strain Symmetry this number reduces to "36".
- For hyper-elastic material (Non-elastic material eg: Rubber) with stress and strain symmetry this reduces to "21". (The material with "21" independent elastic constant is also termed as "Anisotropic/Aelotropic material").
- Further reduction with one plane of material symmetry gives "13" independent elastic constant. These materials are termed as "Monoclinic materials".
- Additional orthogonal plane of symmetry reduces the number of independent constants to "9".
- For a "transverse isotropic material" there are "5" independent elastic constants.
- For an "isotropic material" there are only "2" elastic constants.

Q. >

In a material Under State of plane strain, a $10\text{mm} \times 10\text{mm}$ square centred at a point gets deformed as shown. If the Shear Strain γ_{xy} at this point is expressed as $0.001k$ (in rad). The value of $k = ?$

83

- a) 0.5 b) 0.25 ☒ c) -0.5 d) -0.25

Solution

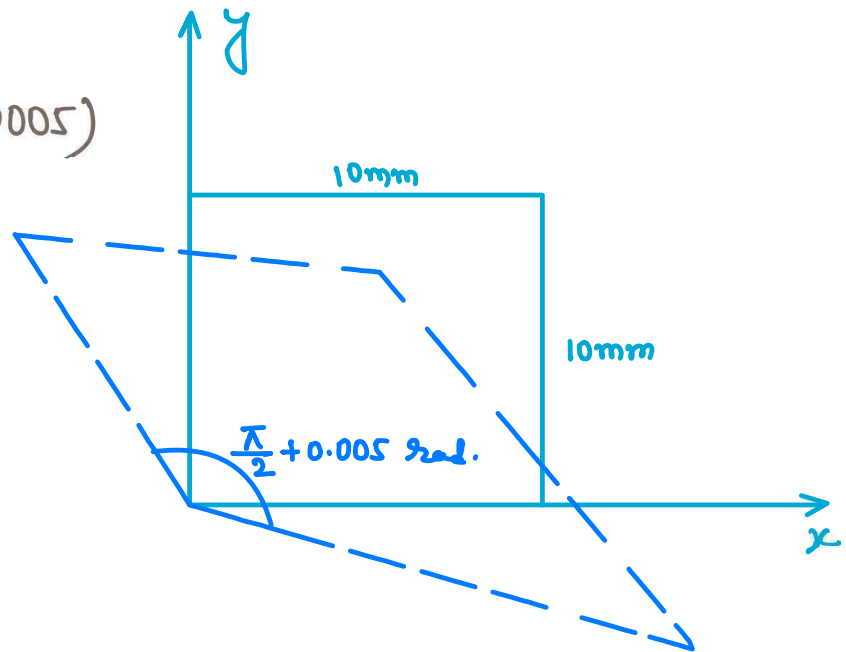
$$\text{Shear Strain} = 90 - (90 + 0.0005)$$

$$= -0.0005 \text{ rad.}$$

As given,

$$0.001K = -0.0005$$

$$K = -0.5$$



Q.7

A square plate $OPQR$ of a linear elastic material with sides 1m is loaded in state of plane stress. Under the given stress condition the plate deforms to new configuration $OP'Q'R'$ as shown. Under the given deformation the edges of the plate remains straight. Find the horizontal displacement of the point $(0.5\text{m}, 0.5\text{m})$ in the plate $OPQR$.

Soln

$$\epsilon_x = \frac{20}{1 \times 10^3}$$

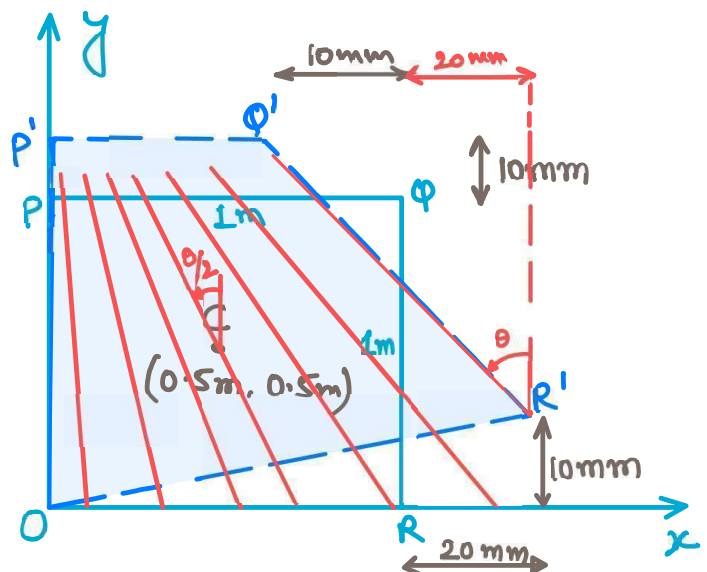
$$\Delta x = \frac{20}{10^3} \times 0.5 \times 10^3 = 10 \text{ mm.}$$

$$\tan \theta = \theta = \frac{30}{1 \times 10^3}$$

$$\delta x = \frac{-30}{1 \times 10^3} \times \frac{1}{2} \times 0.5 \times 10^3$$

$$= -7.5 \text{ mm.}$$

$$\Delta = 10 - 7.5 = \underline{\underline{2.5 \text{ mm}}}$$



Relationship between elastic constants

i) - As discussed earlier, $E = 3K(1-2\mu)$ - - - - - (A)

ii) - $E = 2G(1+\mu)$ - - - - - (B)

Since ϕ is very small,

$$PR = PT$$

$$G = \frac{\tau}{\phi} \Rightarrow \phi = \frac{\tau}{G} \text{ ---- (i)}$$

gn $\Delta \phi RR'$,

$$\tan \phi = \phi = \frac{RR'}{R\phi} \text{ ---- (ii)}$$

from (i) & (ii),

$$\frac{RR'}{R\phi} = \frac{\tau}{G}$$

$$\begin{aligned} \text{Strain in diagonal PR} &= \frac{TR'}{PT} = \frac{TR'}{PR} \\ &= \frac{RR'}{\sqrt{2} \cdot R\phi \sqrt{2}} \\ &= \frac{RR'}{2R\phi} \end{aligned}$$

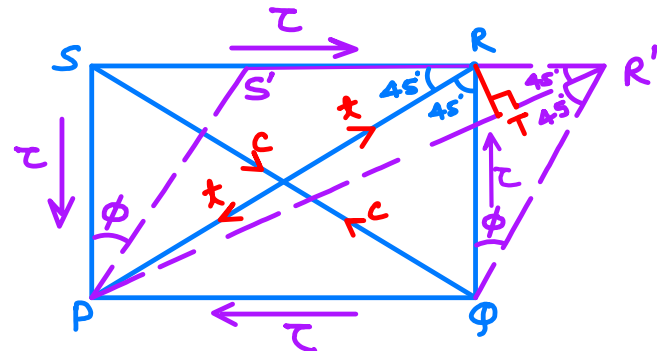
$$\text{Strain in diagonal PR} = \frac{\tau}{2G} \text{ - - - - - (A)}$$

Shear Stress on member leads to development of tensile stress in diagonal PR & compressive stress in diagonal SQ of magnitude $\tau \sin 2\theta = \tau \sin 90^\circ = \tau$

$$\text{Strain in diagonal PR} = \frac{\tau}{E} - \mu \frac{(-\tau)}{E} = \frac{\tau}{E}(1+\mu) \text{ --- (B)}$$

from (A) & (B),

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\mu) \Rightarrow \boxed{E = 2G(1+\mu)}$$



In $\Delta TRR'$,

$$\cos 45^\circ = \frac{TR'}{RR'}$$

$$TR' = \frac{RR'}{\sqrt{2}} \text{ --- (a)}$$

In $\Delta PR\phi$,

$$\sin 45^\circ = \frac{R\phi}{PR}$$

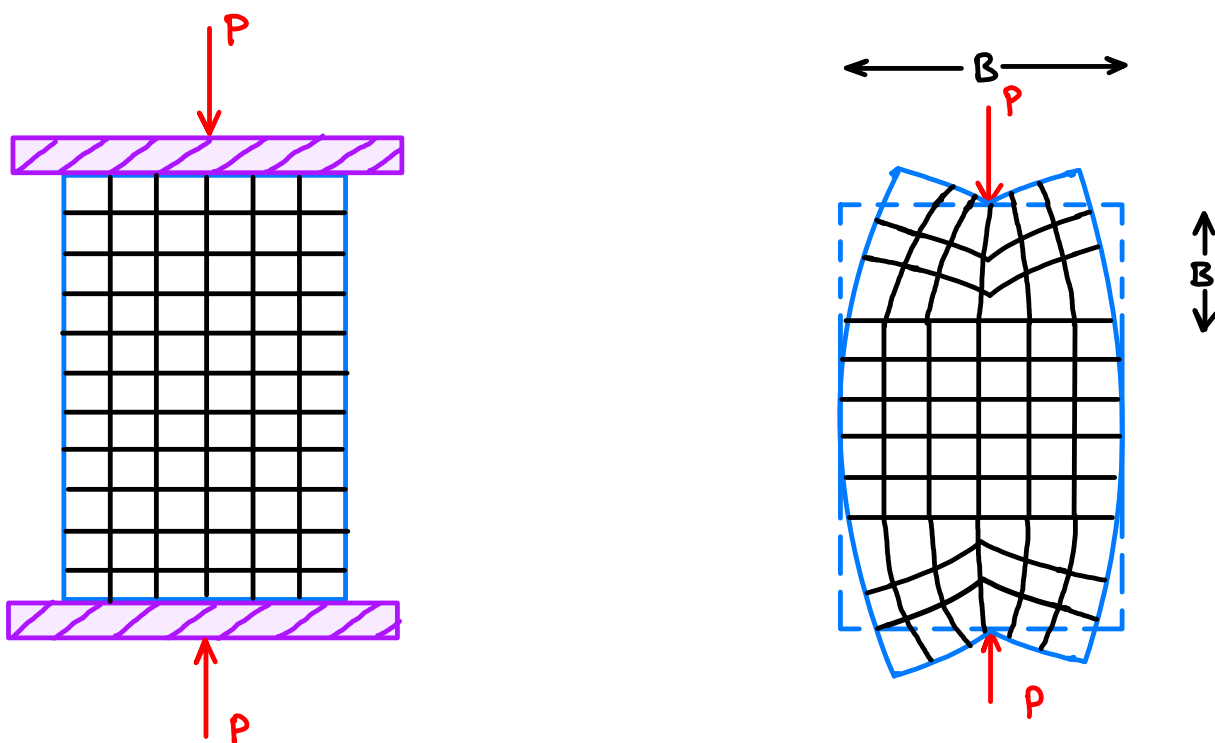
$$PR = R\phi\sqrt{2} \text{ --- (b)}$$

- Also, $E = \frac{9KG}{3k+G}$

- Also, $\mu = \frac{3k-2G}{6k+2G}$

Saint - venant Principle

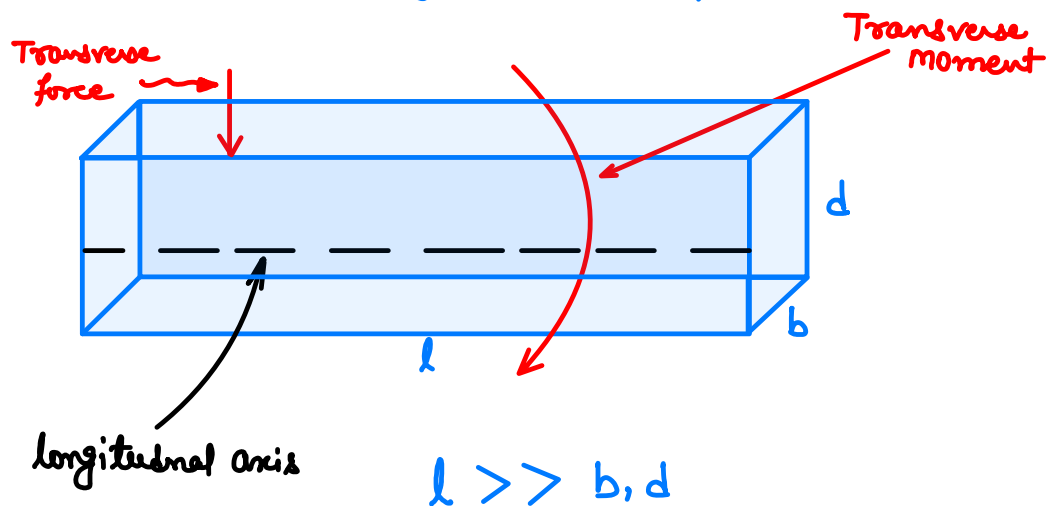
- Except in the immediate vicinity of point of application of loads, the stress distribution may be assumed to be independent of the actual mode of application of loads.
- In Second case, if "B" is the width of the member stress is observed to be constant beyond a depth of "B" from one end.



Shear Force and Bending Moment

★ Beam

- A beam is a structural one dimensional/line element (element whose length is comparatively more than its lateral dimension i.e. width or depth) which is designed to resist shear force and Bending moment due to transverse load/moment (load which is applied perpendicular to the longitudinal axis).

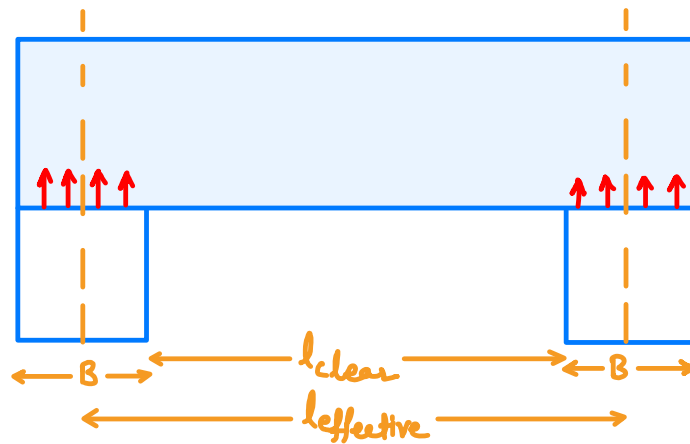


Note:-

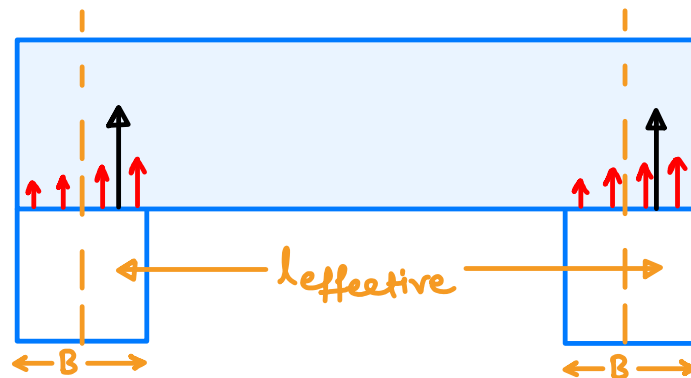
- Structural members like strut, Tiles, Shaft are similar to beams but are designed to resist Compression, tension and torsion respectively and all these members can be of Steel, timber, concrete etc.
- Here clear Horizontal distance between the Supports is called the clear span.
- whereas centre to centre distance between the end supports is termed as effective span (provided if the intensity of reaction is uniform).

- However, if the intensity of reaction is not uniform effective span is C/C distance between the line of actions of the end reactions.

Uniform →



Non-Uniform →

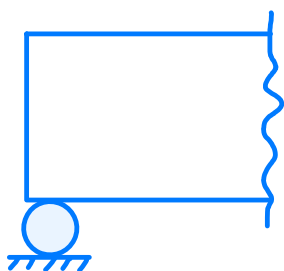


Type of Supports :-

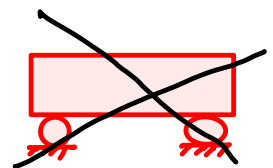
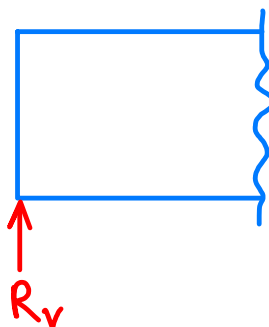
- Supports are classified as follows :-

a> Simple / free / Roller / Rocker Support.

- It is a support on which beam rests freely.

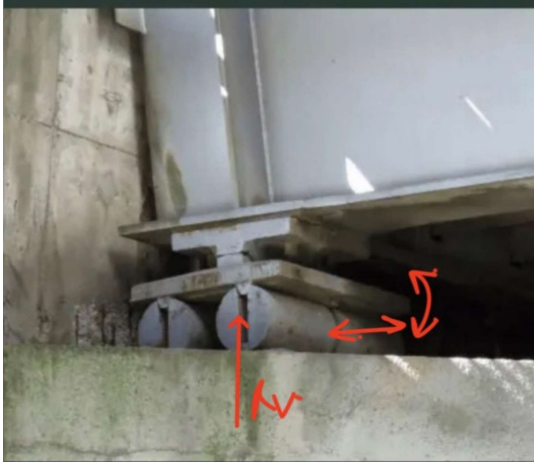
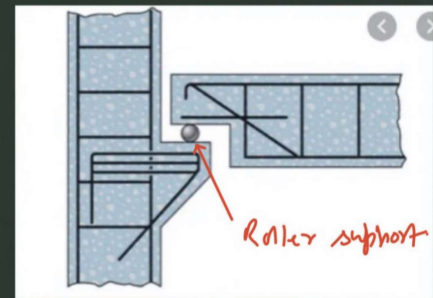


=

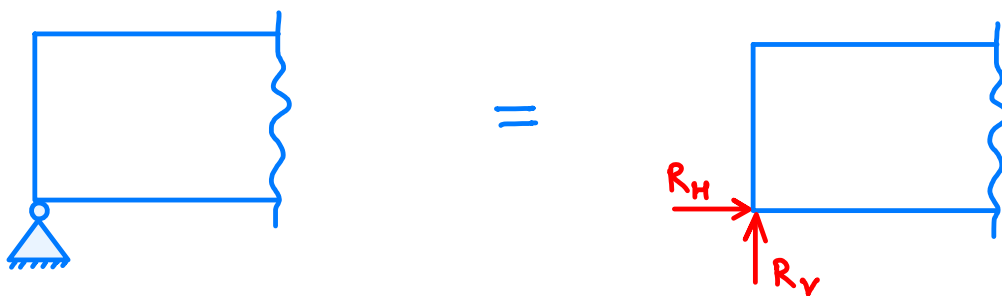


- A Roller Support is the simplest support and gives only one reaction because only one deflection is restrained.

1. SIMPLE OR FREE SUPPORT/ ROLLER SUPPORT/ ROCKER SUPPORT

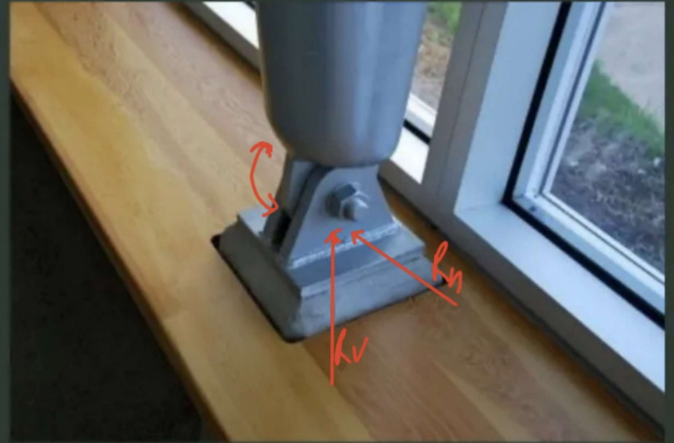
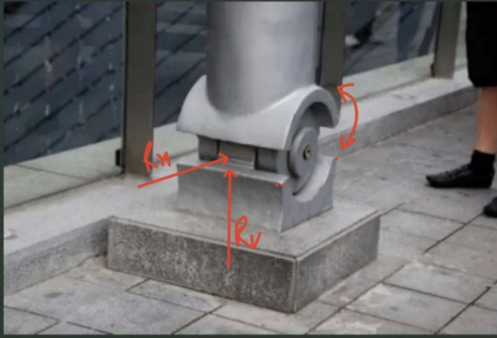


b.> Hinged or pinned Support

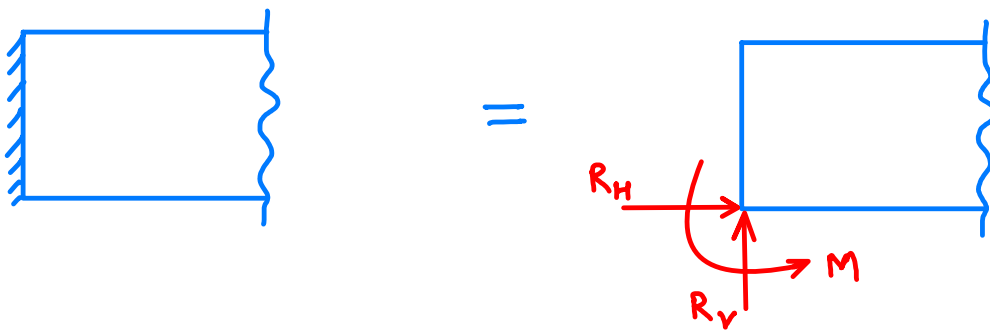


- In this case, beam is hinged or pinned to the support.
- A pin/hinge gives resistance against two deformations or deflections i.e. vertical and Horizontal hence offers two reactions but it does not give resistance against rotation.

2. HINGED OR PINNED SUPPORT

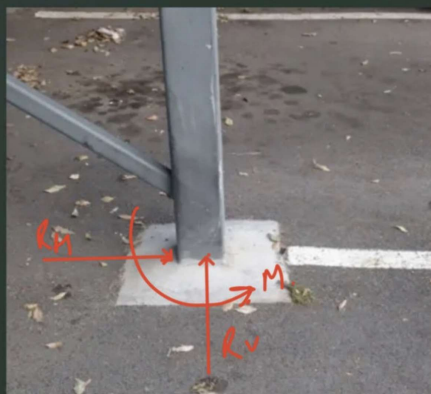


c. > fixed Support / Built in / EnCastre Support

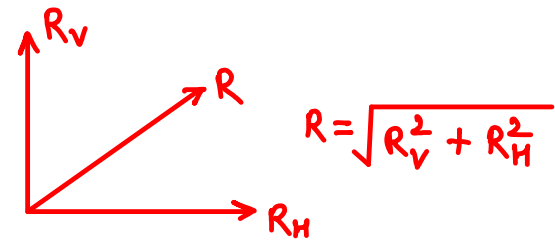
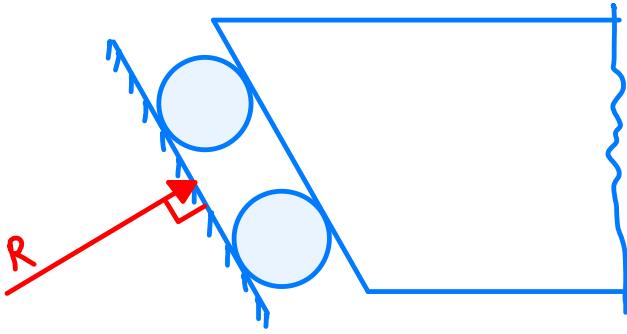


- It is the support which restrains complete movement of the beam both in position as well as direction.
- Hence it offers three reactions.

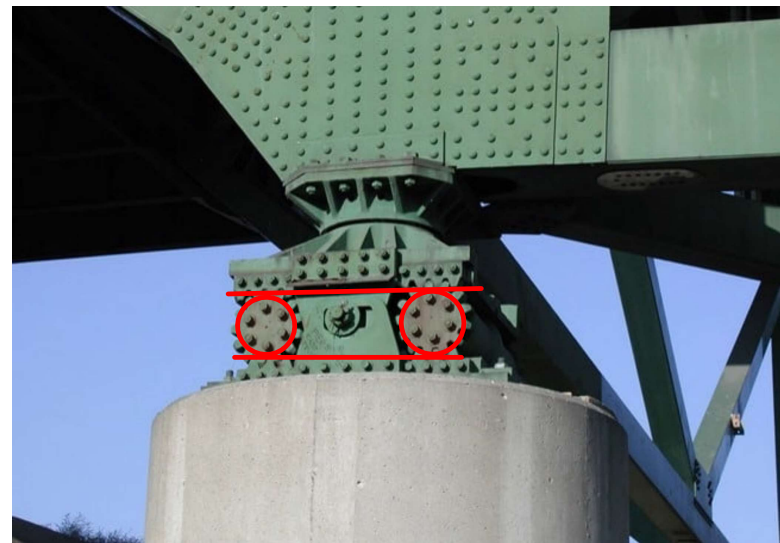
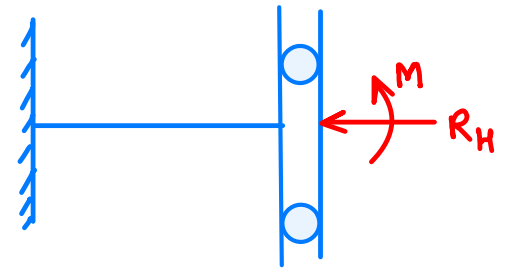
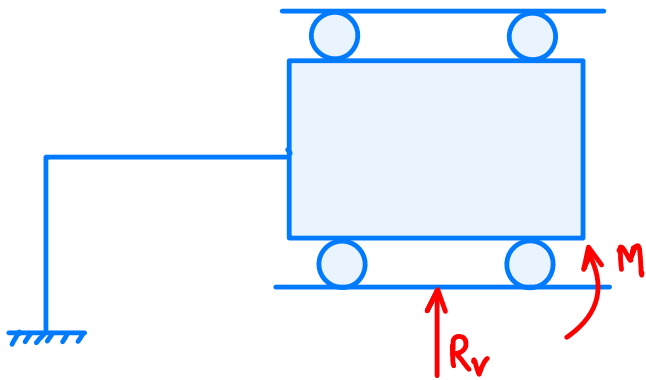
3. BUILT- IN OR FIXED OR ENCASTRE SUPPORT



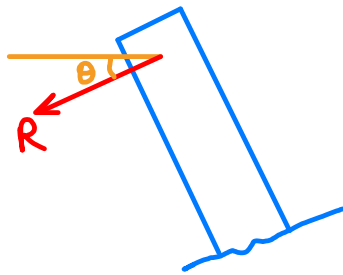
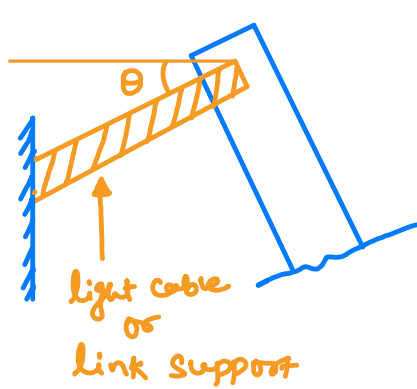
d.> Inclined Roller Support.



e.> Guided Roller Support.

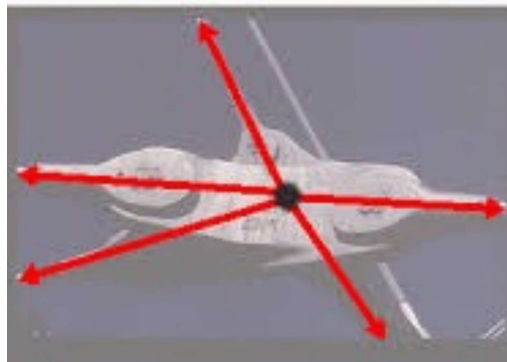
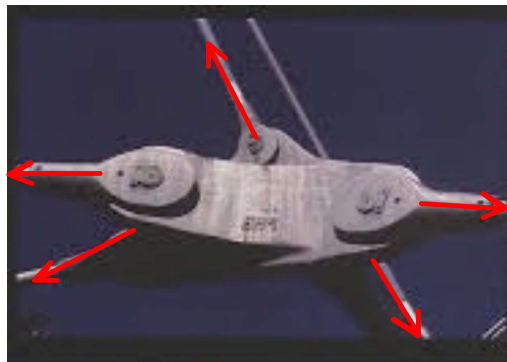
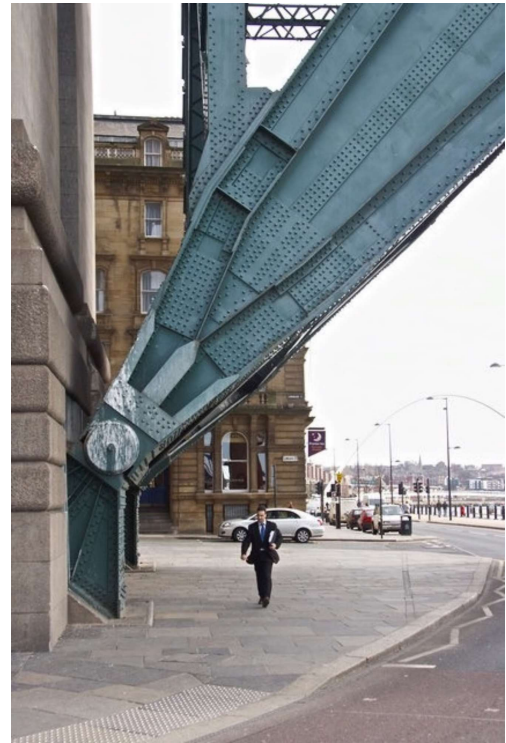


f.> Link Support



- Here only the reaction in the direction of cable or link is offered.

4. LINK SUPPORT



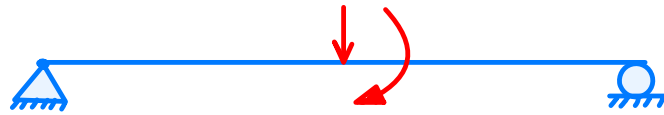
Examples of link support.

Types of beams.

- Beams are classified on the basis of type and number of supports as follows:-

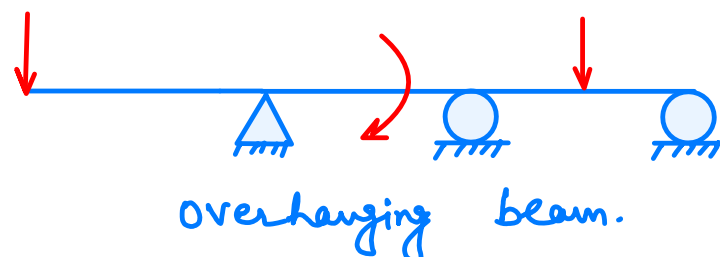
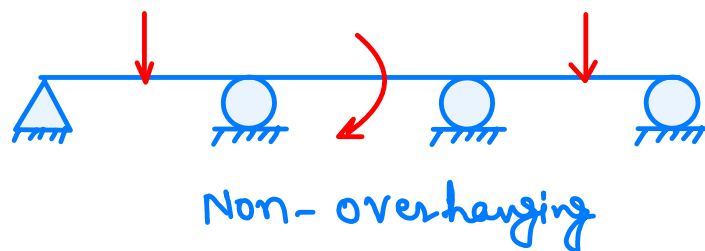
(a) Simply supported beam.

- A beam supported freely on the walls or columns at its both ends is known as simply supported beam.
- This support may be hinged/roller support.
- It is used for shorter span upto 250 feet.



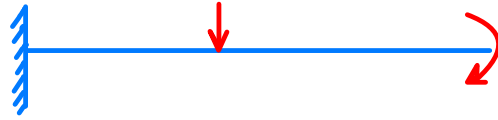
(b) Continuous beam

- A beam which is provided with more than two supports is called continuous beam.
- It may or may not be overhanging beam.
- It is used for long span generally more than 250 feet.

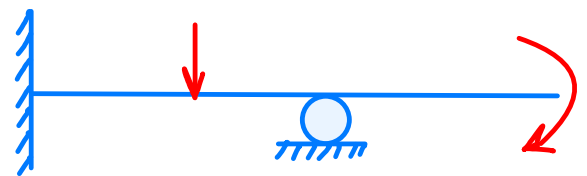
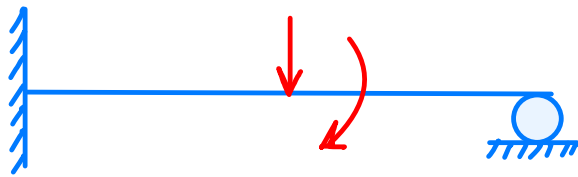


(c) Cantilever beam

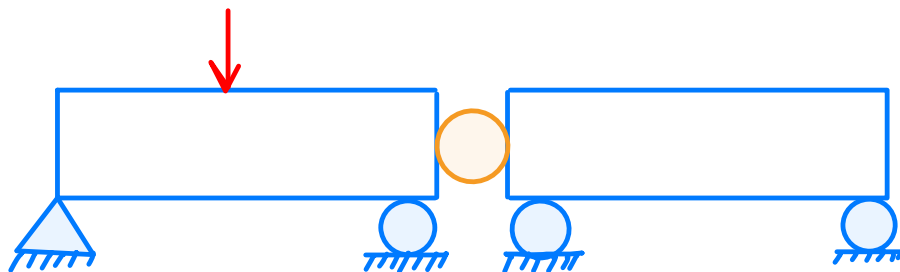
- A beam that fixed at one end and free at other
- It is the most flexible beam amongst the all as it offers high deflection.

(D) Propped Cantilever

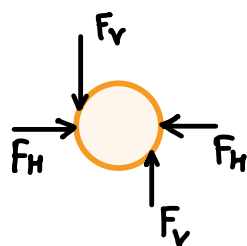
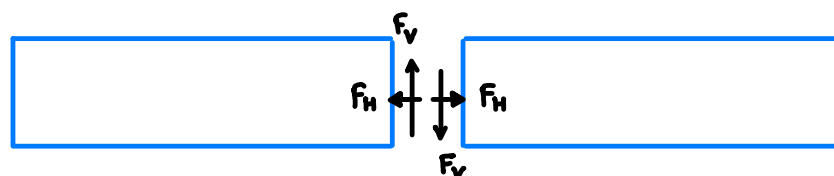
- If a cantilever beam is supported by a Simple Support at free end or in between, it is termed as Propped Cantilever.
- It may or may not be having overhanging portion.
- It is used for large spans.

(e) Beam with internal hinge

- Beam provided with hinge at location anywhere along its span is termed as beam with internal hinge.



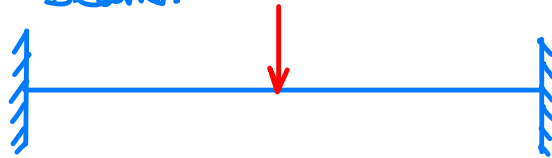
Generally, Hinge is provided in steel bridges.



- It is preferred in Steel bridges (not for concrete) and is provided deliberately to nullify the effect of moment
- As the span increases moment also increases which causes the failure of the hinge prior to that of the structure, thereby safeguarding it against failure (however Hinge is a simple Nut and Bolt connection which can be provided again).

(f) fixed Beam

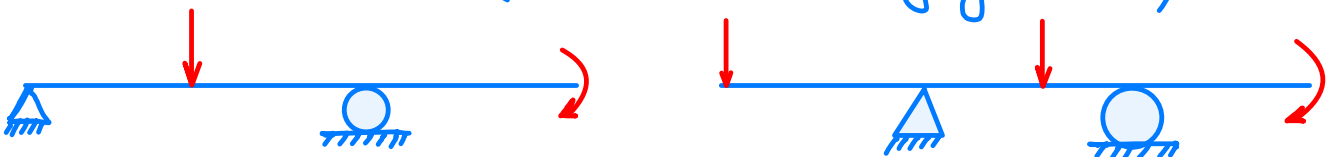
- A beam rigidly fixed at both its ends is termed as fixed or built-in beams.



- It is the most economical beam amongst all as it offers very less deflection/deformation so the requirement of high strength material (E) and bigger dimensions ($I \propto b, d$) is avoided.

(g) Overhanging beam.

- A beam having its end portion/portions extended in the form of cantilever beyond the support is known as over hanging beam.
- It may be overhanging on one (single overhanging beam) or on both sides (Double overhanging beam).



Analysis of beam:-

- Analysis of beam is carried out with the help of number of equilibrium equation available and number of support reactions to which beam is subjected.
- generally, 3 equilibrium conditions are available

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_z = 0$$

Note:-

If internal hinge is present one additional equilibrium condition is also developed $\sum M_{\text{hinge}} = 0$.

- Number of support reaction depends upon type of support.
- On the basis of analysis of the beam, it is further classified as:-

(a) Statically determinate Beam (in 2-D)

A beam is said to be statically determinate when it can be analysed using the equations of statics.

Eg:- cantilever beam, S.S beam, overhanging beams

(b) Statically indeterminate Beam (2-D)

- when the number of unknown reactions or stress components exceeds the number of static equilibrium equations, the beam is termed as statically indeterminate beams.
- In such case for the analysis of the beam additional equations of compatibility are required.

eg:- fixed beams, propped cantilevers, continuous beams.

Here, Degree of indeterminacy (DOI) = $\frac{\text{Number of support reactions}}{\text{Number of equilibrium conditions.}}$

Hence, $DOI > 0 \rightarrow$ Beam is indeterminate

$DOI \leq 0 \rightarrow$ Beam is Determinate

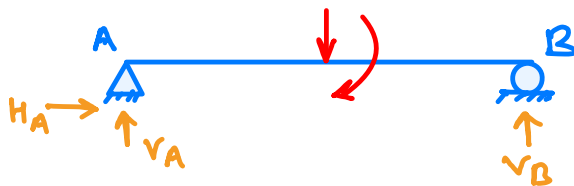
Also,

$DOI \geq 0 \rightarrow$ Beam is stable

$DOI < 0 \rightarrow$ Beam is unstable

eg:-

(i)



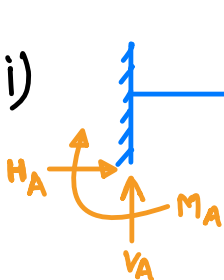
No. of Support rxn = 3

No. of eq^m Condition = 3

$$DOI = 3 - 3 = 0.$$

$\Rightarrow$ Hence, Beam is Determinate
Beam is Stable.

(ii)



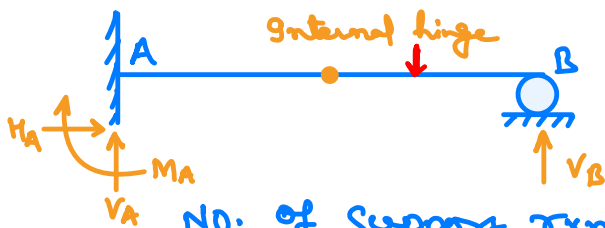
No. of Support rxn = 3

No. of eq^m Condition = 3

$$DOI = 3 - 3 = 0.$$

$\Rightarrow$ Hence, Beam is Determinate
Beam is Stable.

(iii)



No. of Support rxn = 4

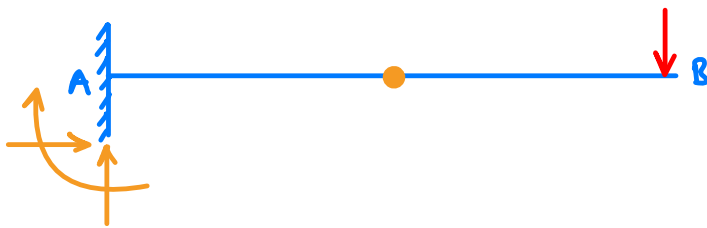
No. of eq^m Condition = $(3 + 1) = 4$

Due to internal hinge

$$DOF = 4 - 4 = 0$$

⇒ Hence, Beam is Determinate
Beam is Stable.

(iv)



No. of Support rxn = 3

No. of eq^m Condition = $(3 + 1) = 4$

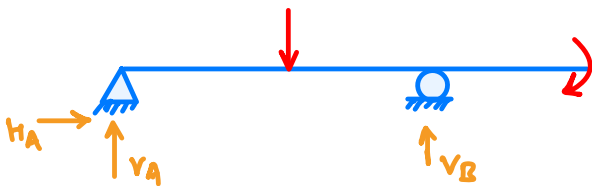
Due to internal hinge

$$DOF = 3 - 4 = -1$$

⇒ Hence, Beam is Determinate
Beam is Unstable.

Determinate beam
may be Stable
or unstable

(v)



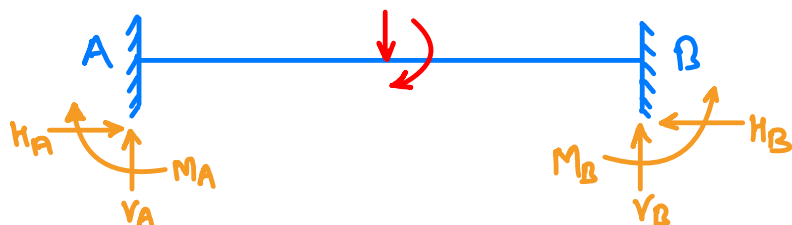
No. of Support rxn = 3

No. of eq^m Condition = 3

$$DOF = 3 - 3 = 0$$

⇒ Hence, Beam is Determinate
Beam is Stable.

(vi)



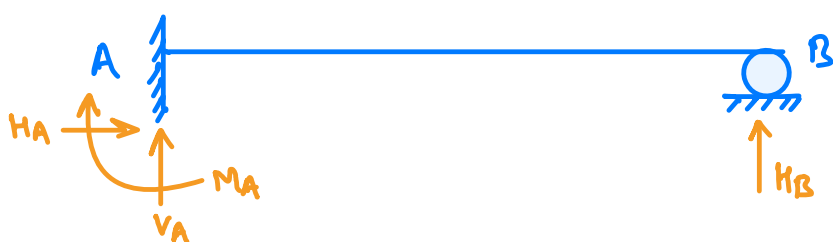
No. of Support rxn = 6

No. of eq^m Condition = 3

$$DOI = 6 - 3 = 3$$

⇒ Hence, Beam is indeterminate
Beam is Stable.

(vii)



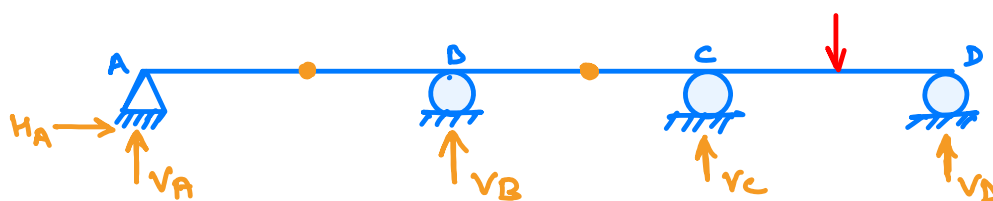
No. of Support rxn = 4

No. of eq^m Condition = 3

$$DOI = 4 - 3 = 1$$

⇒ Hence, Beam is indeterminate
Beam is Stable.

(vii)

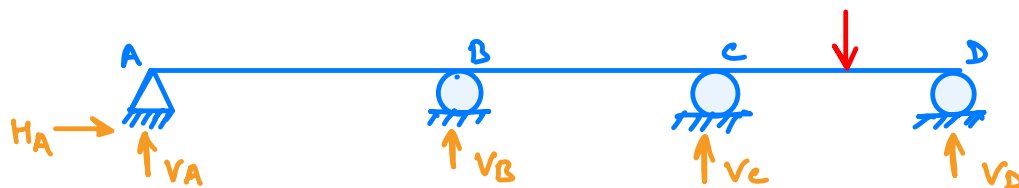


No. of Support reaction = 5, No. of eq^m Condition = 5 (3+1+1)

$$DOI = 5 - 5 = 0$$

Hence, beam is determinate & Stable.

(viii)



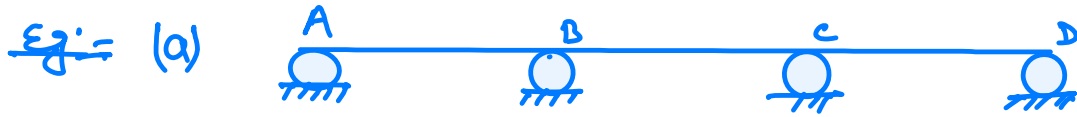
No. of Support reaction = 5, No. of eq^m Condition = 3

$$DOI = 5 - 3 = 2$$

Hence, Beam is indeterminate & Stable.

Note:-

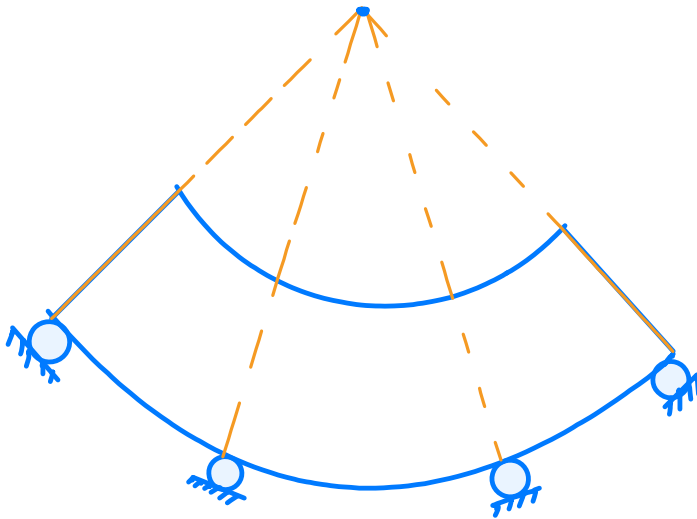
All indeterminate beams are stable (but only mathematically as they can be stable linearly or Angularly)



$$DOI = 4 - 3 = 1$$

Beam is indeterminate and linearly unstable.

(b)



$$DOI = 4 - 3 = 1$$

Beam is indeterminate Angularly Unstable.

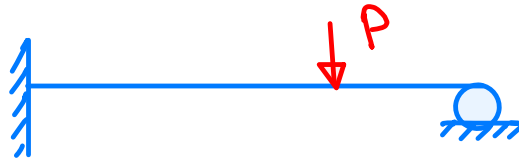
Note:-

- If in a stable structure a member fails, load is redistributed to the remaining/other members & whole structure does not fail.
- But in case of unstable structures if a member fails it results in failure of entire structure and load is not redistributed amongst other members.

Types of Loadings

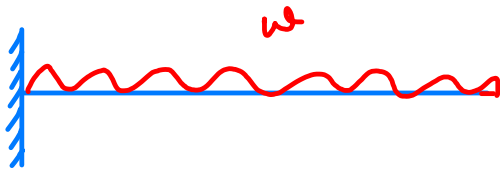
(a) Concentrated / Point load.

- A load acting at a point mathematically on the beam is termed as point load.
- Practically, any type of load acting on beam is distributed over an area, if that area is very small in comparison to length of beam is termed as point load.

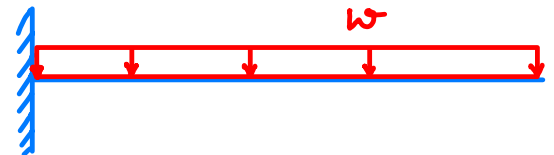


(b) Uniformly Distributed load.

- A load which is spread over a beam in such a manner that each unit length of beam is loaded to same extent, it is termed as UDL.

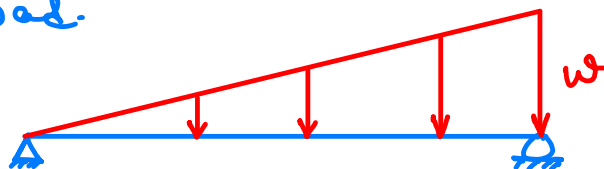


(00)

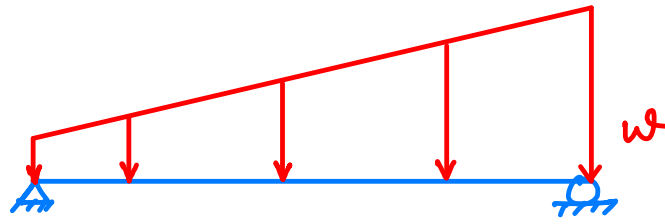


(c) Uniformly Varying load

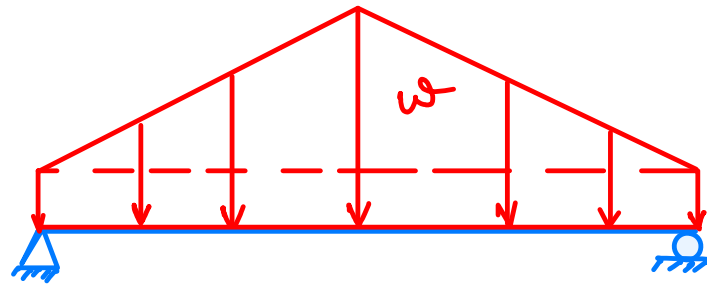
- A load which is spread over a beam in such a manner that its extent varies uniformly on each unit length of the beam, it is termed as uniformly varying load.



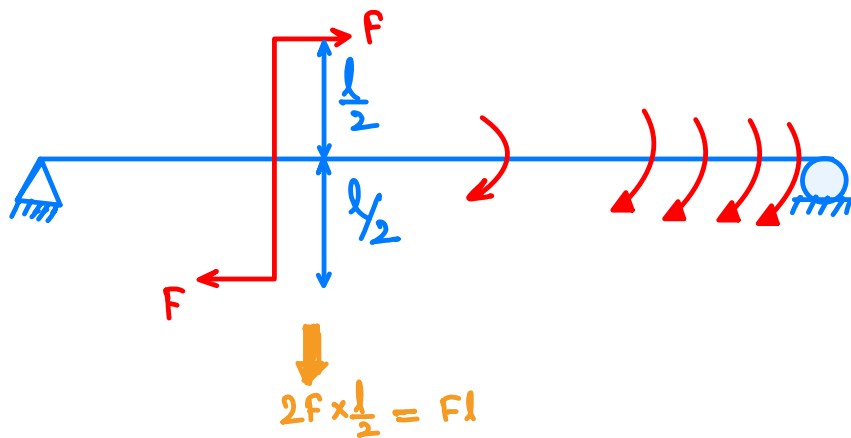
(00,)



(08)

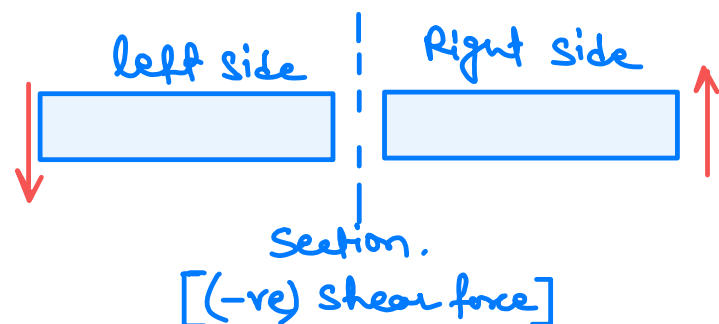
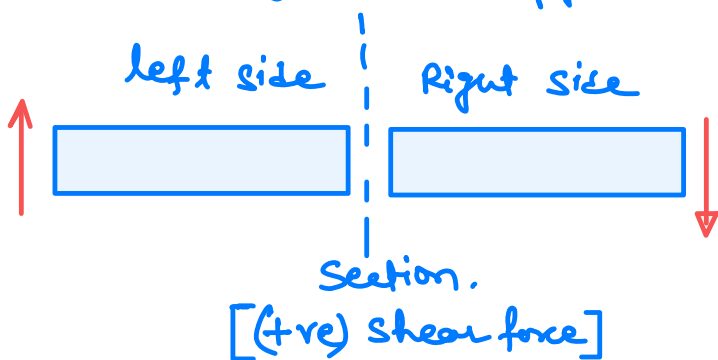


(d) Beam is subjected to moment both Concentrated & Uniformly Distributed moment.

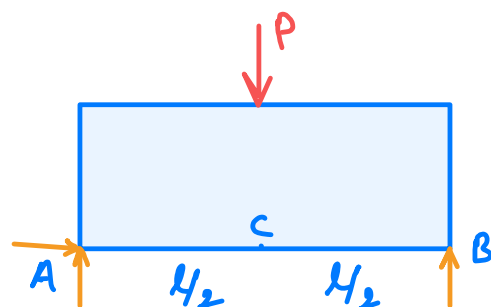
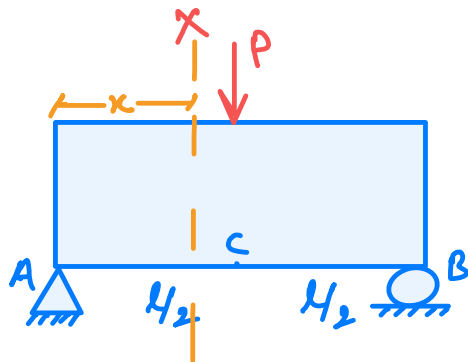


Shear force in beam

- Shear force at a section is algebraic summation or resultant of all the transverse force either to right/left of the section.
- It always acts along plane of cross-section.
- S.f at a section is taken to be positive if the resultant of all transverse forces to the right of section is downward or resultant of transverse forces to the left of the section is upward and for SF to be (-ve) it should be opposite.



eg:-



$$\sum f_x = 0 \Rightarrow H_A = 0$$

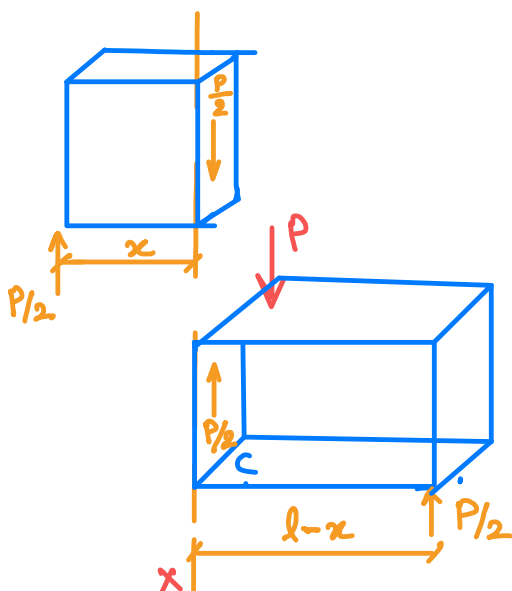
$$\sum f_y = 0 \Rightarrow V_A + V_B - P = 0$$

$$\sum M_A = 0 \Rightarrow V_B \cdot l - P \cdot \frac{l}{2} = 0$$

$$V_A = V_B = \frac{P}{2}$$

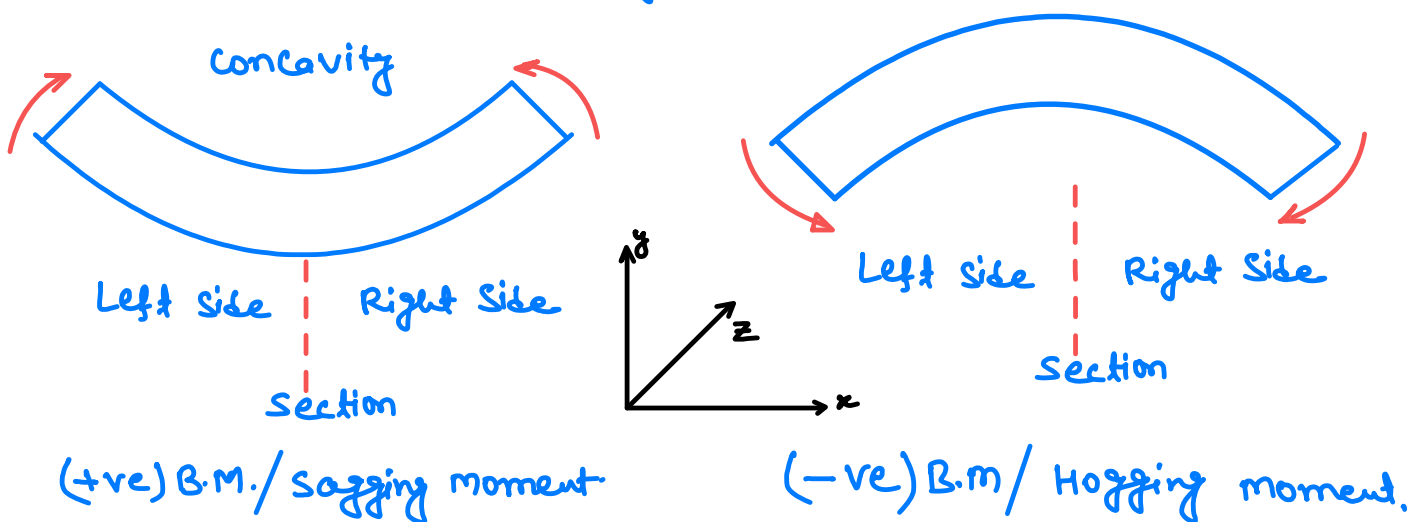
$$SF_{\text{left}} = \frac{P}{2}$$

$$SF_{\text{right}} = \frac{P}{2}$$



Bending Moment in Beam

- Bending moment at any given section along the span of beam is algebraic summation/resultant of moment (force $\times$ lever arm) at that section due to all the transverse load either from left side or right side of the section.
- At sections where the bending moment is such that it tends to bend the beam at that point to a curvature having concavity at top, it is taken as positive (+ve) (It is also termed as sagging moment) & where the Bending moment is such that it tends to bend the beam at that point to a curvature having convexity at the top, it is taken as Negative (It is also termed as Hogging moment).



Eg:- Same as previous case:-

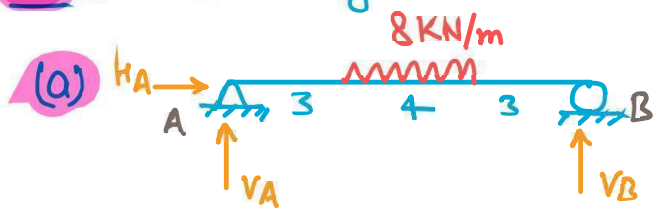
$$BM_{\text{left}} = \frac{P}{2} x$$

$$BM_{\text{right}} = \frac{P}{2}(1-x) - P\left(\frac{1}{2}-x\right) = \frac{Px}{2} - \frac{Px}{2} - \frac{Px}{2} + Px = \frac{Px}{2}$$

Note:-

Bending moment at a section is different from moment at a point where later is summation of moment due to all loadings on the beam produced at that point (for equilibrium moment at a point is zero).

Q> for following cases compute the support reaction:-



$$V_A \times 10 - 8 \times 4 \times \left(\frac{4}{2} + 3\right) = 0$$

$$V_A \times 10 = 160$$

$$V_A = \frac{160}{10} = \underline{16 \text{ kN}}$$

$$V_B = \underline{16 \text{ kN}}$$

$$\sum f_x = 0, \Rightarrow H_A = 0$$

$$\sum f_y = 0 \Rightarrow V_A + V_B - 32 = 0$$

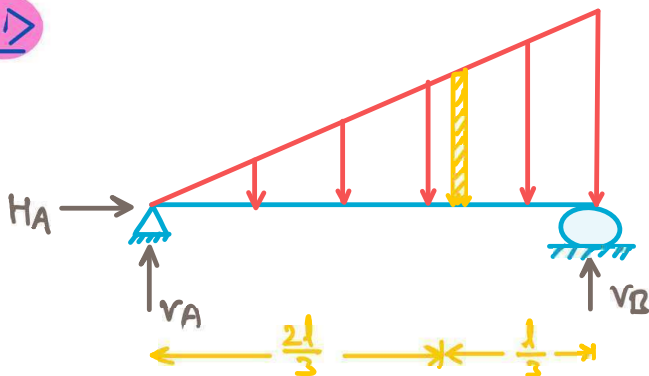
$$\sum M_A = 0 \Rightarrow V_B \cdot 10 - 32 \cdot 5 = 0$$

$$V_A = V_B = 16 \text{ kN.}$$

Note:-

only for moment equilibrium condition all load are assumed to act over its centroid.

b>



$$\sum f_x = 0 \Rightarrow H_A = 0$$

$$\sum f_y = 0 \Rightarrow V_A + V_B - \frac{1}{2} \omega l = 0$$

$$\sum M_A = 0 \Rightarrow V_B \cdot l - \frac{1}{2} \omega l \cdot \frac{2l}{3} = 0$$

$$V_A = \frac{\omega l}{6}, \quad V_B = \frac{\omega l}{3}$$

Note:-

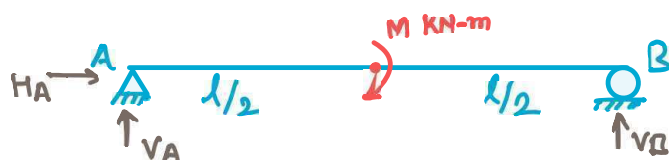
If the beam is symmetrical beam (if we draw a vertical line at mid span of the beam and exact mirror image is observed on both sides, it is termed as symmetrical beam), then vertical reactions at support have equal

magnitude and to compute it moment equilibrium condition is required.

Q. >

$$\sum f_y = 0 \Rightarrow V_A + V_B - 5 - 4 \cdot 4 - 5 = 0.$$

$$V_A = V_B = 13 \text{ kN}.$$

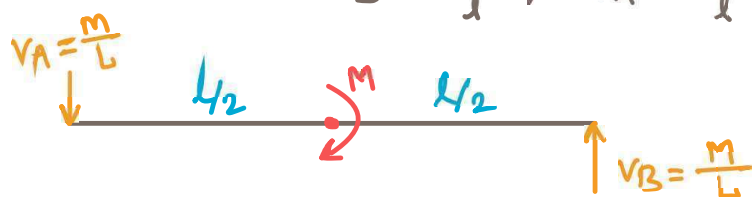
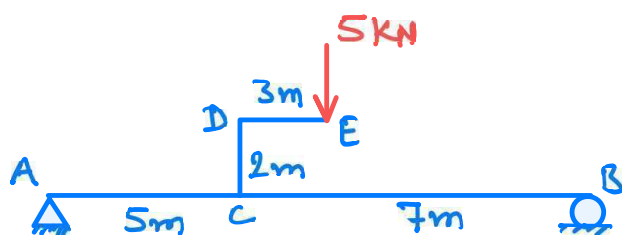
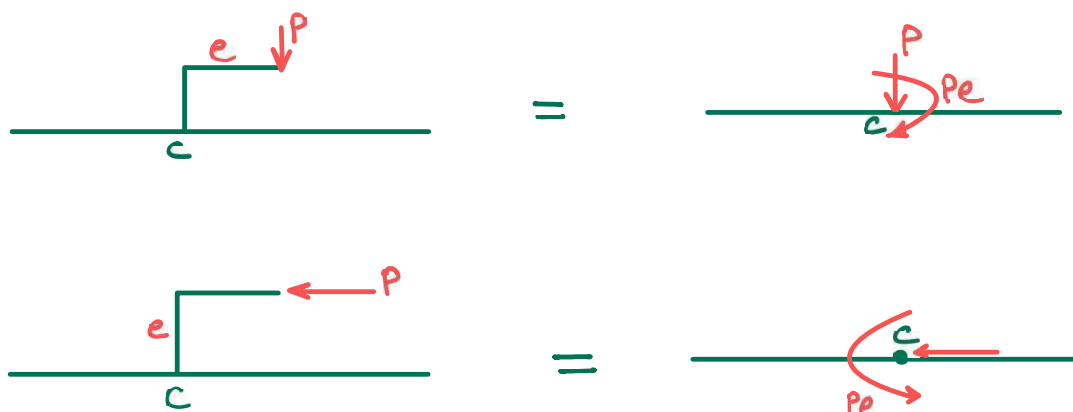
C. >

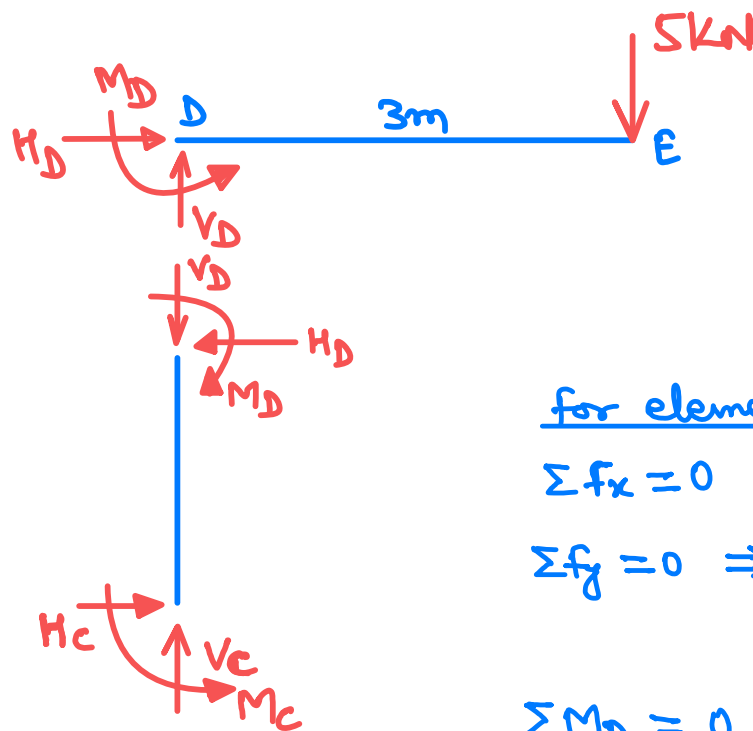
$$\sum f_x = 0 \Rightarrow H_A = 0.$$

$$\sum f_y = 0 \Rightarrow V_A + V_B = 0.$$

$$\sum M_A = 0 \Rightarrow V_B \cdot l - M = 0.$$

$$V_B = \frac{M}{l}, \quad V_A = -\frac{M}{l}$$

d. >Note!:-



for element DC

$$\sum f_x = 0$$

$$H_C - H_D = 0$$

$$H_C = H_D = 0$$

$$\sum f_y = 0$$

$$V_C - V_D = 0$$

$$\Rightarrow V_C = V_D = 5 \text{ kN}$$

$$\sum M_C = 0$$

$$-M_D + M_C = 0$$

$$M_C = M_D = 15 \text{ kN-m.}$$

for element DE

$$\sum f_x = 0 \Rightarrow H_D = 0$$

$$\sum f_y = 0 \Rightarrow V_D - 5 = 0 \\ V_D = 5 \text{ kN.}$$

$$\sum M_D = 0$$

$$-5 \times 3 + M_D = 0.$$

$$M_D = 15 \text{ kN-m}$$

for Beam AB

$$\sum f_x = 0 \Rightarrow H_A - H_C = 0$$

$$H_A = H_C = 0$$

$$\sum f_y = 0$$

$$\Rightarrow V_A + V_B = 5 \text{ kN.}$$

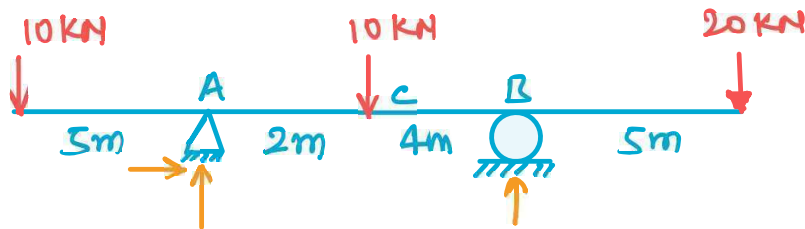
$$\sum M_A = 0$$

$$\Rightarrow V_B (5+7) - 15 - 5 \times 5 = 0$$

$$V_B = \frac{40}{12} = \frac{10}{3} \text{ kN}$$

$$V_A = \frac{5}{3} \text{ kN.}$$

e. >



$$10 \times 5 + V_B \times 6 - 10 \times 2 - 20 \times 11 = 0$$

$$6 \times V_B + 50 - 20 - 220 = 0$$

$$V_B = \frac{190}{6} = \frac{95}{3}$$

$$\sum f_x = 0 \Rightarrow H_A = 0$$

$$\sum f_y = 0 \Rightarrow V_A + V_B = 40 \text{ kN}$$

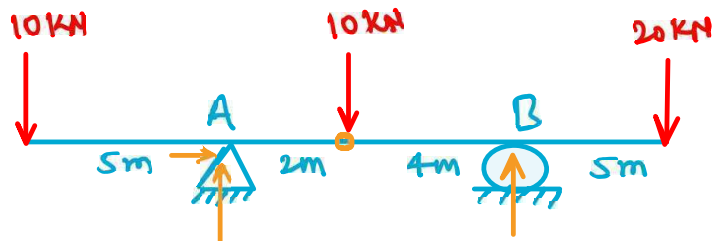
$$\sum M_A = 0 \Rightarrow -10 \times 5 + 10 \times 2 - V_B \times 6 + 20 \times 11 = 0$$

$$V_B = \frac{95}{3} \text{ kN}$$

$$V_A = 25/3 \text{ kN}$$

$$\text{or, } \sum M_B = 0 \Rightarrow +20 \times 5 - 10 \times 4 + V_A \times 6 - 10 \times 11 = 0$$

$$V_A = 25/3$$



$$\sum f_x = 0, \quad \sum f_y = 0, \quad \sum M_{\text{anywhere}} = 0, \quad M_C = 0$$

$$-10 \times 7 + V_A \times 2 = 0 \Rightarrow V_A = 35 \text{ kN}$$

$$\text{or, } 20 \times 9 - V_B \times 4 = 0 \Rightarrow V_B = 45 \text{ kN.}$$

Due to internal hinge

Beam is Determinate, but unstable (-1).

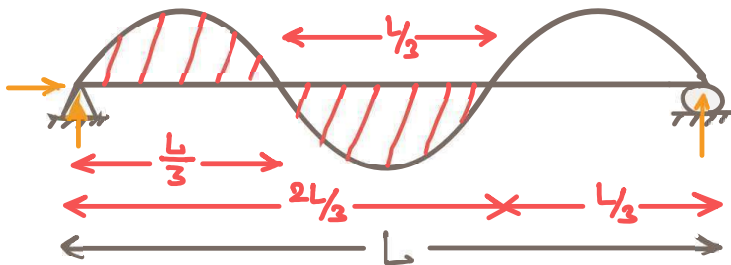
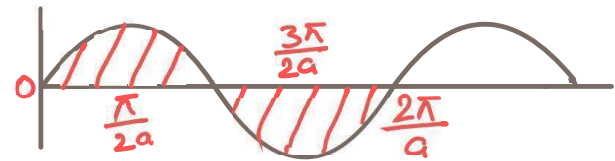
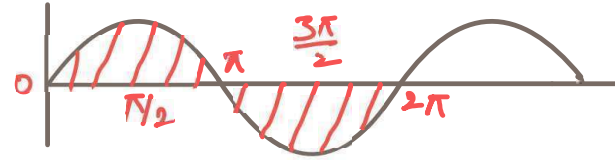
$$\text{So, } V_A + V_B \neq 40$$

Q.7 calculate the support reaction for simply supported beam subjected to uniformly varying load given as $y = \sin\left(\frac{3\pi x}{L}\right)$, where L is effective span of beam.

Soln

$$y = \sin \theta, \text{ Time Period} = 2\pi$$

$$y = \sin a\theta, \text{ Time period} = \frac{2\pi}{a}$$



$$\sum F_x = 0 \Rightarrow H_A = 0$$

Since beam is symmetric $V_A = V_B$

$$\sum F_y = 0 \Rightarrow V_A + V_B = W$$

W = Area of plane section

$$W = \int_0^L y \, dx \quad \text{or,} \quad \int_0^{L/3} y \, dx.$$

$$W = \int_0^L \frac{\sin 3\pi x}{L} \, dx = -\frac{L}{3\pi} \left| \cos \frac{3\pi x}{L} \right|_0^L$$

$$= -\frac{L}{3\pi} (\cos 3\pi - \cos 0)$$

$$= -\frac{L}{3\pi} (-1 - 1) = \frac{2L}{3\pi}$$

$$V_A = V_B = \frac{2L}{3\pi} \cdot \frac{1}{2} = \frac{L}{3\pi}$$

Relation between SF, BM and loading.

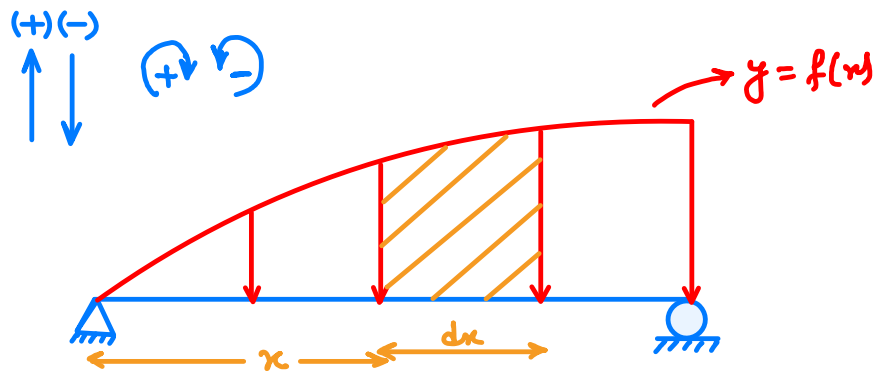
for equilibrium :-

$$\sum f_y = 0$$

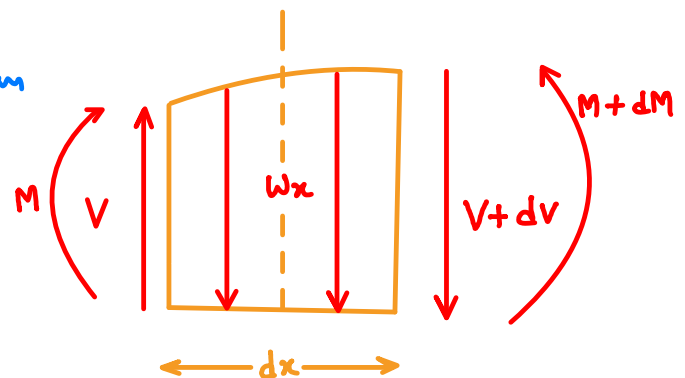
$$\Rightarrow V - w dx - (V + dV) = 0$$

$$\Rightarrow -w dx = dV$$

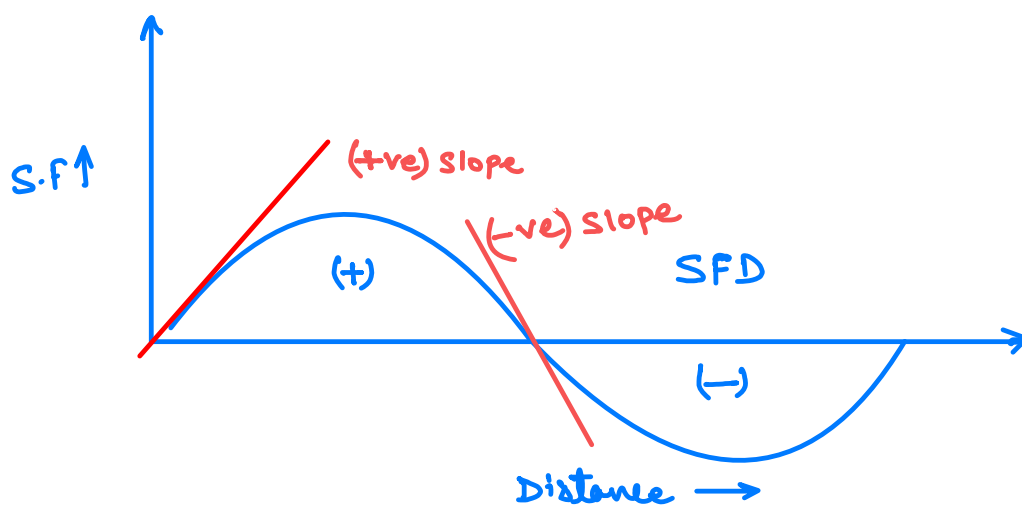
$$w = -\frac{dV}{dx} \quad \text{--- (A)}$$



- Hence, Slope of Shear force diagram is equal to the intensity of distributed load.



- If Slope of Shear force diagram is positive this implies that loading intensity at that point is (-ve) and if slope of S.F.D is (-ve) this implies that loading intensity at that point is positive (+ve).



for equilibrium, $\Sigma M_o = 0$

$$M + Vdx - \cancel{w dx \cdot \frac{dx}{2}}^{\text{Neglected}} - (M + dM) = 0$$

$$Vdx = dM$$

$$V = \frac{dM}{dx} \quad \text{--- (B)}$$

— Hence, Slope of B.M.D at any section is equal to S.F. at that Section.

from equation (A),

$$-\frac{dV}{dx} = w$$

$$-\int dV = \int w dx$$

$$\Rightarrow \text{change in Shear force, } (V_{\text{final}} - V_{\text{initial}}) = \text{Area Under distributed loading diagram between two points.}$$

from equation (B),

$$\frac{dM}{dx} = V$$

$$\int dM = \int V dx$$

$$\Rightarrow \text{Change in B.M between two Section } (BM_{\text{final}} - BM_{\text{initial}}) = \text{Area Under S.F.D between two points}$$

Note!:-

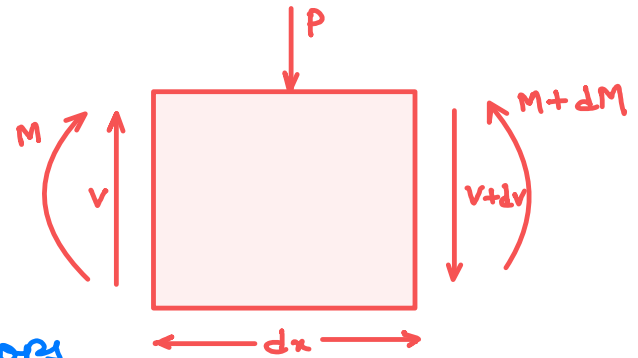
The first relation (A) does not hold good for point load and the second relation (B) does not hold good for Concentrated Moment hence, their effect is taken separately as follows:-

$$\Sigma f_y = 0$$

$$\Rightarrow V - P - (V + dV) = 0$$

$$\boxed{dV = -P}$$

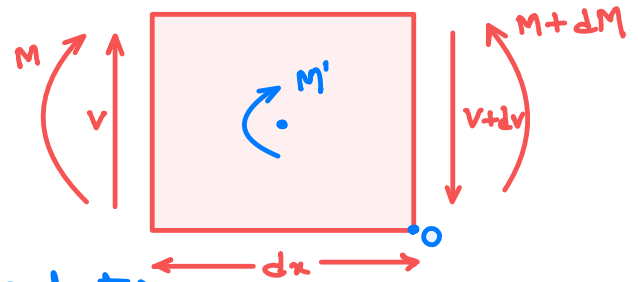
- Hence, if point load acts downward at a section, SF drops by amount "P" at that section and if point load act upward at the section SF increases by amount "P" at that section.



$$\Sigma M_o = 0$$

$$\Rightarrow M + Vdx + M' - (M + dM) = 0$$

$$dM = M' + Vdx$$

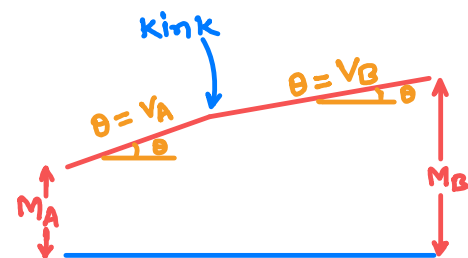
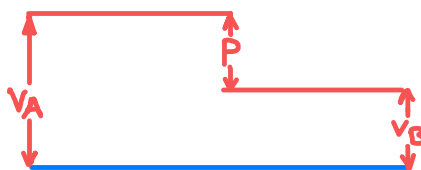
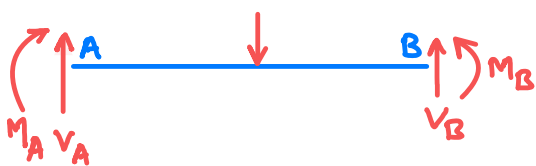


- Hence, due to clockwise concentrated moment, Bending moment suddenly increases at that section and due to anticlockwise moment, Bending moment suddenly decreases at that section.

Loading

SFD

BMD

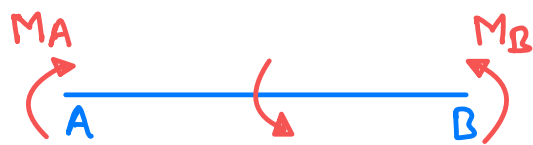


- point load causes sudden change in SFD and kink in the BMD.
- for point load in any position of beam, SFD has zero slope and BMD is a linear/ 1° curve.

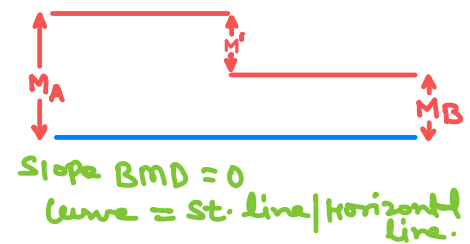
Loading

SFD

BMD



0

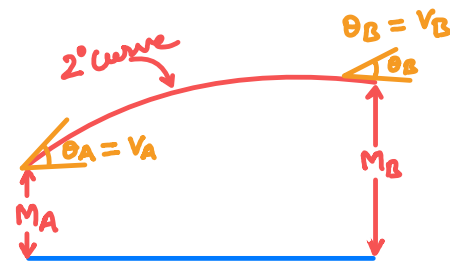
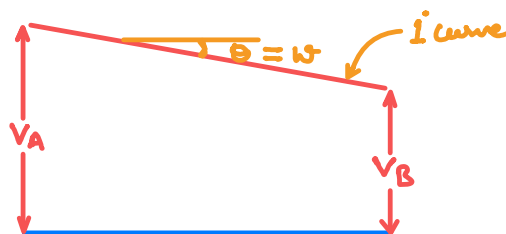
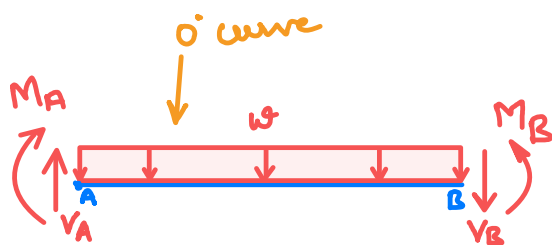


- for pure Bending, $SF = 0$, Slope of BMD is also zero.

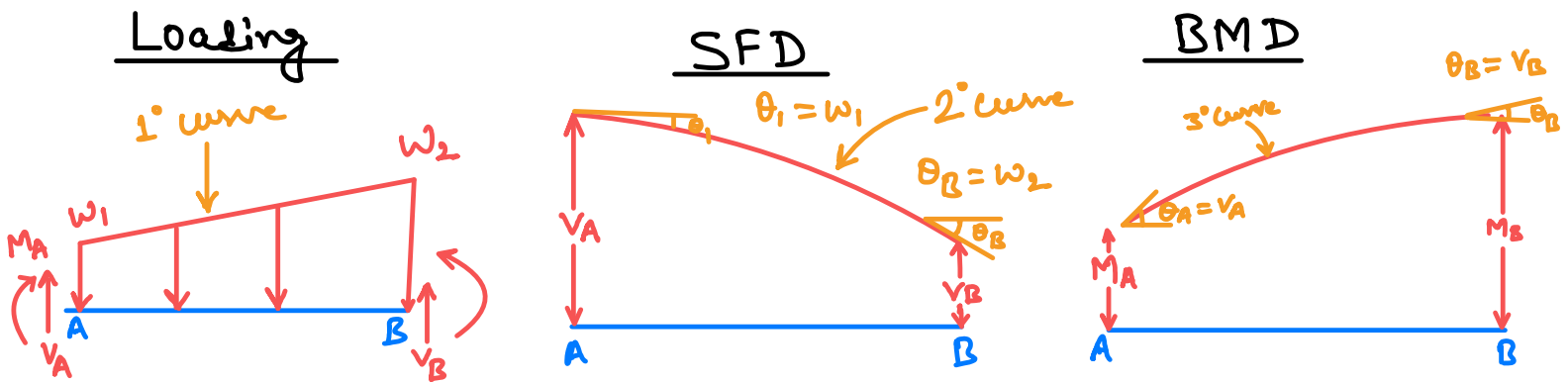
Loading

SFD

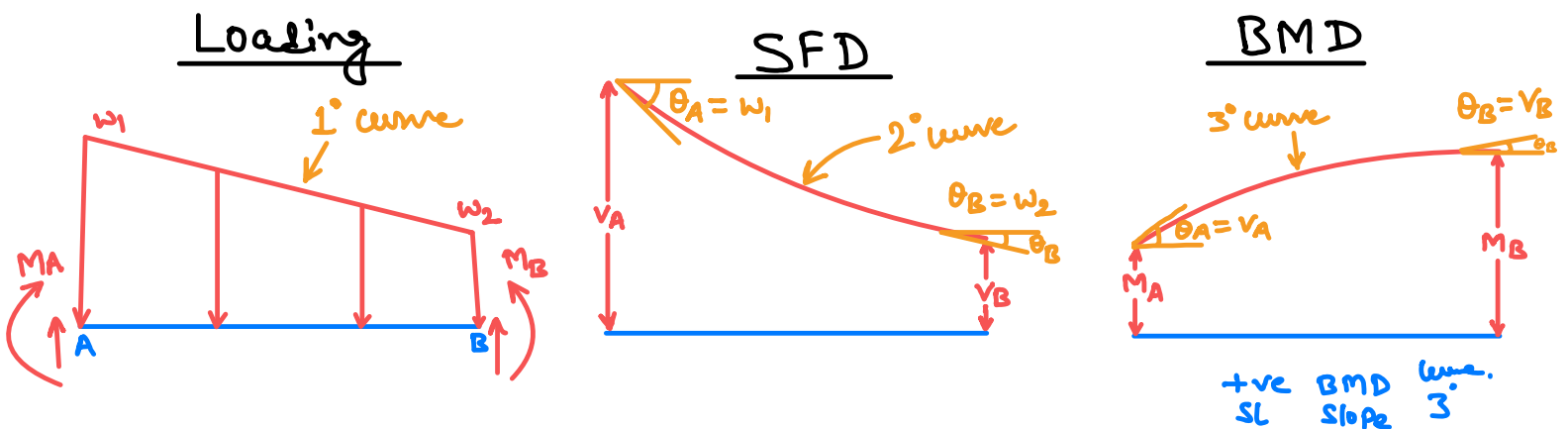
BMD



- Here, loading is Negative and constant hence, Slope of SFD is negative & constant.
- Here, SF is positive & decreasing hence Slope of BMD is positive & decreasing.
- If load intensity is uniform (UDL) then SFD is linear and BMD is parabolic.



- Here loading is Negative & increasing hence, Slope of SFD is Negative & increasing.
- Here SF is positive and decreasing hence, Slope of BMD is also positive and decreasing.
- If loading intensity is UVL then SFD is parabolic and BMD is Cubical.



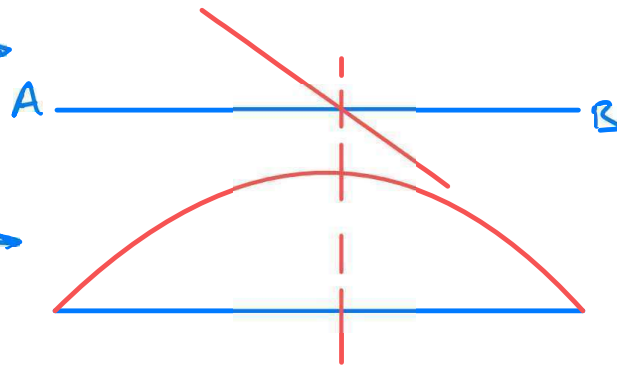
- Here loading is Negative & decreasing, hence Slope of SFD is Negative & decreasing
- Here, SF is positive and decreasing, hence Slope of BMD positive & decreasing.

Note:-

- A point where SF changes sign Bending moment is either maximum or minimum wrt its neighbourhood.

- For BM to be maximum or minimum, $\frac{dM}{dx} = 0$
hence, $V = 0$.

SFD: →



BMD: →

Note:-

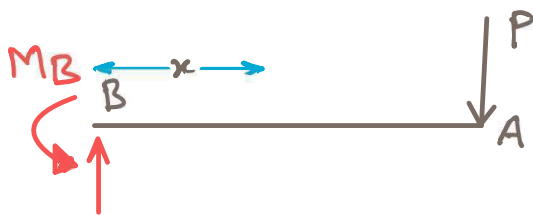
A point where BM changes the sign is termed as "point of contraflexure" and the same point in the deflected curve shape is termed as "point of inflection."

★ While drawing SFD & BMD following critical points are to be analysed properly :-

- (a) Starting and end point of beam
- (b) point of concentrated load and moment
- (c) Starting and end point of UDL & UVL.
- (d) point where SF & BM changes sign.

Q> Draw the SFD & BMD for following cases:-

A>



$$\sum f_y = 0 \Rightarrow V_B - P = 0$$

$$V_B = P$$

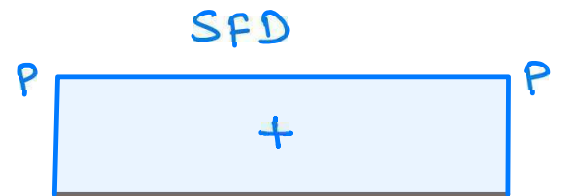
$$\sum M_B = 0 \Rightarrow Pl - M_B = 0$$

$$M_B = Pl$$

Soln

1st Approach.

$$V_x = V_B = P$$

2nd Approach

At point B & point A due to point load V_B and P in upward & downward direction

SF increases suddenly at B by $V_B = P$

SF decreases suddenly at A by P

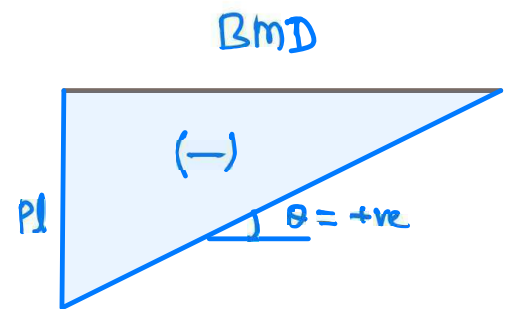
- Intensity of loading between $B \rightarrow A$, $w=0 \Rightarrow$ Slope of SFD = 0

for BMD, 1st Approach.

$$BM_x = V_B x - M_B = Px - Pl$$

$$\text{At point B, } x=0 \Rightarrow BM_B = -Pl$$

$$\text{At point A, } x=l \Rightarrow BM_A = 0$$

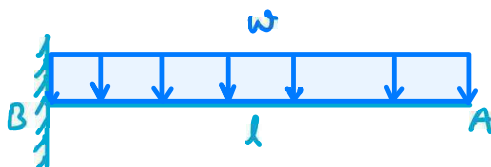
2nd Approach

$$BM_B = -M_B = -Pl$$

$$\Delta BM = BM_A - BM_B = \text{Area of SFD}$$

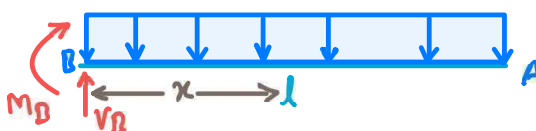
$$BM_A - (-Pl) = Pl$$

$$BM_A = 0$$

B.>Solution.

$$\Sigma f_y = 0 \Rightarrow V_B - wl = 0$$

$$V_B = wl$$



$$\Sigma m_B = 0 \Rightarrow wl \cdot \frac{l}{2} + M_B = 0$$

$$M_B = -\frac{wl^2}{2}$$

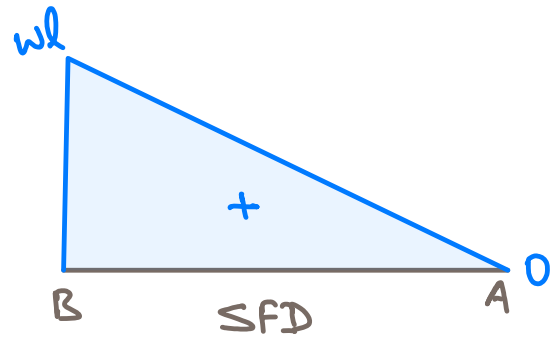
1st Approach

$$V_x = V_B - wx$$

$$= wl - wx.$$

At point B, $x=0 \Rightarrow V_B = wl$

At point A, $x=l \Rightarrow V_A = 0$

2nd Approach.

Due to upward point load at B (V_B), SF increases suddenly by $V_B = wl$.

$$\Delta V = V_A - V_B = \text{Area of loading diagram}$$

$$V_A - wl = -wl$$

$$V_A = 0$$

Since loading is negative and constant (w), slope of SFD is also Negative and constant (w).

BMD1st Approach

$$Bm_x = V_B \cdot x + m_B - wx \cdot \frac{x}{2}$$

$$= wl \cdot x + \left(\frac{wl^2}{2} \right) - \frac{wx^2}{2}$$

At pt. B, $x=0$, $Bm_B = -\frac{wl^2}{2}$

At pt. A, $x=l$, $Bm_A = wl - \frac{wl^2}{2} - \frac{wl^2}{2}$

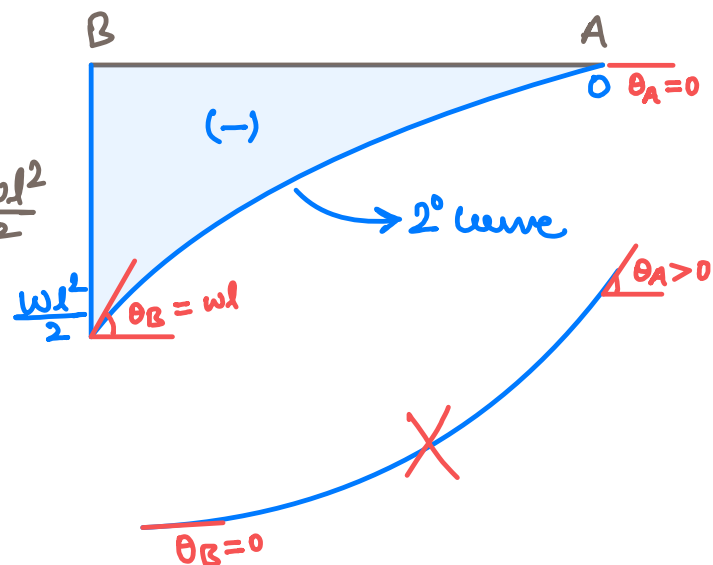
$$Bm_A = 0$$

2nd Approach.

$$Bm_B = -\frac{wl^2}{2}$$

$\therefore$ SF is positive and decreasing

Slope of BMD is also positive & decreasing

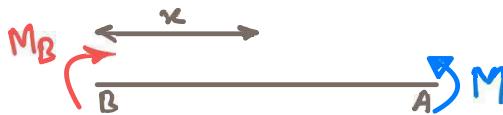


$$\Delta BM = BM_A - BM_B = \text{area of SFD}$$

$$BM_A - \left(-\frac{wl^2}{2}\right) = \frac{1}{2} wl \cdot l$$

$$BM_A = 0$$

C>

Solution:-

$$\sum M_B = 0, \Rightarrow -M + M_B = 0, \Rightarrow M_B = M$$

SFD

1st Approach.

$$V_x = 0$$

2nd Approach.

Since loading intensity $w=0$,
Slope of SFD = 0

BMD

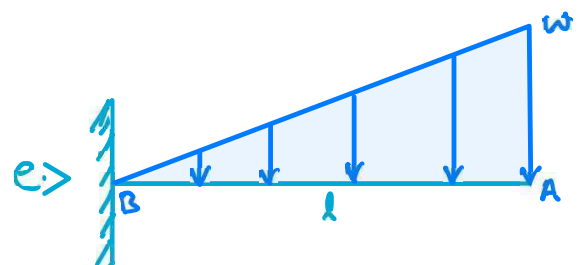
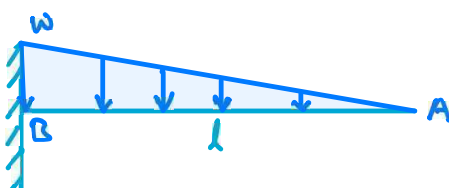
1st Approach

$$BM_x = M$$

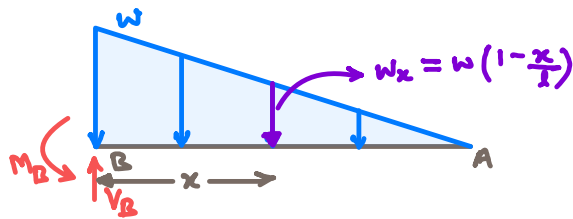


2nd Approach.

- Due to concentrated moment
 M_B clockwise at B and M anticlockwise at A,
 BM increases by M at B & decreases by M at A.
- Since $SF=0$, Slope of BMD is 0.



D>

Solution:-

$$\sum f_y = 0 \Rightarrow V_B - \frac{1}{2}wl = 0$$

$$V_B = \frac{wl}{2}$$

$$\sum M_B = 0 \Rightarrow \frac{1}{2}wl \cdot \frac{l}{3} - M_B = 0$$

$$M_B = \frac{wl^2}{6}$$

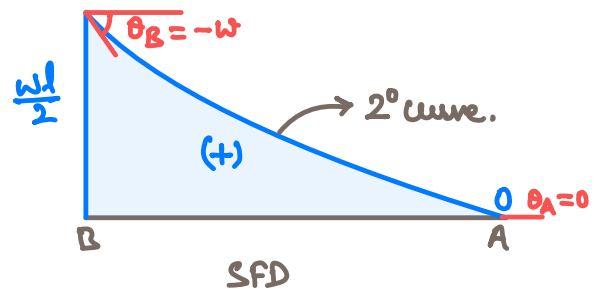
SFD1st Approach

$$V_x = V_B - \frac{1}{2}\{w + w(1 - \frac{x}{l})\}x$$

$$= \frac{wx^2}{2l} - wx + \frac{wl}{2}$$

$$\text{At Point B, } x=0 \Rightarrow V_x = V_B = \frac{wl}{2}$$

$$\text{At point A, } x=l \Rightarrow V_x = V_A = 0$$

2nd Approach

- Due to point load V_B at point B, SF increases suddenly by $V_B = \frac{wl}{2}$
- Since loading is decreasing & Negative slope of SFD is also Negative and decreasing.
- $\Delta V = V_A - V_B = \text{Area of loading diagram.}$

$$V_A - \frac{wl}{2} = -\frac{wl}{2}$$

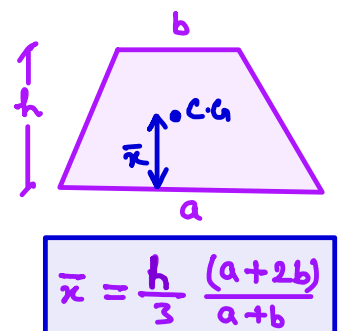
$$V_A = 0$$

BMD1st Approach

$$BM_x = V_B x - M_B - \frac{1}{2}\{w + w(1 - \frac{x}{l})\}x \cdot \bar{x}$$

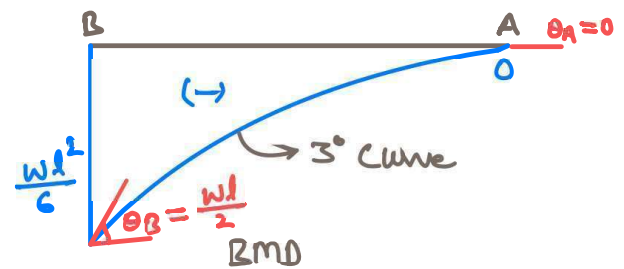
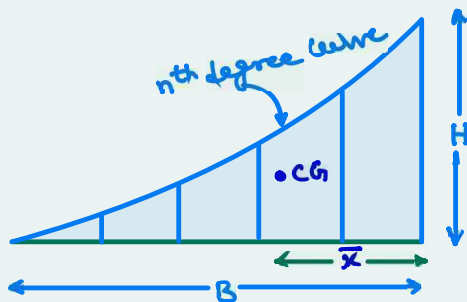
$$\text{At point B, } x=0 \Rightarrow BM_B = -M_B = -\frac{wl^2}{6}$$

$$\text{At point A, } x=l \Rightarrow BM_A = BM_A = 0$$



2nd Approach.

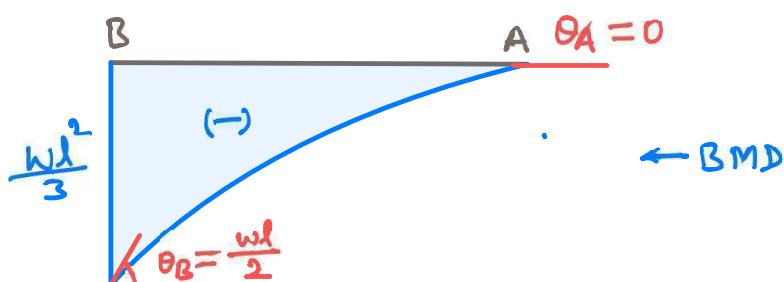
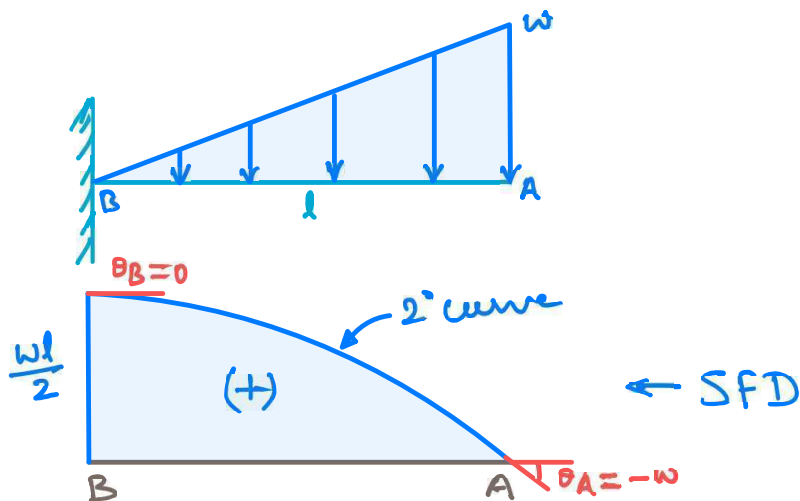
- $Bm_B = -M_B = -\frac{wl^2}{6}$
- Since SF is (+ve) & decreasing, slope of BM is also positive & decreasing.
- $\Delta BM = Bm_A - Bm_B = \text{Area of SFD}$
 $Bm_A - \left(-\frac{wl^2}{6}\right) = \frac{1}{2} \cdot \frac{wl}{2} \cdot l$
 $Bm_A = 0.$

Note :-

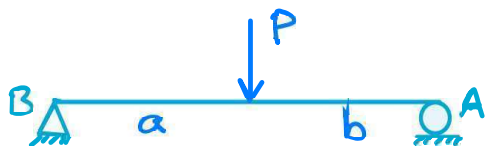
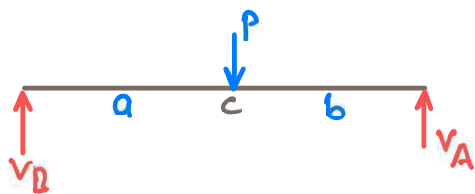
$$\text{Area} = \frac{BH}{n+1}$$

$$\bar{x} = \frac{B}{n+2}$$

e.>



f >

Solution:-

$$\sum f_y = 0 \Rightarrow V_A + V_B - P = 0$$

$$\Rightarrow V_A + V_B = P \quad \text{--- (i)}$$

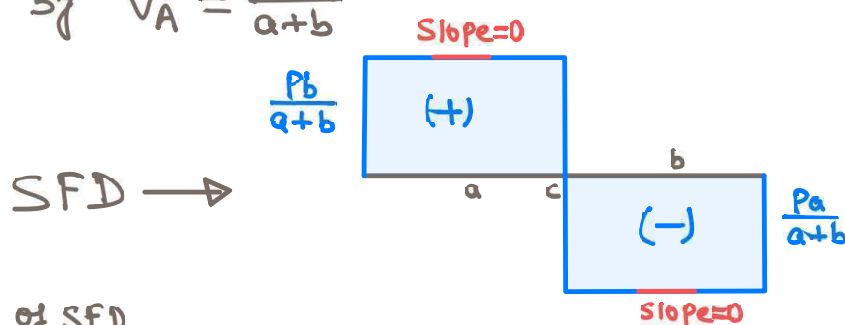
$$\sum M_B = 0 \Rightarrow -V_A \times (a+b) + P \times a = 0 \quad \text{--- (ii)}$$

$$V_A = \frac{Pa}{a+b}$$

$$V_B = \frac{Pb}{a+b}$$

SFD:-

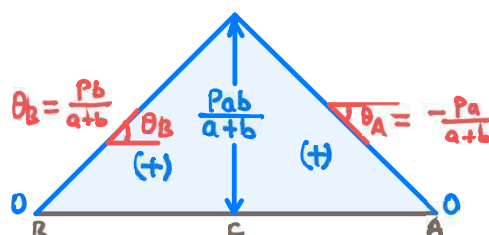
- Due to point load at B, " V_B " in upward direction, SF increases suddenly by $V_B = \frac{Pb}{a+b}$
- As loading is zero from B to C, Slope of SFD is also zero
- Due to point load at C, P in downward direction SF decreases suddenly by P
- Hence SF at right side of C = $\frac{Pb}{a+b} - P = \frac{-Pa}{a+b}$
- As loading is zero from C to A, Slope of SFD is also zero.
- Due to point load at A, " V_A " in upward direction SF increases suddenly by $V_A = \frac{Pa}{a+b}$



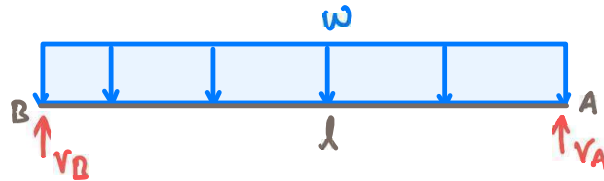
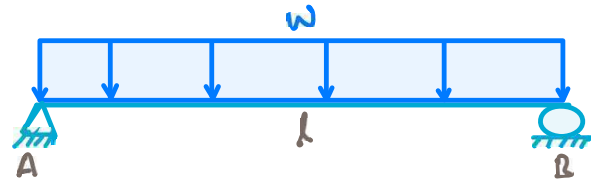
$$\Delta BM = BM_C - BM_B = \text{Area of SFD}$$

$$BM_C - 0 = \frac{Pb}{a+b} \cdot a$$

$$BM_C = \frac{Pab}{a+b}$$

BMD $\rightarrow$ 

Q.7



$$V_A = V_B \text{ --- (i)}$$

$$\sum f_y = 0 \Rightarrow V_A + V_B - wl = 0 \text{ --- (ii)}$$

$$V_A = V_B = \frac{wl}{2}$$

$$\Delta V = V_C - V_B = \text{Area of loading diagram}$$

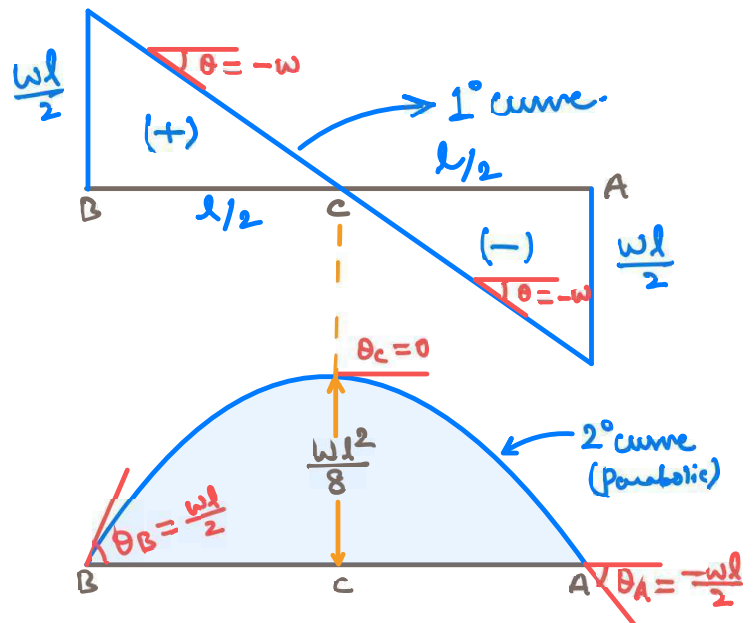
$$0 - \frac{wl}{2} = \frac{1}{2} w x$$

$$x = \frac{l}{2}$$

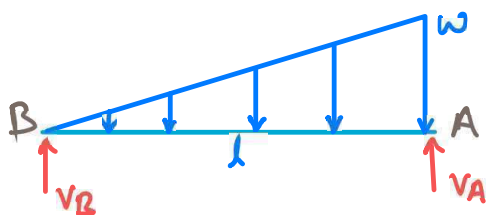
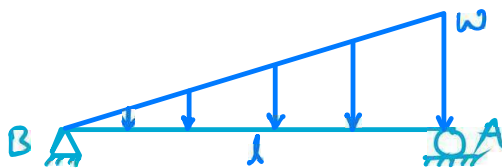
$$\Delta BM = BM_C - BM_A = \text{Area of SFD}_{AC}$$

$$BM_C - 0 = \frac{1}{2} \cdot \frac{wl}{2} \cdot \frac{l}{2}$$

$$BM_C = \frac{wl^2}{8}$$



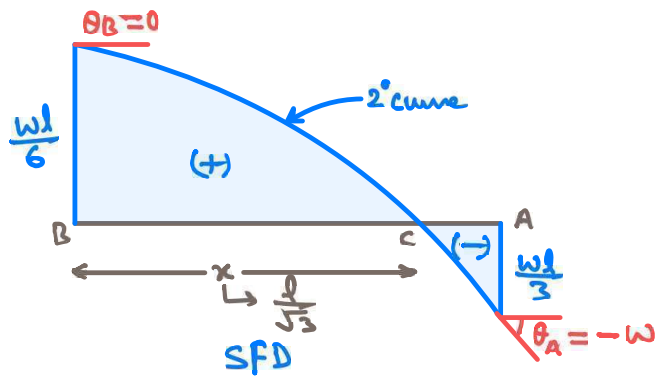
H.7



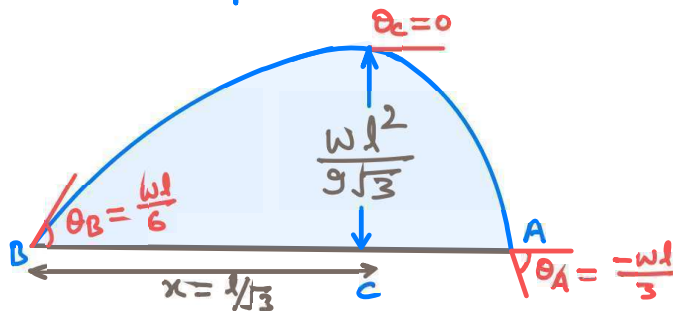
$$\sum f_y = 0, \quad V_B + V_A - \frac{wl}{2} = 0 \text{ --- (i)}$$

$$122 \sum M_B = 0, \quad -V_A \cdot l + \frac{wl}{2} \cdot \frac{2l}{3} = 0 \text{ --- (ii)}$$

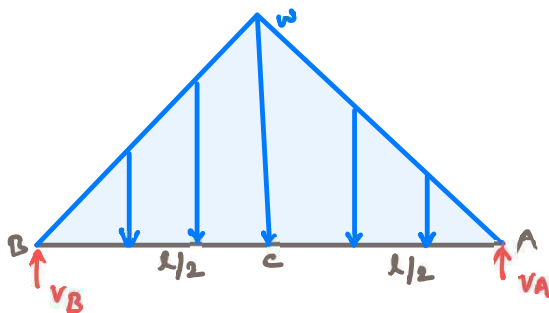
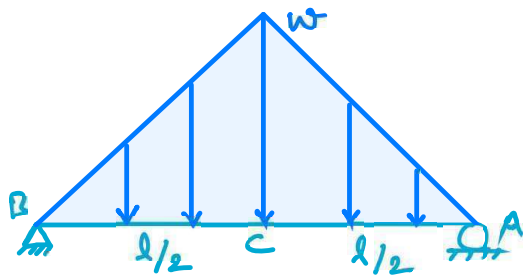
$$V_A = \frac{wl}{3}, \quad V_B = \frac{wl}{6}$$



$$\begin{aligned} \Delta V &= V_C - V_B = \text{Area of loading diag.} \\ &= 0 - \frac{wl}{6} = \frac{1}{2} \cdot x \cdot \frac{wx}{1} \\ x &= \frac{l}{3} \end{aligned}$$



$$\begin{aligned} \Delta BM &= BM_C - BM_B = \text{Area of SFD} \\ BM_C - 0 &= \frac{2}{2} \frac{wl}{6} \cdot \frac{l}{3} \\ BM_C &= \frac{wl^2}{9\sqrt{3}} \end{aligned}$$

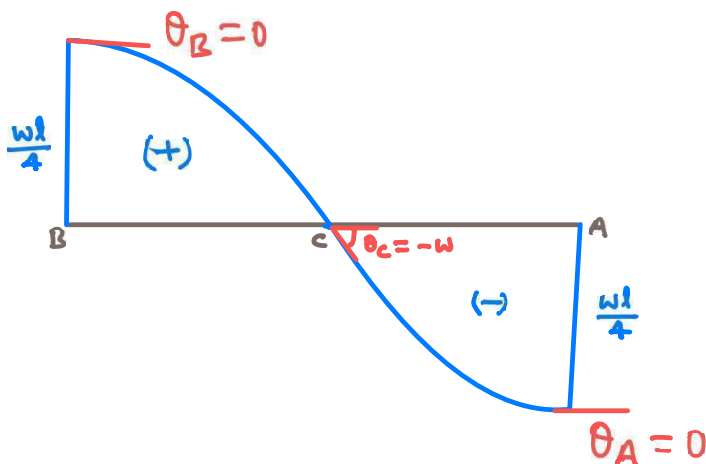


As beam is Symmetric

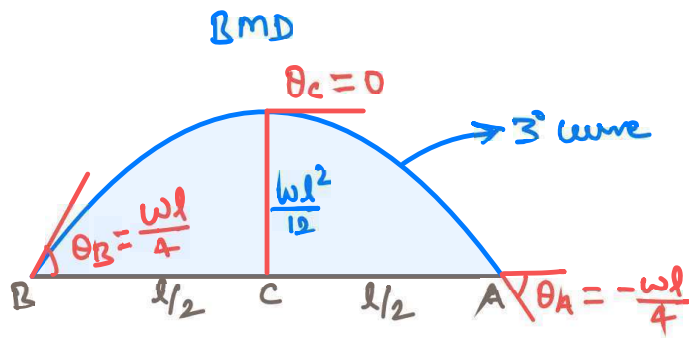
$$V_A = V_B$$

$$\sum F_y = 0, \Rightarrow V_A + V_B = \frac{1}{2} l \cdot w$$

$$V_A = V_B = \frac{wl}{4}$$



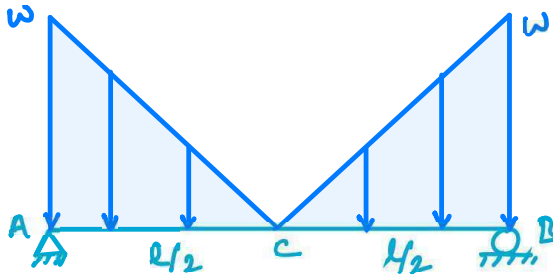
$$\begin{aligned} \Delta V &= V_C - V_B = \text{Area of loading diag.} \\ &= V_C - \left(\frac{wl}{4}\right) = -\frac{1}{2} \cdot \frac{l}{2} \times w \\ V_C &= 0 \end{aligned}$$



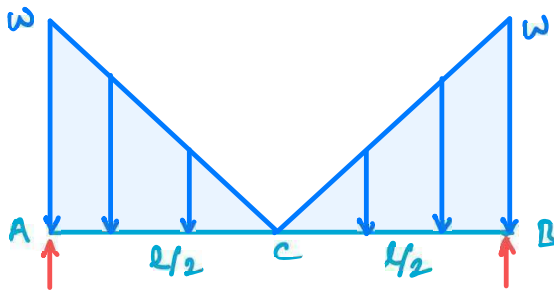
$$\Delta BM = BM_C - BM_B = \text{Area of SFD}$$

$$= BM_C - 0 = \frac{2}{3} \left(\frac{wl}{4} \cdot \frac{l}{2} \right)$$

$$BM_C = \frac{wl^2}{12}$$



Soln

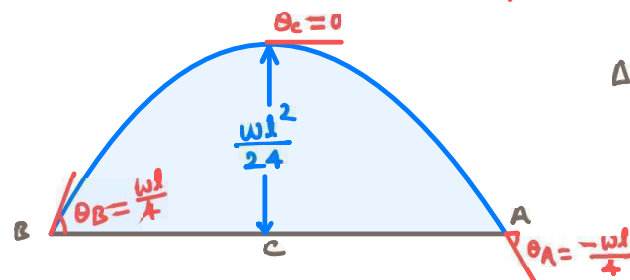
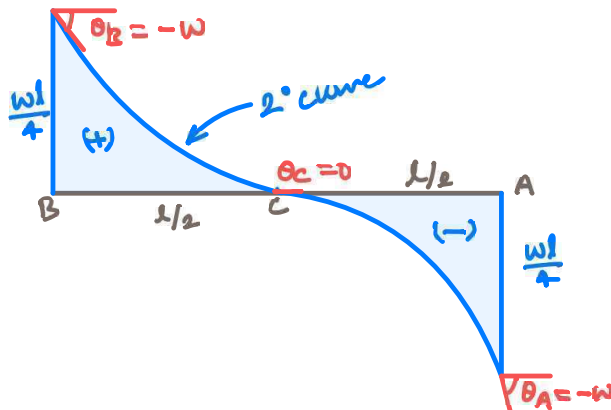


Since beam is symmetric,

$$V_A = V_B$$

$$\Sigma f_y = 0 \Rightarrow V_A + V_B - \frac{1}{2} wl \cdot 2 = 0$$

$$V_A = V_B = \frac{wl}{4}$$



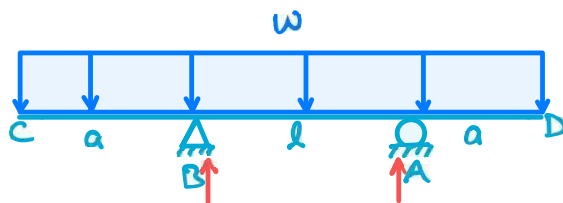
$$\Delta BM = BM_C - BM_B = \text{Area of SFD}_{BC}$$

$$= BM_C - 0 = \frac{1}{2} \frac{wl}{4} \cdot \frac{l}{2}$$

$$BM_C = \frac{wl^2}{24}$$

Note:-

- SFD & BMD gives the location of the section along the span of the beam where S.F & B.M is maximum thereby gives the idea about the safety of the beam.
- If the section at which SF is maximum (i.e. shear stress is maximum) shear strength is less than maximum shear stress redesigning is required and if the section at which B.M is maximum (i.e. Bending stress is max^m) Bending stresses are more than yield strength, redesigning is required.

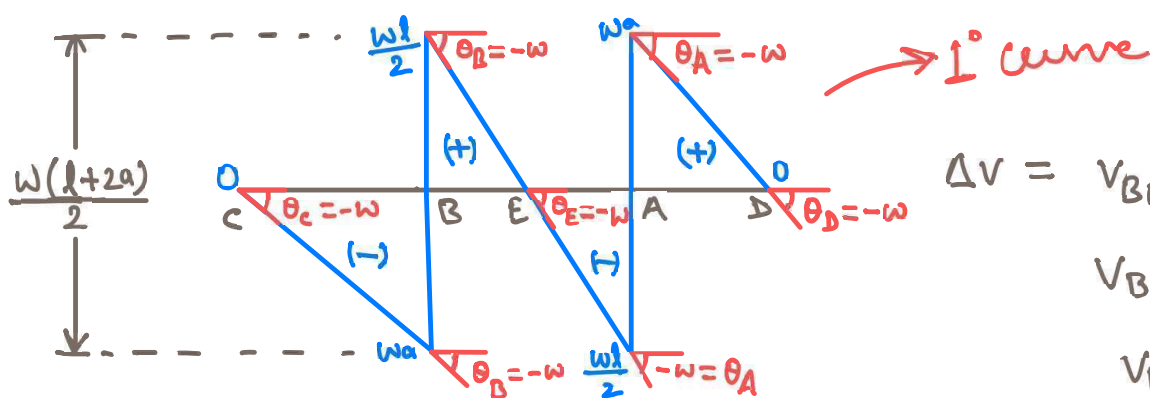
Soln

Since the beam is symmetric

$$V_B = V_A$$

$$\sum F_y = 0 \Rightarrow V_A + V_B - w(l+2a) = 0.$$

$$V_A = V_B = \frac{w(l+2a)}{2}$$



$$\Delta V = V_{B_L} - V_C = \text{Area of load triangle}$$

$$V_{B_L} - 0 = -wa$$

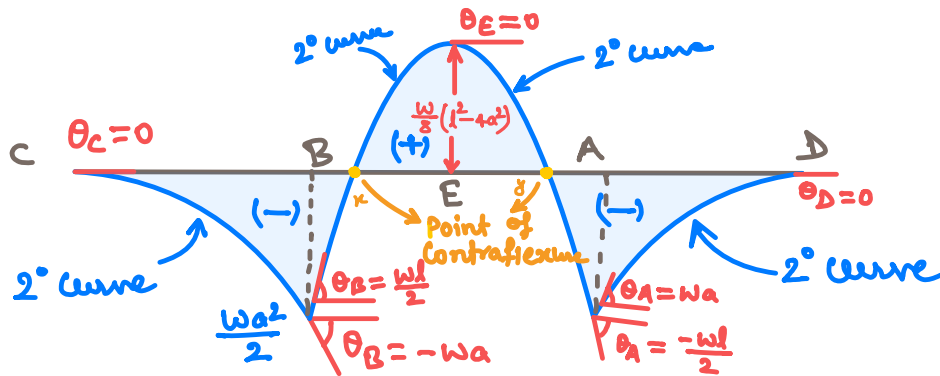
$$V_{B_L} = -wa$$

Due to point load at B, V_B in upward direction, SF increases suddenly by V_B

$$V_{B_{right}} = -wa + V_B = -wa + \frac{w(l+2a)}{2}$$

$$V_{BR} = \frac{wl}{2}$$

BMD



$$\Delta Bm = Bm_B - Bm_C = \text{Area SFD}$$

$$Bm_B - 0 = -\frac{1}{2} (wa) a$$

$$Bm_B = -\frac{wa^2}{2}$$

$$\Delta Bm = Bm_E - Bm_B = \text{Area of SFD}$$

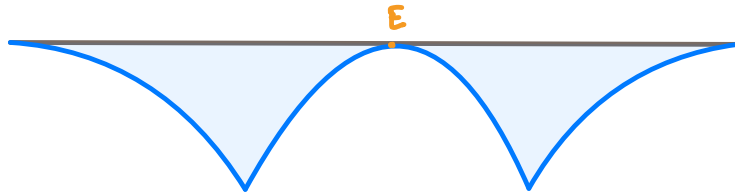
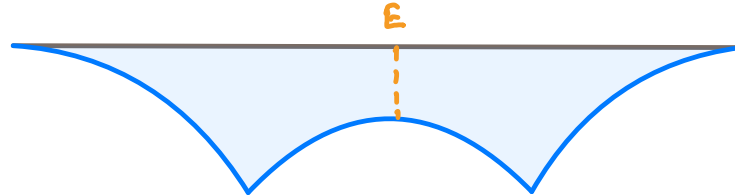
$$= Bm_E - \left(-\frac{wa^2}{2}\right) = \frac{1}{2} \frac{wl}{2} \times \frac{l}{2} \Rightarrow Bm_E = \frac{wl^2}{8} - \frac{wa^2}{2}$$

$$Bm_E = \frac{w}{8} (l^2 - 4a^2)$$

Case - (i) $l > 2a$ i.e. $l^2 > 4a^2$, Bm_E (+ve) (2 POC)

Case - (ii) $l = 2a$ i.e. $l^2 = 4a^2$, $Bm_E = 0$ (0 POC)

Case - (iii) $l < 2a$ i.e. $l^2 < 4a^2$, Bm_E (-ve) (0 POC)

Case - 2Case - 3Note:-

- for B.M to be zero at center of the beam, % of overhang required on both sides.

$$\% \text{ of overhang (on each side)} = \frac{a}{l+2a} \times 100 = \frac{a}{4a} \times 100 = 25\%$$

- for 2 point of Contraflexure, % of overhang required on each side : 25%.
- for 0 point of Contraflexure, % of overhang required on each side : $\geq 25\%$.
- when supports are near the end of the beam then Sagging B.M. at centre is large and if support move towards each other then Sagging B.M. decreases but Hogging B.M. at support increases.
- Hence Condition of maximum B.M to be minimum will occur when maximum Sagging B.M is equal to maximum hogging B.M and that the beam will be the most Economical Beam.

for most economical Beam.

Maximum Sagging B.M = Maximum Hogging B.M

$$\frac{w}{8}(l^2 - 4a^2) = \frac{wa^2}{2}$$

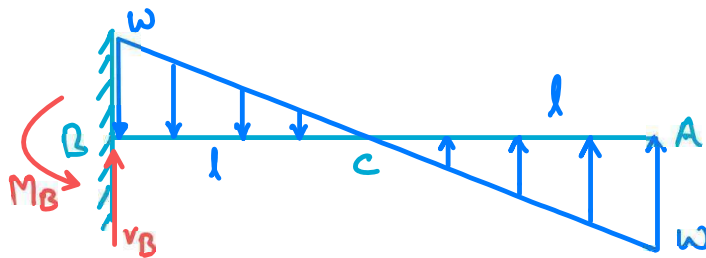
$$l^2 - 4a^2 = 4a^2$$

$$l^2 = 8a^2$$

$$l = 2\sqrt{2}a$$

• % overhang = $\frac{a}{l+2a} \times 100 = \frac{a}{2\sqrt{2}a+2a} \times 100 = \underline{20.7\%}$

■ Distance between supports = $100 - 20.7 \times 2 = \underline{58.6\%}$



Solution.

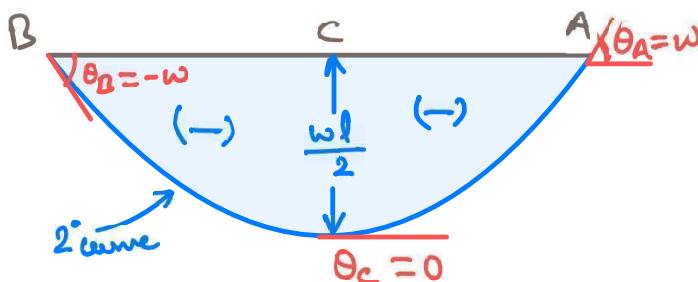
$$\sum f_y = 0 \Rightarrow V_B - \frac{1}{2}wl + \frac{1}{2}wl = 0$$

$$V_B = 0.$$

$$\sum M_B = 0 \Rightarrow -M_B + \frac{1}{2}wl \cdot \frac{l}{3} - \frac{1}{2}wl \left(\frac{2l}{3} + l \right) = 0$$

$$M_B = -\frac{2}{3}wl^2$$

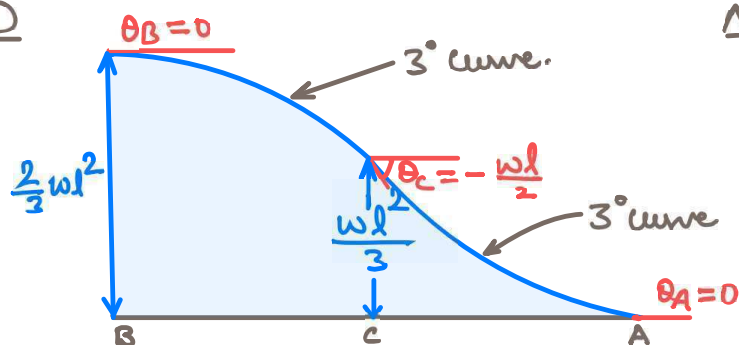
SFD.



$$\Delta V = V_C - V_B = \text{Area of load diagram}$$

$$V_C - 0 = -\frac{1}{2}l \cdot w$$

$$V_C = -\frac{wl}{2}$$

BMD

$$\Delta BM = BM_c - BM_B = \text{Area of SFD}$$

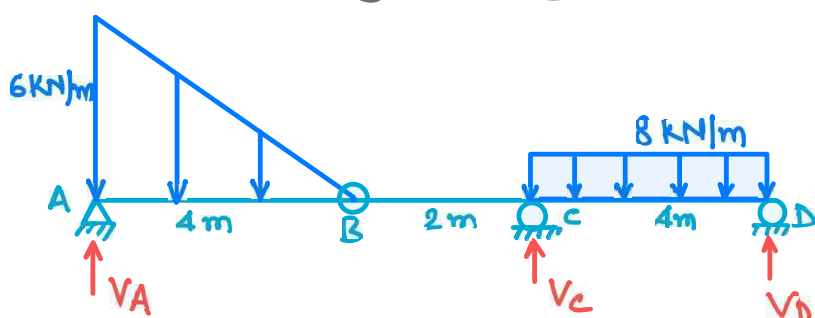
$$BM_c - \frac{2}{3}wl^2 = -\frac{2}{3}\frac{wl}{2} \cdot l$$

$$BM_c = \frac{wl^2}{3}$$

$$\Delta BM = BM_A - BM_c = \text{Area of SFD}$$

$$BM_A - \frac{wl^2}{3} = -\frac{2}{3}\frac{wl}{2} \cdot l \Rightarrow \underline{\underline{BM_A = 0}}$$

M >

Solution

$$\Sigma f_y = 0 \Rightarrow V_A + V_C + V_D - \frac{1}{2} \cdot 6 \cdot 4 - 8 \cdot 4 = 0$$

$$V_A + V_C + V_D = 44 \quad \dots (i)$$

$$M_B = 0 \Rightarrow V_A \times 4 - \frac{1}{2} \cdot 6 \cdot 4 \cdot \frac{2}{3} \cdot 4 = 0 \quad (\text{Left to right})$$

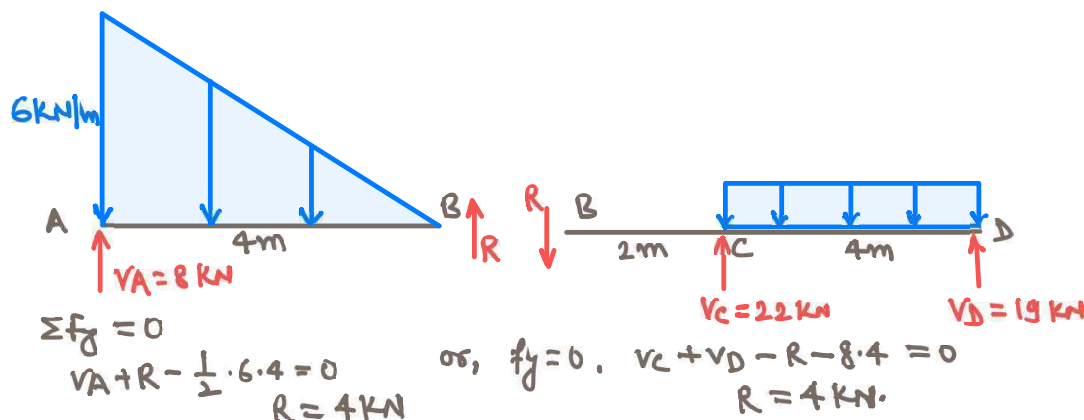
$$V_A = 8 \text{ kN}$$

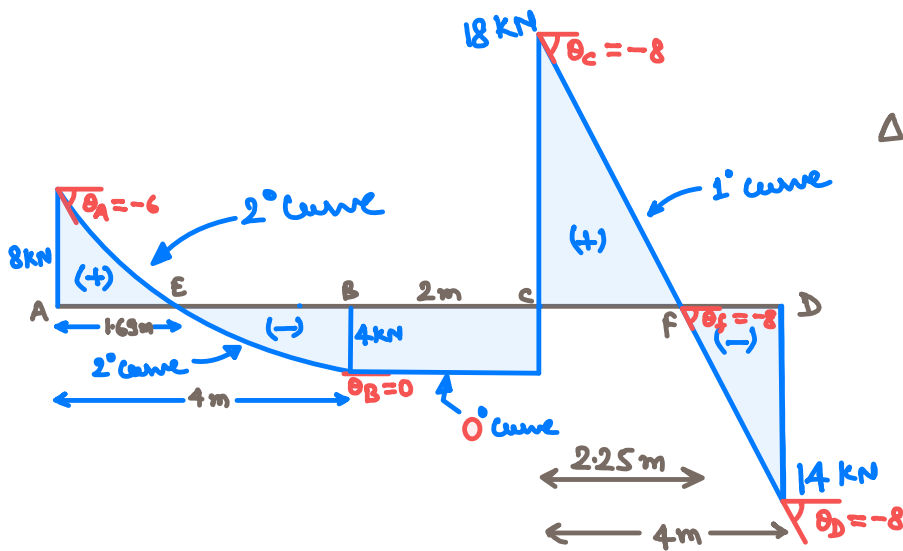
$$M_B = 0 \Rightarrow -V_D \times 6 - V_C \times 2 + 8 \times 4 \times \left(\frac{4}{2} + 2\right) = 0 \quad (\text{Right to left})$$

$$V_C + 3V_D = 64 \quad \dots (ii)$$

$$V_C = 22 \text{ kN}$$

$$V_D = 14 \text{ kN}$$

Note:-

SFD

$$\Delta V = V_B - V_A = \text{Area of loading dial}$$

$$= V_B - 8 = -\frac{1}{2} \cdot 6 \cdot 4$$

$$V_B = -4 \text{ kN}$$

$$\frac{6}{4} = \frac{w}{4-x}$$

$$w = \frac{6}{4} (4-x)$$

Let at distance x m from A, i.e. at point E, $SF = 0$

$$\Delta V = V_E - V_A = \text{Area of loading diagram.}$$

$$= 0 - 8 = -\frac{1}{2} \left(6 + \frac{6}{4} (4-x) \right) x$$

$$x = 1.69 \text{ m}$$

$$\Delta V = V_D - V_C = \text{Area of loading diagram.}$$

$$V_D - 18 = -8 \times 4$$

$$V_D = -14 \text{ kN}$$

Let at distance x m from C, i.e. at point F, $SF = 0$

$$\Delta V = V_F - V_C = \text{Area of loading diagram}$$

$$0 - 18 = -8 \times x$$

$$x = 2.25 \text{ m}$$

BMD

$$\Delta Bm = Bm_E - Bm_A = \text{Area of SFD}$$

$$Bm_E - 0 = \frac{1}{3} \times 8 \times 1.69 \Rightarrow Bm_E = 4.51 \text{ kNm}$$

$$\Delta Bm = Bm_B - Bm_E = \text{Area of SFD}$$

$$= Bm_B - 4.51 = -\frac{2}{3} \cdot 4 (4 - 1.69)$$

$$Bm_B = -1.65$$

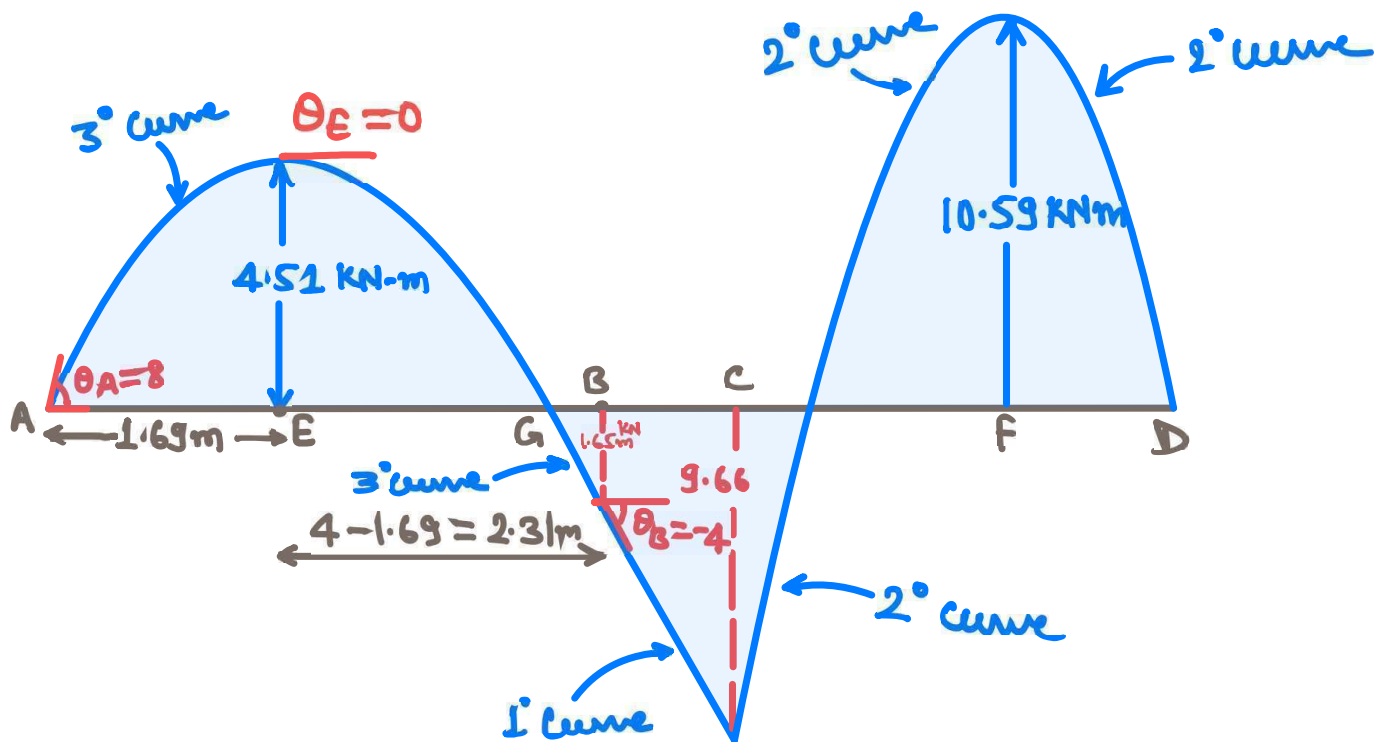
Wrong solution

Hinge without $\rightarrow$ B.m 0
Rotation

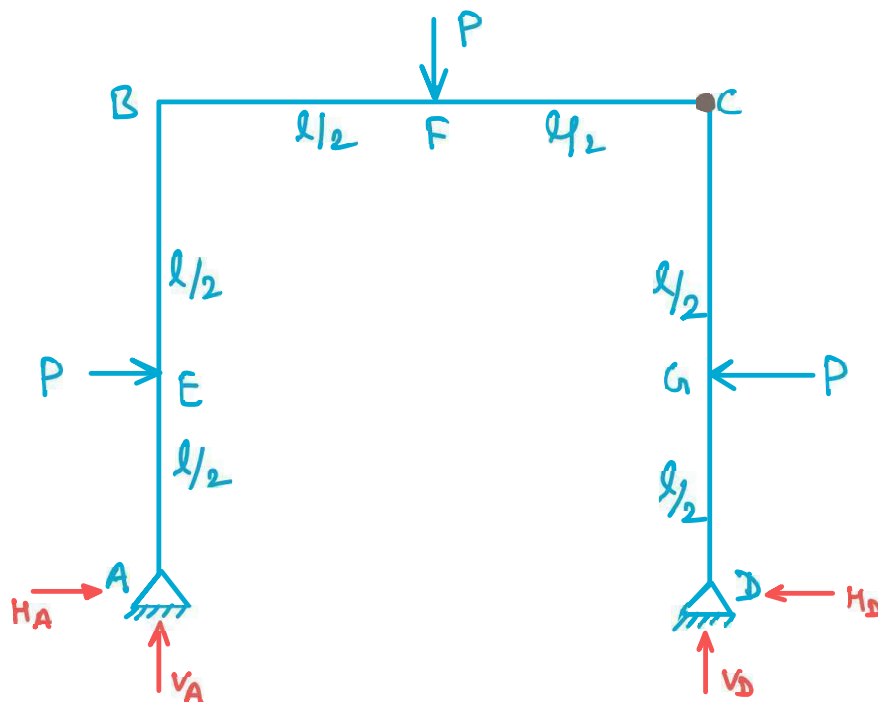
Let BM is at x m from point E equals to zero

$$\Delta BM = BM_G - BM_E = \text{Area of SFD}$$

$$= 0 - 4.51 = -\frac{2}{3} \cdot x \cdot y_G$$



N >



Solution.

$$\sum F_x = 0 \Rightarrow H_A - H_D + P - P = 0$$

$$H_A = H_D \text{ ---- (i)}$$

$$\sum F_y = 0 \Rightarrow V_A + V_D - P = 0$$

$$V_A + V_D = P \text{ ---- (ii)}$$

$$\sum M_A = 0 \Rightarrow P \times \frac{l}{2} + P \times \frac{l}{2} - P \times \frac{l}{2} - V_D \times l = 0$$

$$V_D = \frac{P}{2}$$

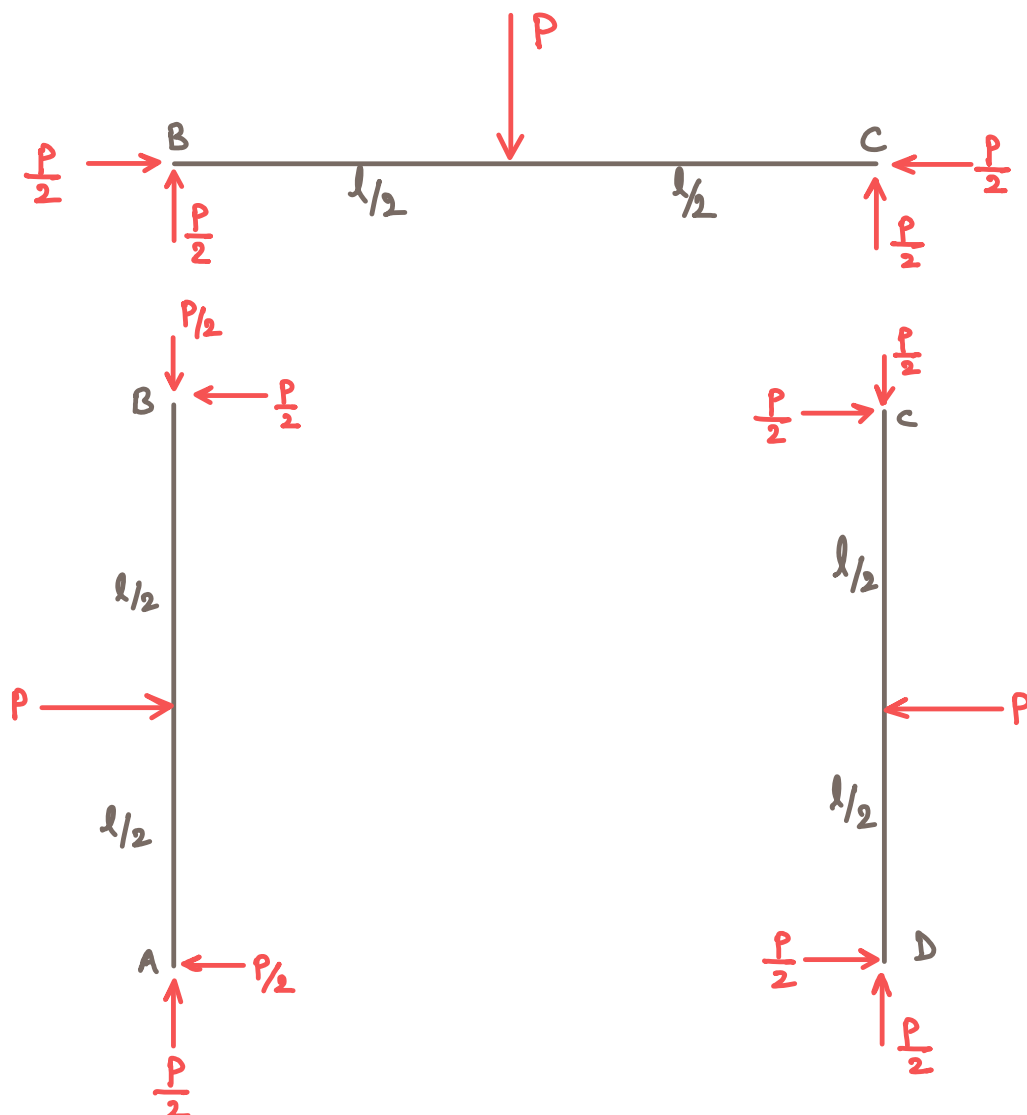
$$V_A = \frac{P}{2}$$

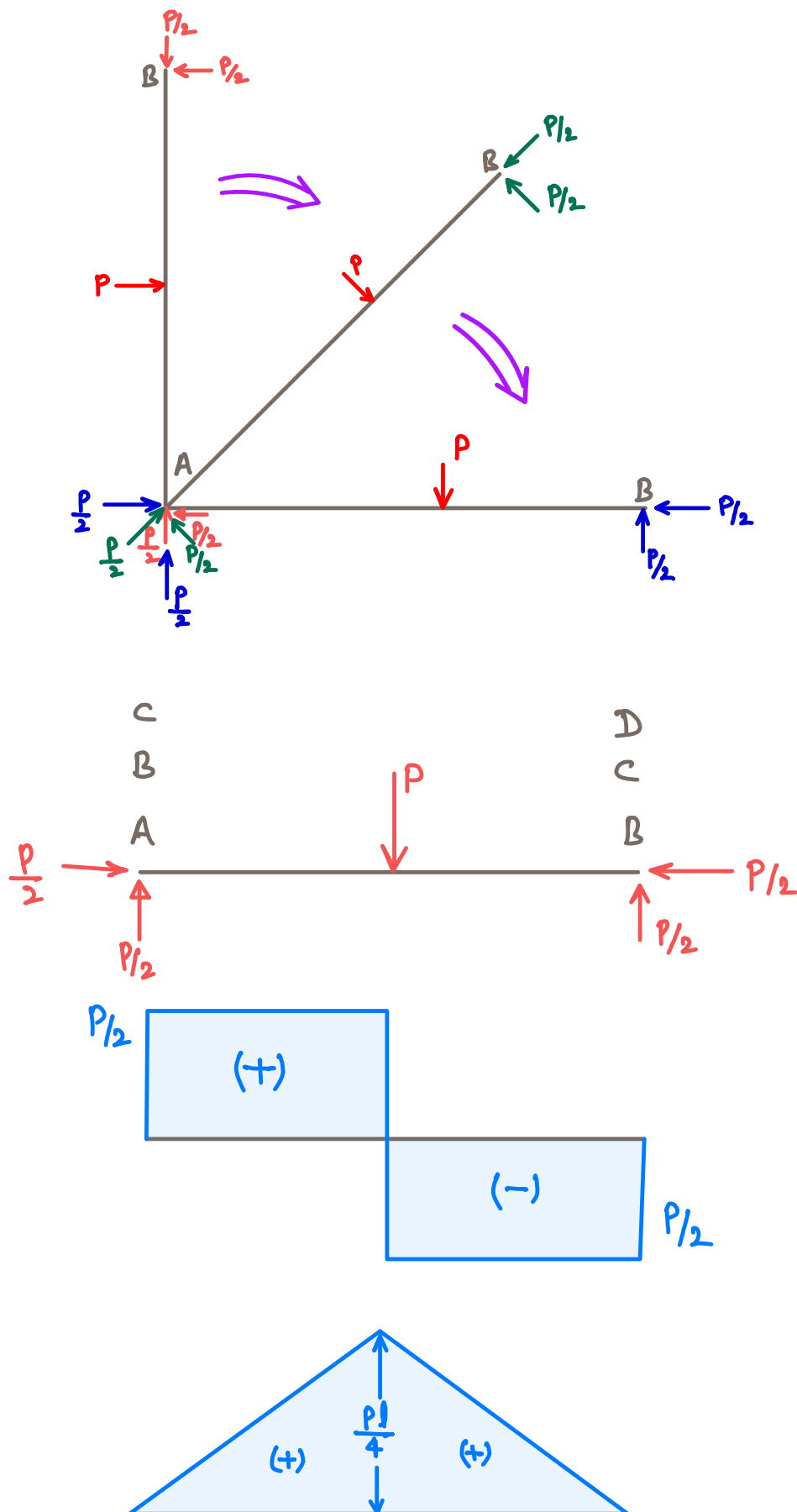
from L.H.S

$$\sum M_C = 0 \Rightarrow V_A \times l - H_A \times l - P \times \frac{l}{2} - P \times \frac{l}{2} = 0$$

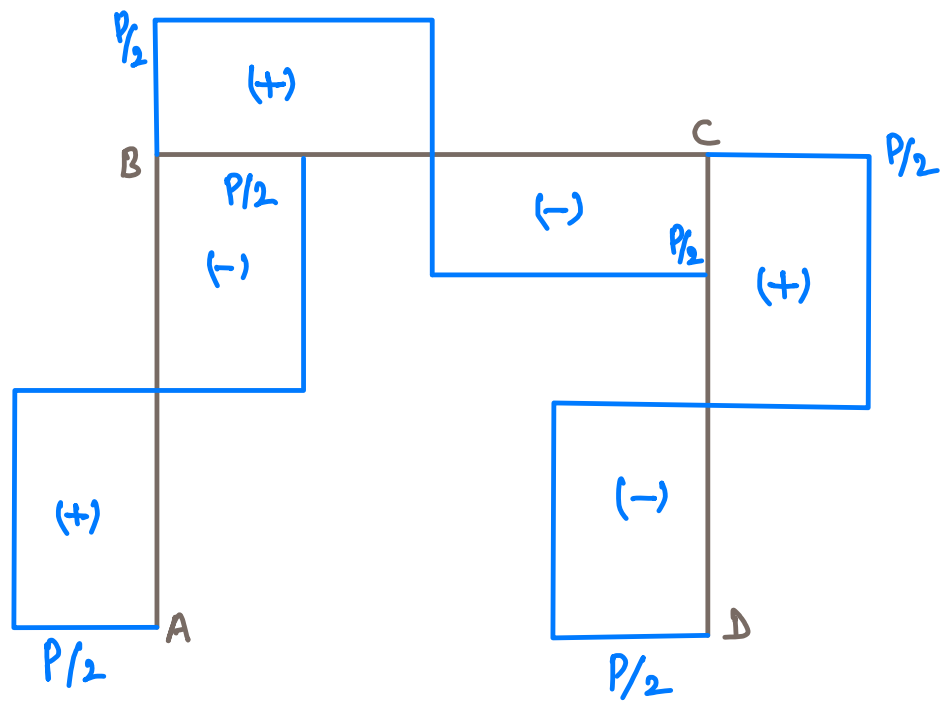
$$H_A = -\frac{P}{2} = H_D$$

Symmetry is
wrt Loading
~~Not geometry~~

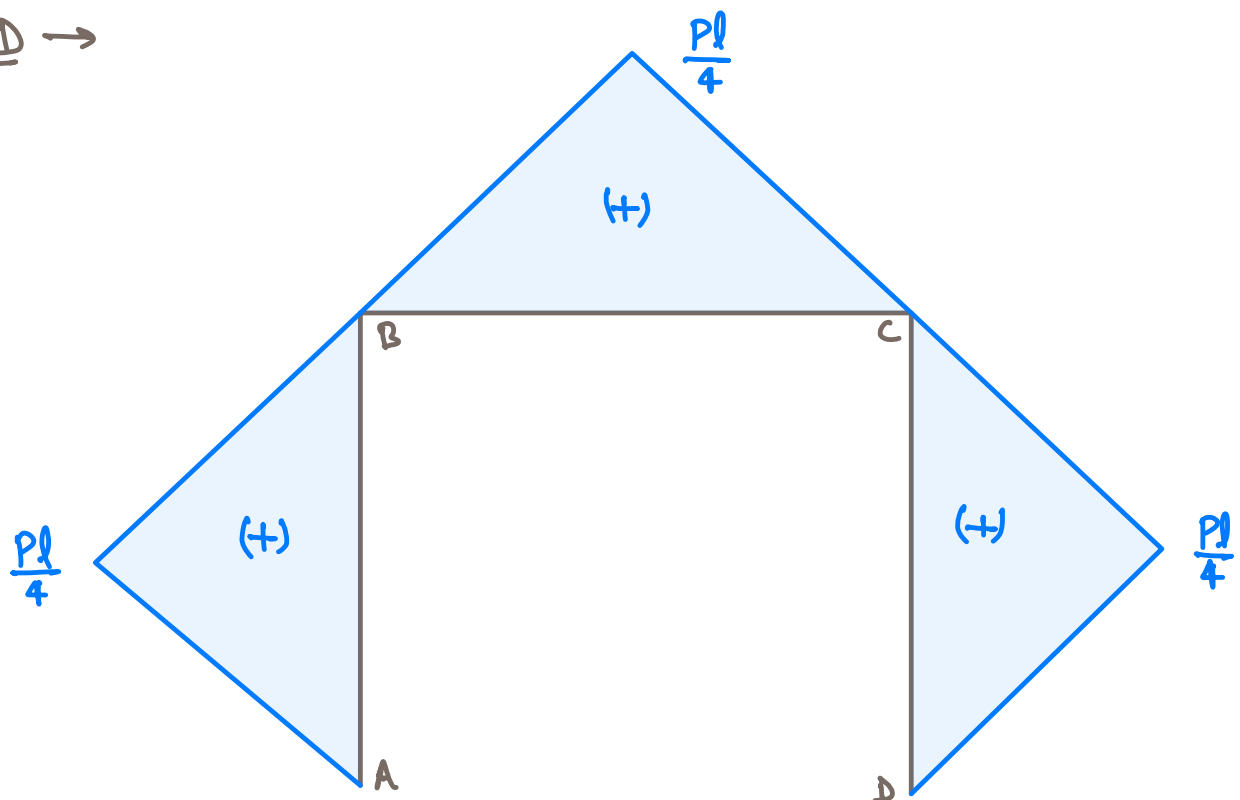




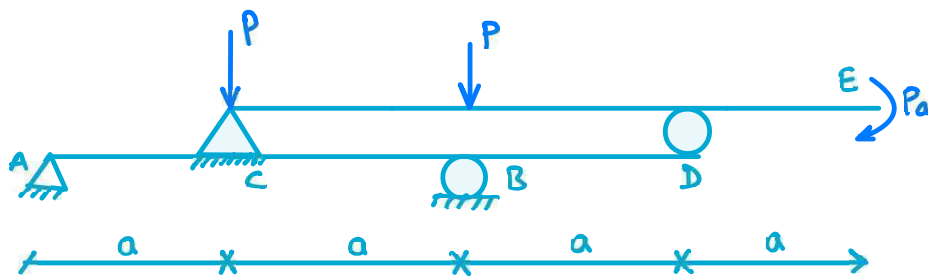
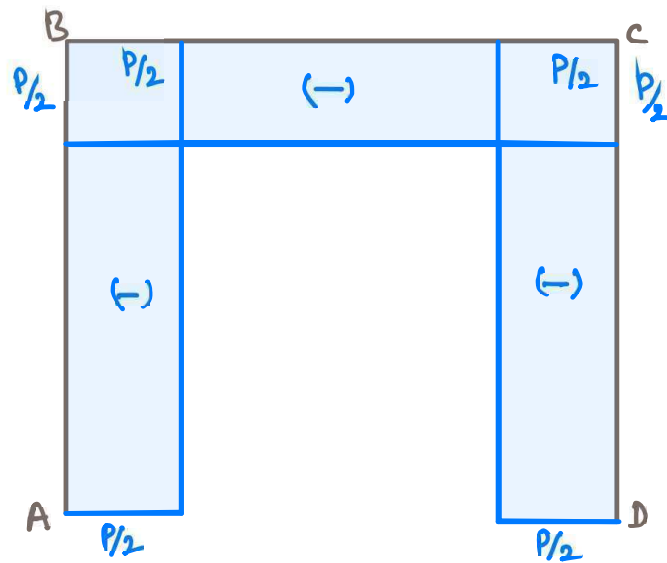
SFD:-



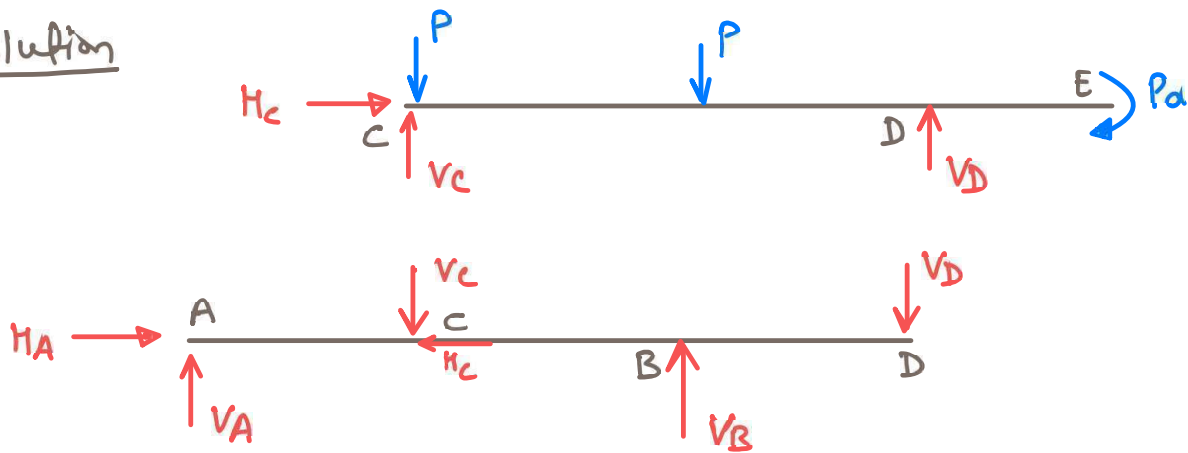
BMD →



Area load diagram.



Solution



$$\sum f_x = 0 \Rightarrow H_c = 0 \text{ — for beam CDE}$$

$$\Rightarrow H_A - H_c = 0$$

$$H_A = H_c = 0 \text{ — for beam ABD}$$

$$\sum f_y = 0 \Rightarrow V_C + V_D - P - P = 0$$

$$V_C + V_D = 2P \text{ — for beam CDE}$$

$$\Rightarrow V_A + V_B - V_C - V_D = 0 \text{ — for beam ABD}$$

$$V_A + V_B = V_C + V_D$$

$$V_A + V_B = 2P$$

$$\sum m_C = 0 \Rightarrow Pa - V_D \cdot 2a + Pa = 0$$

$$V_D = P, \quad V_C = P \text{ — for beam CDE}$$

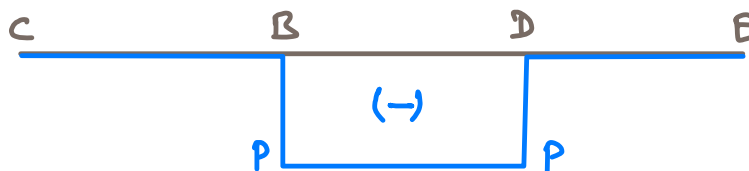
$$\sum m_A = 0 \Rightarrow V_C \cdot a - V_B \cdot 2a + V_D \cdot 3a = 0$$

$$V_B = 2P$$

$$V_A = 0$$



SFD:



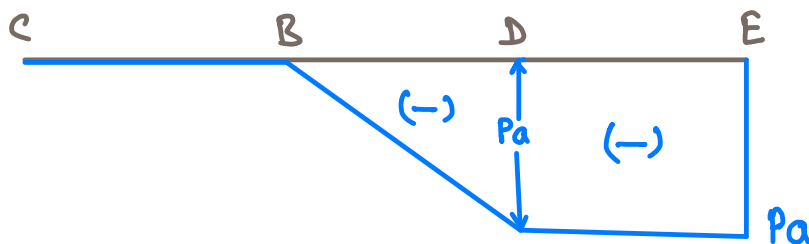
$$\Delta Bm = Bm_D - Bm_B = Pa$$

$$Bm_D = -Pa$$

$$Bm_E - Bm_D = 0$$

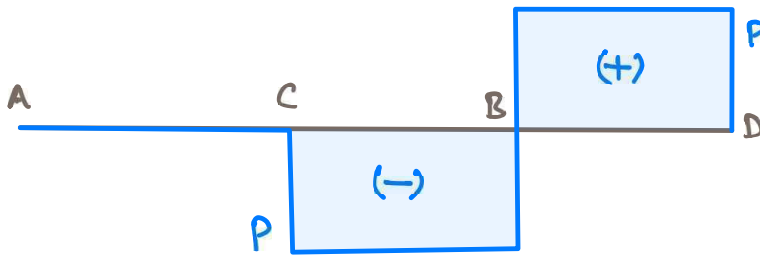
$$Bm_E = -Pa$$

BMD:-

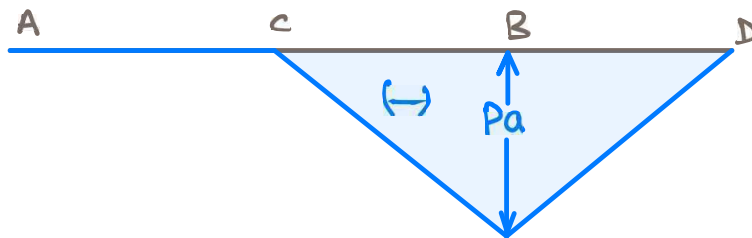


A C B D

SFD:-



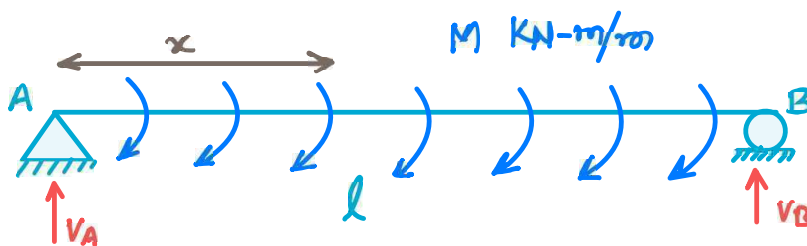
BMD:-



$$BM_D - (-Pa) = Pa$$

$$BM_D = 0$$

P. >



Solution:-

$$\sum f_y = 0 \Rightarrow V_A + V_B = 0$$

$$\sum m_A = 0 \Rightarrow -V_B \times l + Ml = 0$$

$$V_B = M$$

$$V_A = -M$$

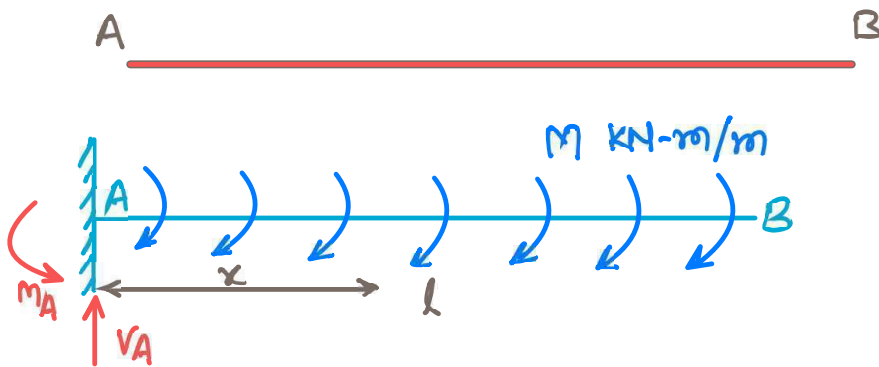
SFD:-



BMD

$$Bm_x = V_A x + Mx =$$

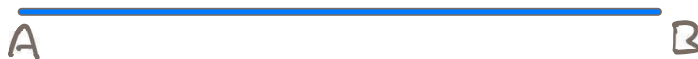
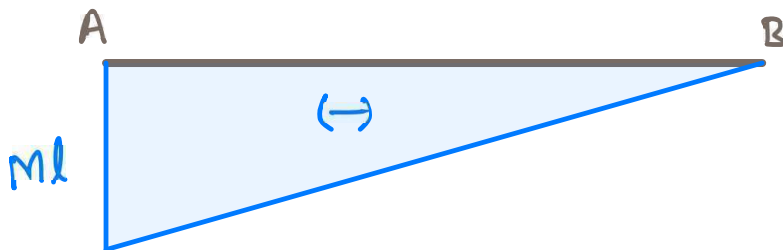
$$Bm_x = -m_x + m_x = 0$$

Q. >Solution:-

$$\sum f_y = 0 \Rightarrow V_A = 0$$

$$\sum M_A = 0 \Rightarrow -M_A + Ml = 0$$

$$M_A = ml$$

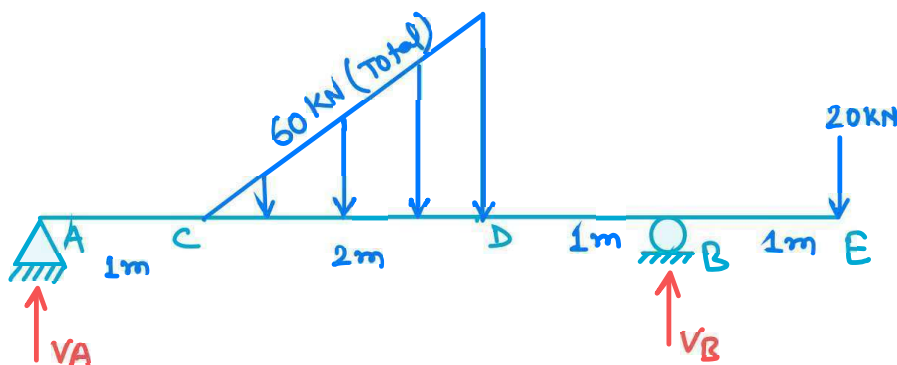
SFDBMD

$$Bm_x = -M_A + mx$$

$$= mx - ml$$

At point B, $x = l$

$$Bm_B = 0$$

R. >Solution:-

$$\sum f_y = 0 \Rightarrow V_A + V_B - 60 - 20 = 0$$

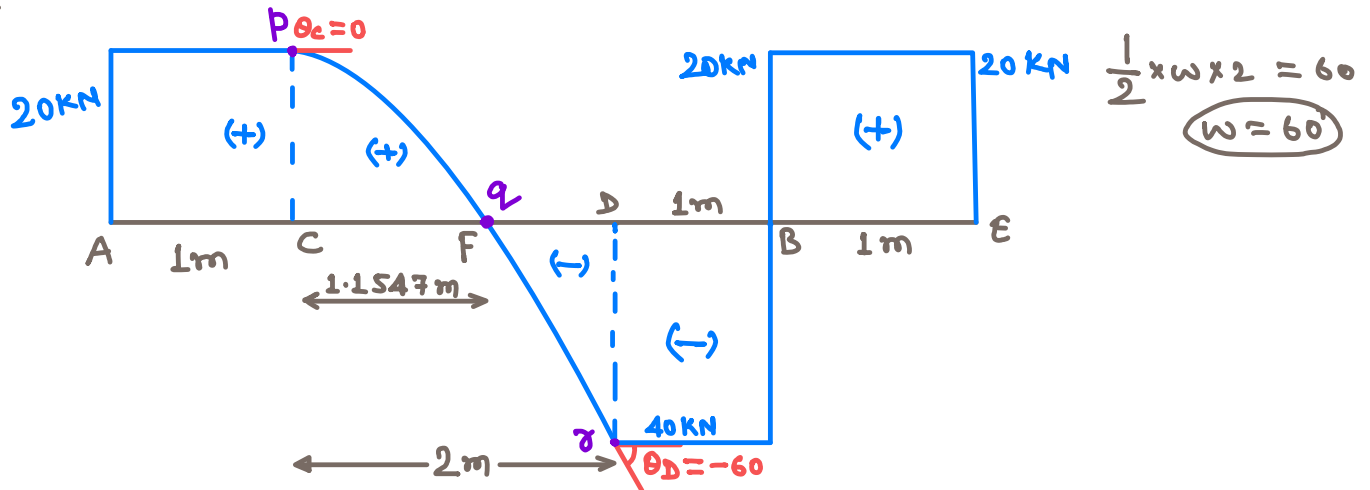
$$V_A + V_B = 80 \text{ kN} \dots (i)$$

$$\Sigma m_A = 0 \Rightarrow 60 \left(1 + \frac{2}{3} \times 2\right) - V_B \cdot 4 + 20 \times 5 = 0$$

$$V_B = 60 \text{ kN}$$

$$V_A = 20 \text{ kN.}$$

SFD



$$\Delta V = V_D - V_C = \text{Area of loading diag}$$

$$= V_D - 20 = \frac{1}{2} \times 2 \times (-60)$$

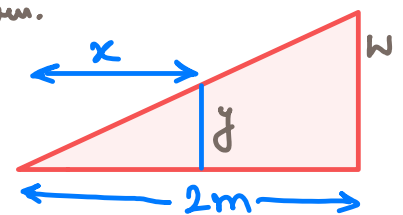
$$V_D = -40 \text{ kN}$$

Let at distance x from C, i.e. at point F, S.F = 0

$$\Delta V = V_F - V_C = \text{Area of loading diagram.}$$

$$0 - 20 = -\frac{1}{2} x \cdot \frac{60}{2} x$$

$$x = 1.1547 \text{ m}$$



$$\frac{w}{2} = \frac{y}{x} \Rightarrow y = \frac{wx}{2}$$

$$\text{Area} = \frac{1}{2} y x$$

BMD:-

$$\Delta Bm = Bm_C - Bm = \text{Area of SFD}$$

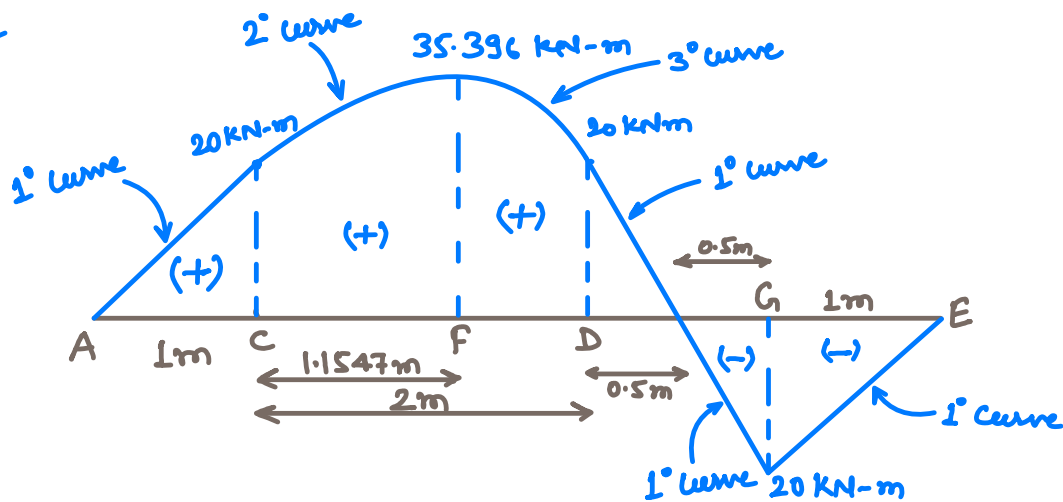
$$Bm_C - 0 = 20 \times 1$$

$$Bm_C = 20 \text{ kN-m}$$

$$\Delta Bm = Bm_F - Bm_C = \text{Area of SFD}$$

$$= Bm_F - 20 = \frac{2}{3} \times 20 \times 1.147$$

$$Bm_F = 35.396 \text{ kN-m.}$$

BMD

$$\Delta BM = BM_D - BM_E = \text{Area of SFD}$$

$$BM_D - 20 = \left(\frac{2}{3} \times 2 \times 60\right) - 2 \times 40$$

$$BM_D = 20 \text{ kN-m}$$

or

Equation of parabola pqr,

$$y = ax^2 + bx + c$$

$$p(0, 20), q(1.1547, 0), r(2, -40)$$

$$y = -15x^2 + 20$$

$$\begin{aligned} \text{Area} &= \int y \cdot dx = \int_0^2 (-15x^2 + 20) dx \\ &= \left[-\frac{15x^3}{3} + 20x \right]_0^2 \\ &= -5(8-0) + 20(2) = 0 \end{aligned}$$

$$\text{Hence, } BM_D - BM_E = 0$$

$$BM_D = BM_E = 20 \text{ kN-m}$$

$$\Delta Bm = Bm_B - Bm_D = \text{Area of SFD}$$

$$Bm_B - 20 = -40 \times 1$$

$$Bm_B = -20 \text{ kN-m.}$$

Let at distance x from D, at point G, $Bm = 0$

$$\Delta Bm = Bm_G - Bm_D = \text{Area of SFD}$$

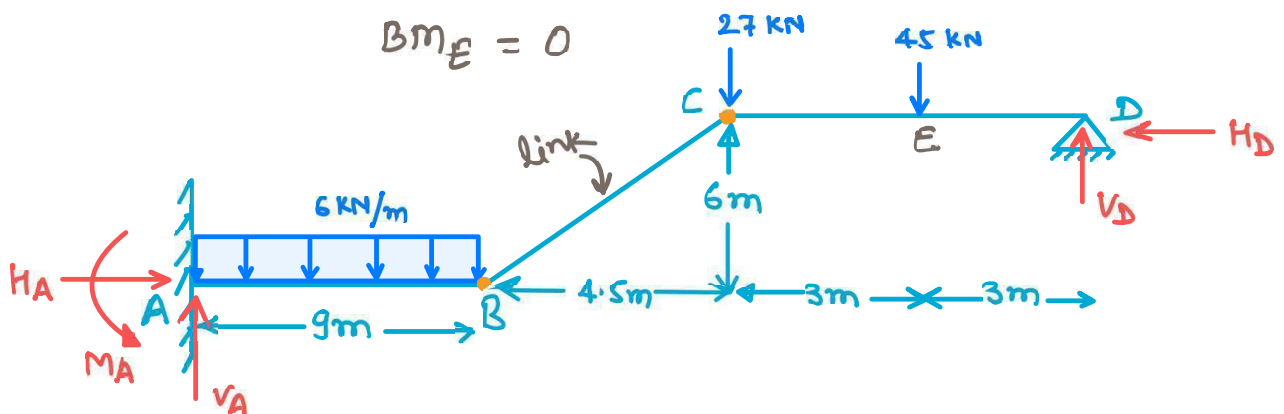
$$0 - 20 = -40x$$

$$x = 0.5 \text{ m}$$

$$\Delta Bm = Bm_E - Bm_B = \text{Area SFD}$$

$$Bm_E - (-20) = 20 \times 1$$

$$Bm_E = 0$$



Soln

$$\sum F_x = 0 \Rightarrow H_A - H_D = 0 \Rightarrow H_A = H_D \quad \dots (i)$$

$$\sum F_y = 0 \Rightarrow V_A + V_D - 6 \times 9 - 27 - 45 = 0$$

$$V_A + V_D = 126 \text{ kN} \quad \dots (ii)$$

$$\sum M_A = 0 \Rightarrow -M_A + 6 \times 9 \times 4.5 + 27 \times (9 + 4.5) + 45 \times (16.5) - V_D \times 19.5 - H_D \times 6 = 0$$

$$M_A + 6H_D + 19.5V_D = 1350 \quad \dots (iii)$$

$$\sum M_C = 0 \Rightarrow -V_D \times 6 + 45 \times 3 = 0$$

(R.H.S)

$$V_D = 22.5 \text{ kN} \quad \dots (iv)$$

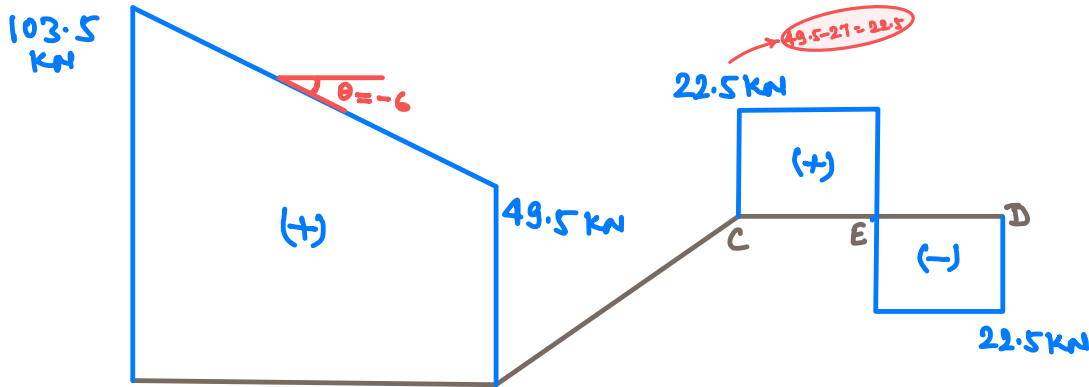
$$V_A = 103.5 \text{ kN} \quad \dots (v)$$

$$\sum M_B = 0 \Rightarrow 27 \times 4.5 + 45 \times 7.5 - V_D \times 10.5 - H_D \times 6 = 0$$

(R.H.S)

$$H_D = H_A = 37.125 \text{ kN} \quad \text{--- (vi)}$$

$$M_A = 688.5 \text{ kNm}$$



$$\Delta V = V_B - V_A = \text{Area of loading diagram}$$

$$V_B - 103.5 = -6 \times 9$$

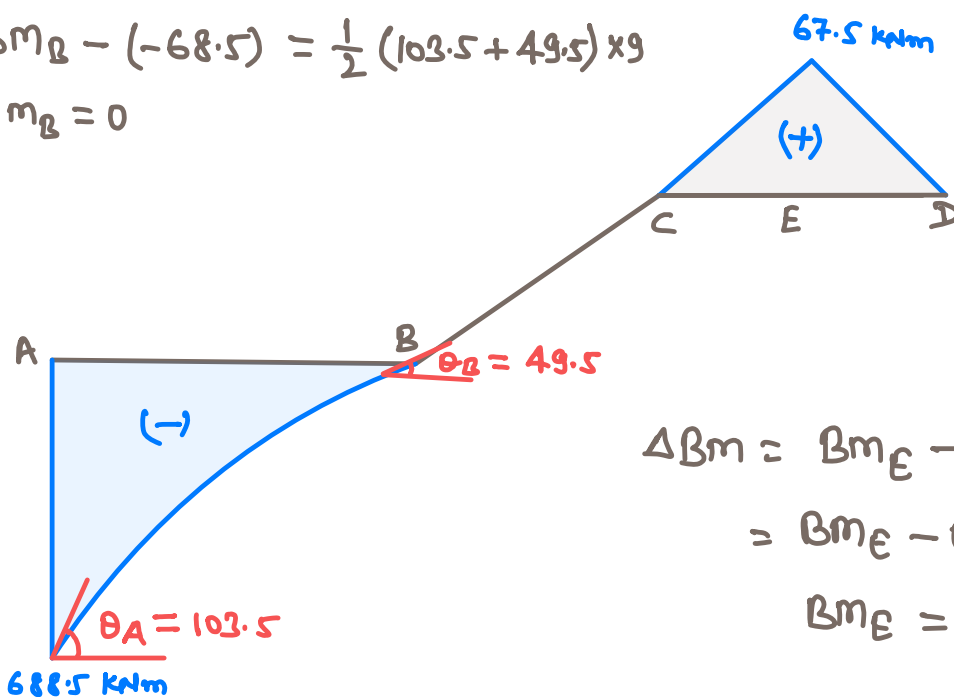
$$V_B = 49.5 \text{ kN.}$$

BMD:-

$$\Delta Bm = Bm_B - Bm_A = \text{Area of SFD}$$

$$= Bm_B - (-68.5) = \frac{1}{2} (103.5 + 49.5) \times 9$$

$$= Bm_B = 0$$



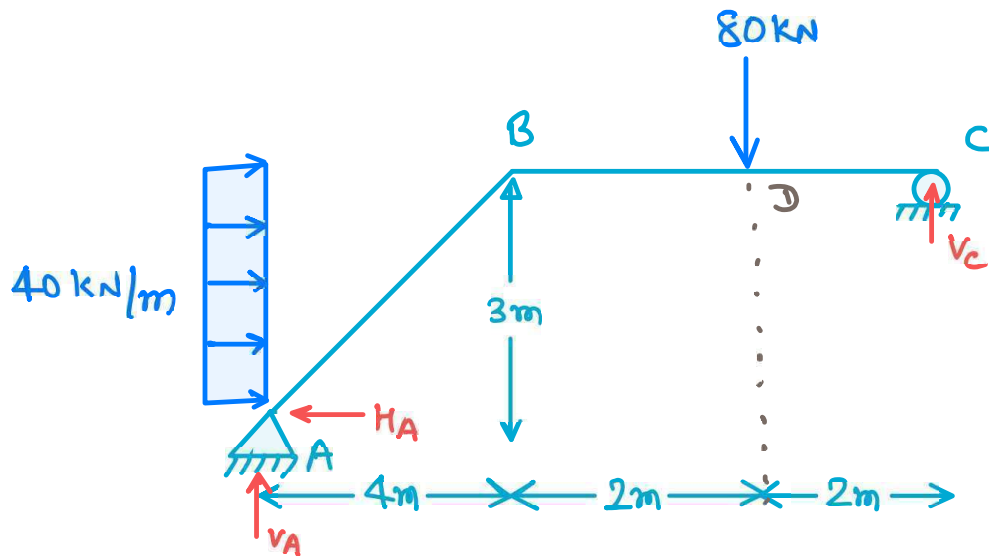
$$\Delta Bm = Bm_E - Bm_C = \text{Area SFD}$$

$$= Bm_E - 0 = 22.5 \times 2.$$

$$Bm_E = 66.5 \text{ kNm}$$

Slope
+ve = dec.

T. >

Solution

$$\sum f_x = 0 \Rightarrow 40 \times 3 - H_A = 0$$

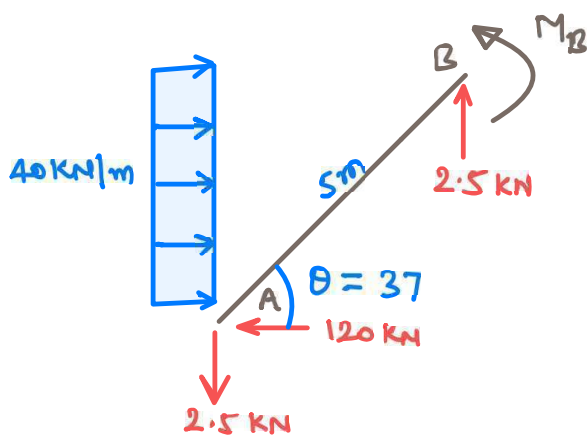
$$H_A = 120 \text{ kN} \quad \dots \quad (i)$$

$$\sum f_y = 0 \Rightarrow V_A + V_C = 80 \text{ kN} \quad \dots \quad (ii)$$

$$\sum M_A = 0 \Rightarrow 40 \times 3 \times 1.5 + 80 \times (4+2) - V_C (4+2+6) = 0$$

$$V_C = 82.5 \text{ kN}$$

$$V_A = -2.5 \text{ kN}$$

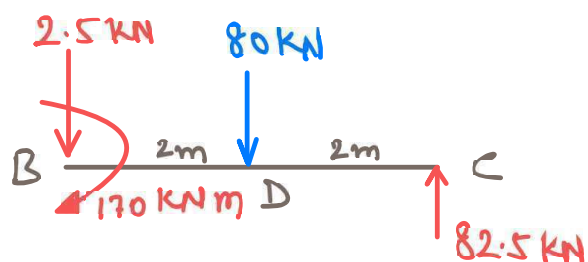


$$\theta = \sin^{-1}\left(\frac{3}{5}\right) = 37$$

$$\sum M_B = 0$$

$$-2.5 \times 4 + 120 \times 3 - 40 \times 3 \times 1.5 - M_B = 0$$

$$M_B = 170 \text{ kN-m}$$

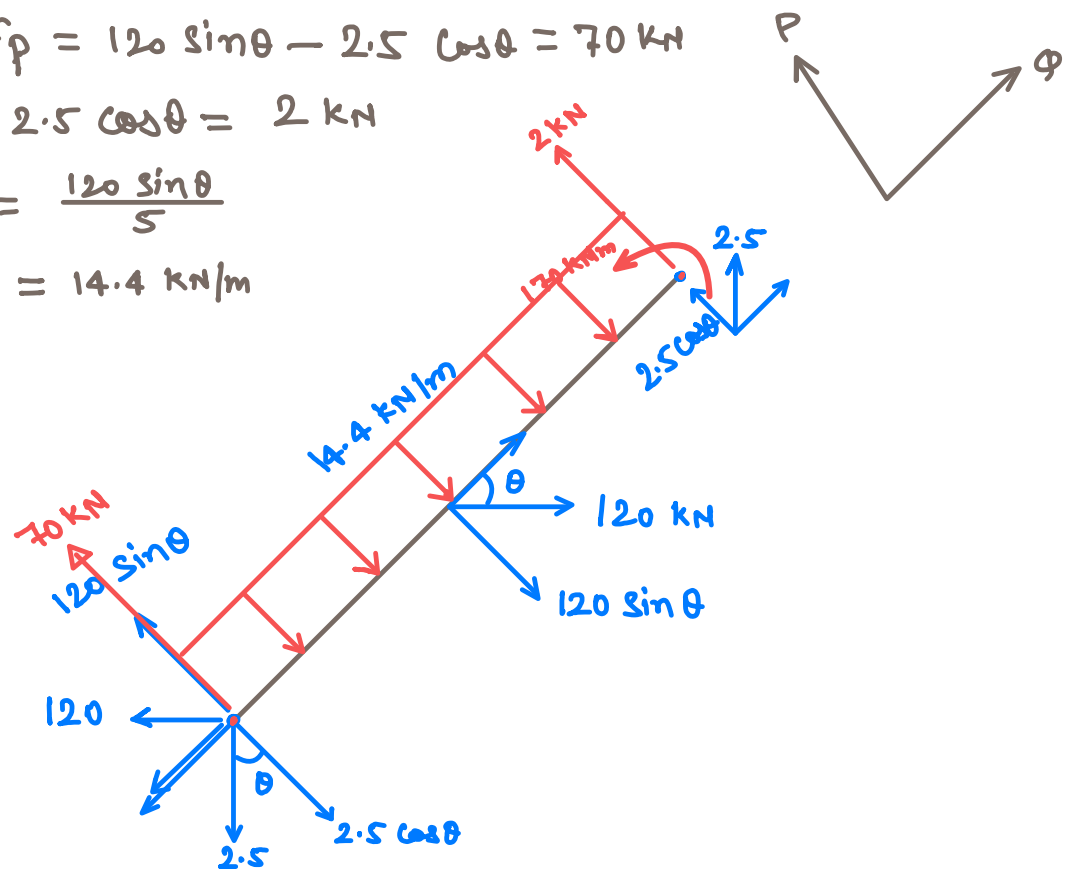


At point A, $\sum f_p = 120 \sin \theta - 2.5 \cos \theta = 70 \text{ kN}$

At point B, $\sum p = 2.5 \cos \theta = 2 \text{ kN}$

overlength AB, $w = \frac{120 \sin \theta}{5}$

$$w = 14.4 \text{ kN/m}$$



SFD

$\Delta V = V_B - V_A = \text{Area of loading diag.}$

$$V_B - 70 = -14.4 \times 5$$

$$V_B = -2 \text{ kN}$$

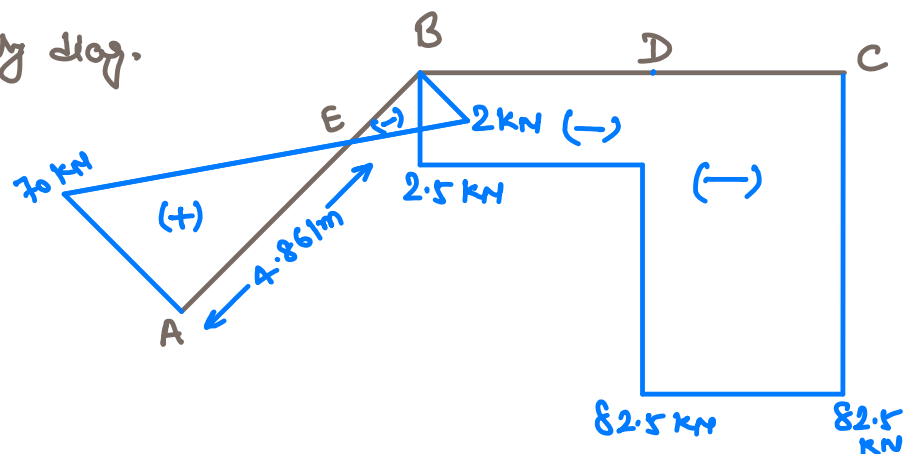
Let at distance x from A,

at point E, $SF = 0$

$\Delta V = V_E - V_A = \text{Area of loading diag.}$

$$0 - 70 = -14.4 \times x$$

$$x = 4.861 \text{ m.}$$



BMD

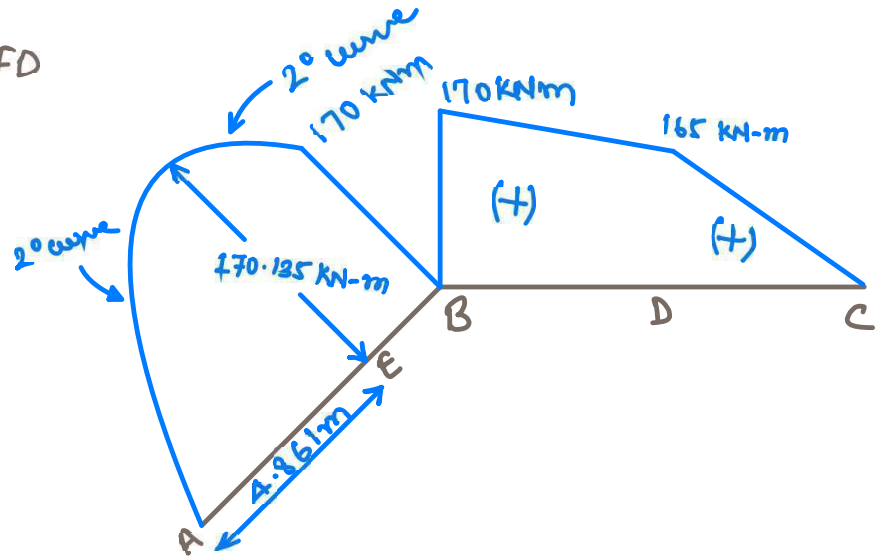
$$\Delta Bm = Bm_E - Bm_A = \text{Area of SFD}$$

$$= Bm_E - 0 = \frac{1}{2} \times 70 \times 4.861$$

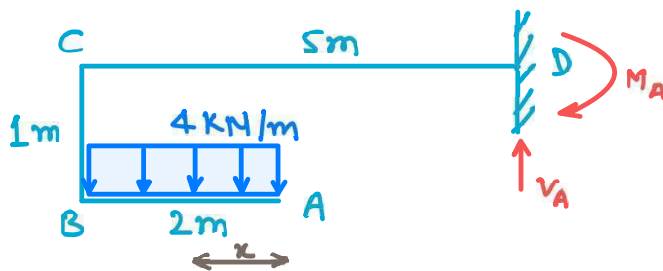
$$Bm_E = 170.135 \text{ KN-m}$$

$$Bm_D - 170 = -2.5 \times 2$$

$$Bm_D = 165 \text{ KN-m}$$



u>

Solution

$$\sum f_y = 0, \Rightarrow V_D - 4 \cdot 2 = 0 \Rightarrow V_D = 8 \text{ kN}$$

$$\sum M_D = 0 \Rightarrow 4 \cdot 2 \cdot \left(5 - \frac{2}{2}\right) + M_D$$

$$M_D = 32 \text{ kNm}$$

SFD:-

for member AB,

$$V_A = 0$$

$$\Delta V = V_B - V_A = \text{Area of loading diag.}$$

$$V_B - 0 = 4 \times 2 \Rightarrow V_B = +8 \text{ kN}$$

for member BC,

$$V_x = 0$$

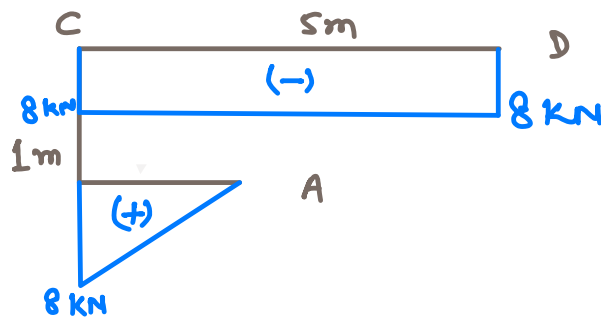
for member CD,

$$V_D = 8 \text{ kN}$$

$$\Delta V = V_C - V_D = \text{Area of loading diag.}$$

$$= V_C - (-8) = 0$$

$$= V_C = -8 \text{ kN.}$$



BMD

For member AB,

$$BM_A = 0$$

$$\Delta BM = BM_B - BM_A = \text{Area of SFD}$$

$$= BM_B - 0 = \frac{1}{2} \times 2 \times 8$$

$$BM_B = 8 \text{ kN-m.}$$

for member BC.

$$BM_B = 8 \text{ kN-m}$$

$$\Delta BM = BM_C - BM_B = \text{Area of SFD}$$

$$BM_C - 8 = 0 \Rightarrow BM_C = 8 \text{ kNm}$$

for member CD,

$$BM_C = 8 \text{ kN-m}$$

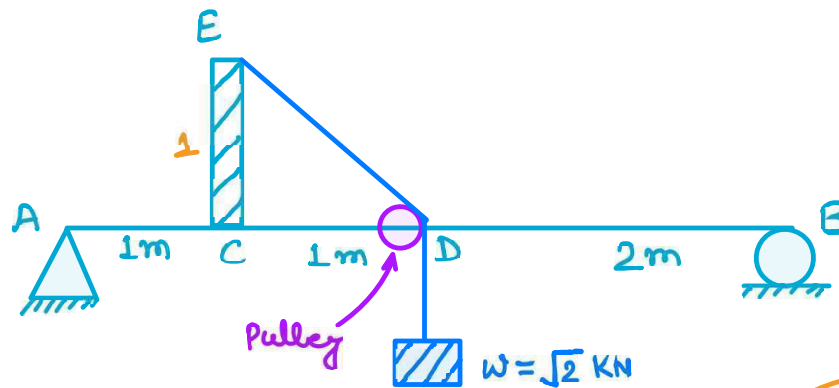
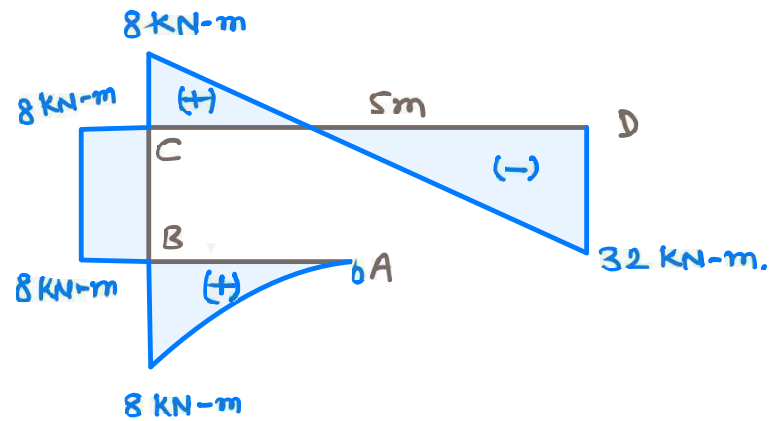
$$BM_D = 32 \text{ kN-m.}$$

let at a distance x from point C
i.e. point E, $BM = 0$

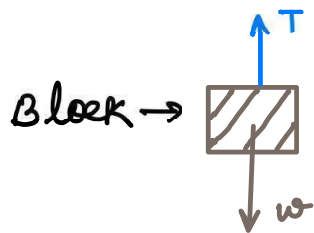
$$BM_E - BM_C = \text{Area of SFD}$$

$$0 - 8 = -8 \times x$$

$$x = 1\text{m}$$



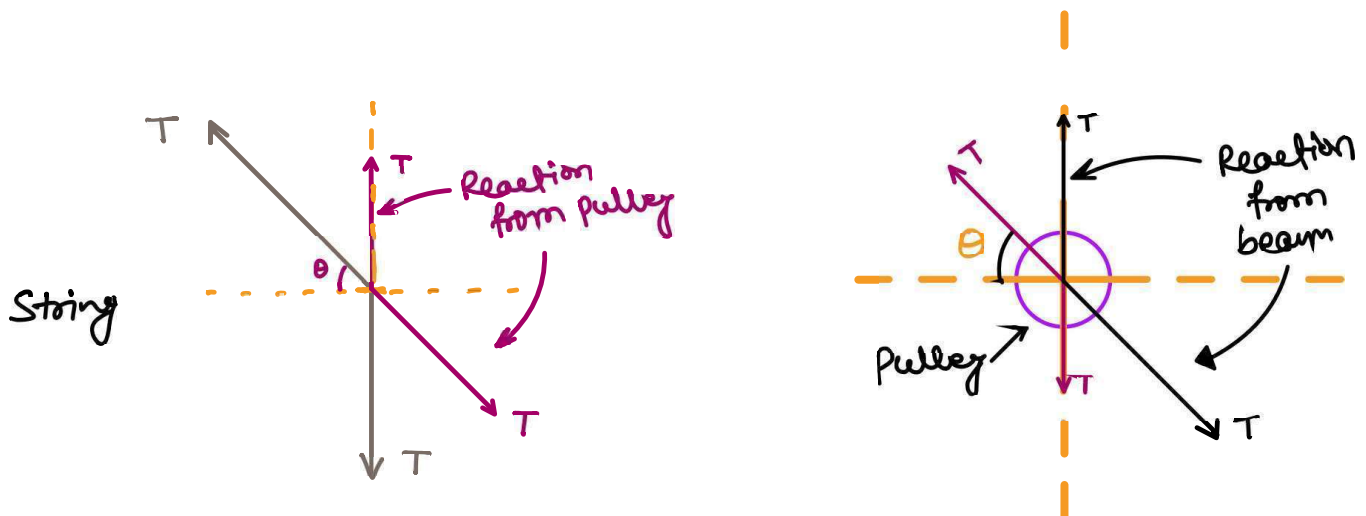
Solution.

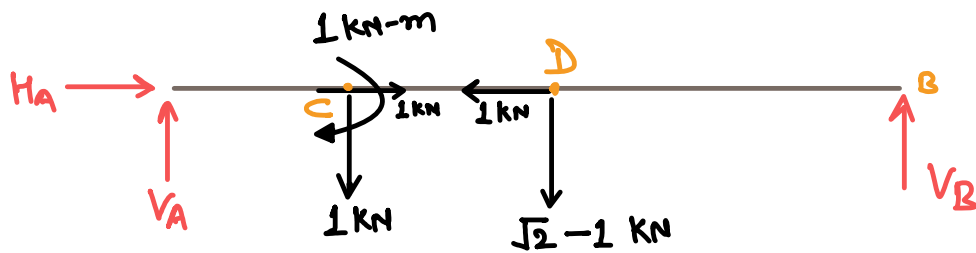
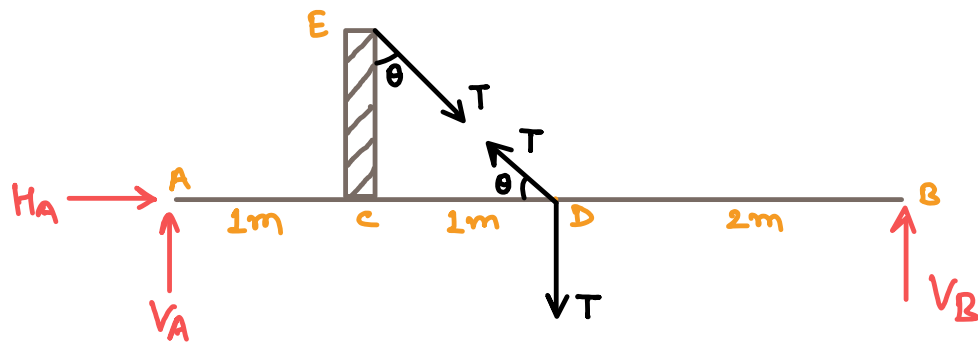


$$\sum f_y = 0, \quad T = W$$

$$\tan \theta = \frac{1}{1} = 1$$

$$\theta = 45^\circ$$





$$\begin{aligned} T &= \frac{T}{\sqrt{2}} \\ \sqrt{2} &= \frac{\sqrt{2}}{\sqrt{2}} \\ \sqrt{2} - 1 \end{aligned}$$

$$\sum f_x = 0 \Rightarrow H_A + 1 - 1 = 0, \quad H_A = 0 \quad \text{--- (i)}$$

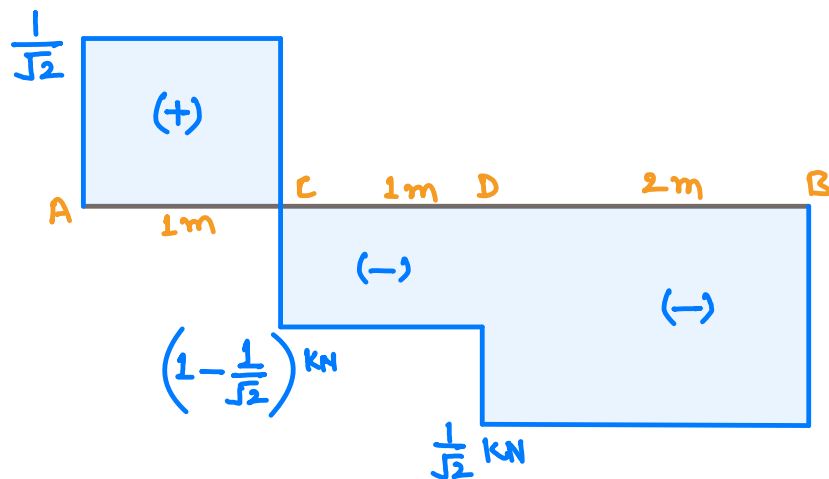
$$\sum f_y = 0 \Rightarrow V_A + V_B - 1 - (\sqrt{2} - 1) = 0$$

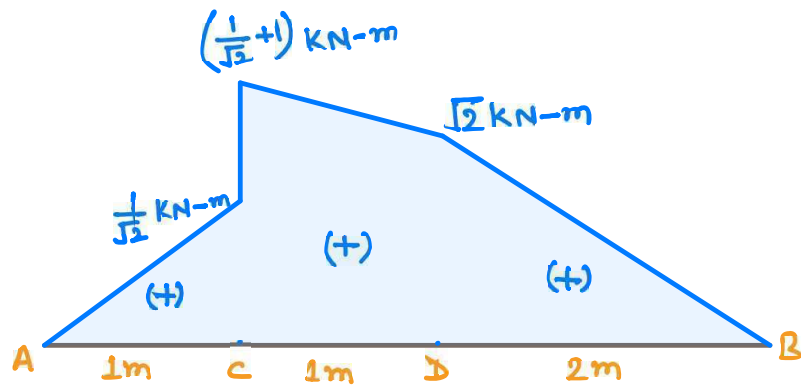
$$V_A + V_B = \sqrt{2} \text{ kN} \quad \text{--- (ii)}$$

$$\sum M_A = 0 \Rightarrow 1 \cdot 1 + 1 + (\sqrt{2} - 1) \cdot 1 - V_B (1 + 1 + 2) = 0$$

$$\left. \begin{aligned} V_B &= \frac{1}{\sqrt{2}} \\ V_A &= \frac{1}{\sqrt{2}} \end{aligned} \right\} \text{ kN.}$$

SFD



BMD

$$BM_A = 0$$

$$\Delta BM = BM_C - BM_A = \text{Area SFD}$$

$$BM_C - 0 = \frac{1}{\sqrt{2}} \cdot 1$$

$$BM_C = \frac{1}{\sqrt{2}}$$

$$\Delta BM = BM_D - BM_C = \text{area of SFD}$$

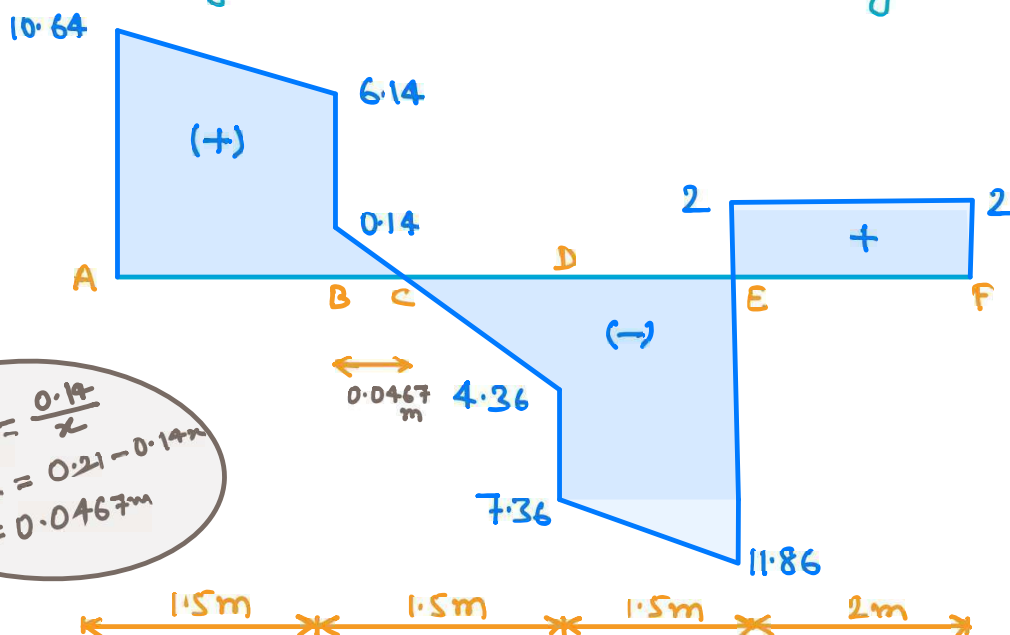
$$BM_D - \left(\frac{1}{\sqrt{2}} + 1\right) = -\left(1 - \frac{1}{\sqrt{2}}\right) \cdot 1$$

$$BM_D = \sqrt{2}$$

$$\Delta BM = BM_B - BM_D = \text{Area of SFD}$$

$$BM_B - \sqrt{2} = -\frac{1}{\sqrt{2}} \times 2 \Rightarrow BM_B = 0$$

Q. > from the given S.F.D draw the loading & B.m. diagram

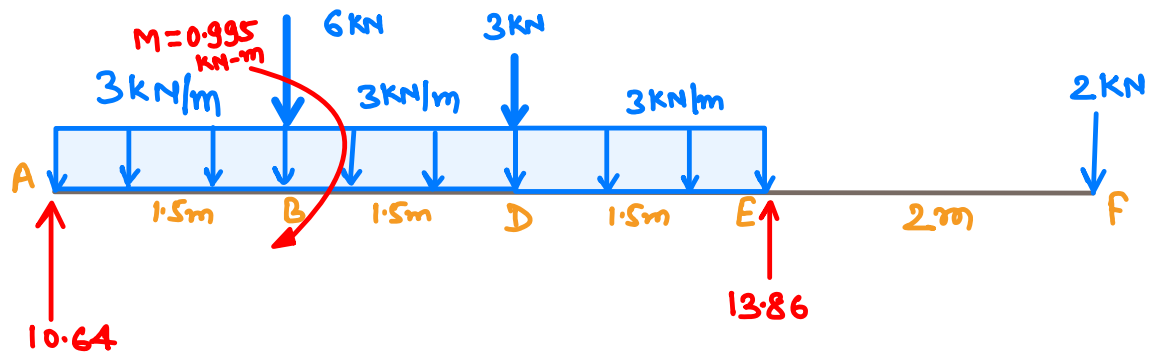


$$\frac{4.36}{1.5 - x} = \frac{0.14}{x}$$

$$4.36x = 0.21 - 0.14x$$

$$x = 0.0467 \text{ m}$$

Loading
Diagram



- At point A, there is an upward load of 10.64 kN
from A to B, $\frac{dV}{dx} = \frac{6.14 - 10.64}{1.5} = -3 \text{ kN/m} = w$
- At point B, there is a load of $0.14 - 6.14 = -6 \text{ kN} (\downarrow)$
from B to D, $\frac{dV}{dx} = \frac{-4.36 - 0.14}{1.5} = -3 \text{ kN/m} = w$
- At point D, there is a load of $-7.36 - (-4.36) = -3 \text{ kN} (\downarrow)$
from D to E, $\frac{dV}{dx} = \frac{-11.86 - (-7.36)}{1.5} = -3 \text{ kN/m}$
- At point E, There is a load of $2 - (11.86) = 13.86 (\uparrow)$
from E to F, $\frac{dV}{dx} = 0 = w$, there is no loading.
- At point F, there is a load of $0 - 2 = -2 \text{ kN} (\downarrow)$

Note:-

- Loading diagram obtained from SFD, will satisfy force equilibrium condition but might not satisfy moment equilibrium conditions.

$$M_A = 6 \times 1.5 + 3 \times 3 + 2 \times 6.5 + 3 \times 4.5 \times \left(\frac{4.5}{2}\right) - 13.8 \times 4.5 = -0.995 \neq 0$$

- Hence for moment equilibrium, there must be clockwise moment at point A which may exist at any point on beam

$$\Delta Bm = Bm_B - Bm_A = \text{area of SFD}$$

$$Bm_B - 0 = \frac{1}{2} (10.64 + 6.14) 1.5$$

$$Bm_B = 12.585 \text{ KN}\cdot\text{m}$$

$$12.585 + 0.995 = 13.58 \text{ KN}\cdot\text{m} \quad \text{moment at B}$$

$$\Delta Bm = Bm_C - Bm_B = \text{Area of SFD}$$

$$Bm_C - 13.58 = \frac{1}{2} (0.14 + 0) 0.0467$$

$$Bm_C = 13.583 \text{ KN}\cdot\text{m}.$$

$$\Delta Bm = Bm_D - Bm_C = \text{Area of SFD}$$

$$Bm_D - 13.583 = -\frac{1}{2} \times 4.36 \times (1.5 - 0.0467)$$

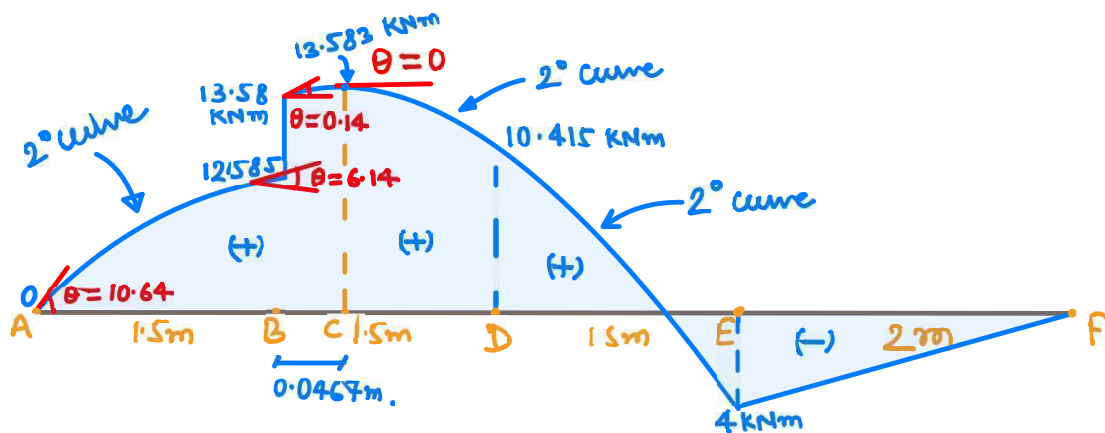
$$Bm_D = 10.415 \text{ KN}\cdot\text{m}.$$

$$\Delta Bm = Bm_E - Bm_D = \text{area of SFD}$$

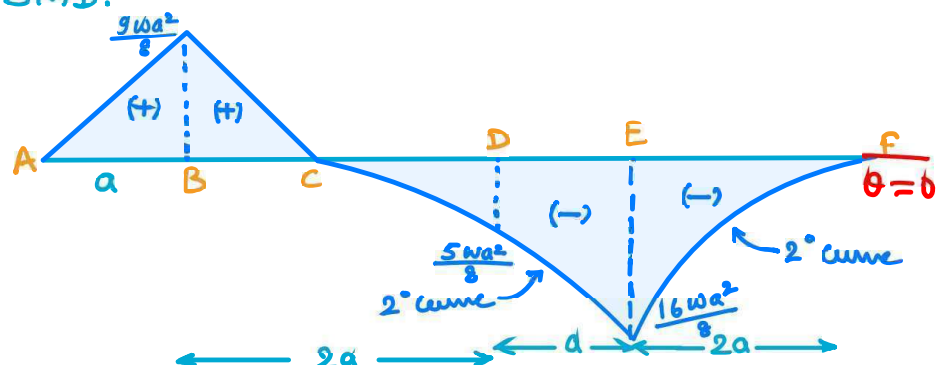
$$Bm_E - 10.415 = -\frac{1}{2} (7.36 + 11.86) 1.5$$

$$Bm_E = -4 \text{ KN}\cdot\text{m}$$

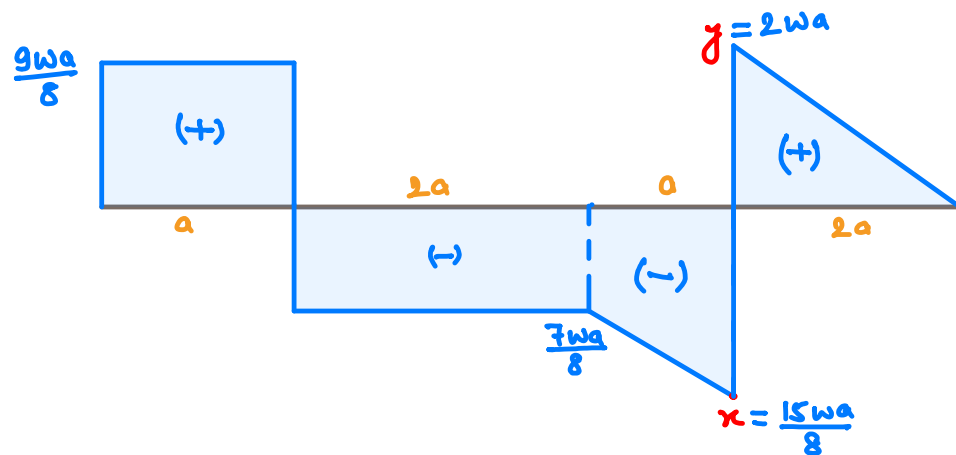
BMD



Q.) Draw SFD and loading diagram & deflected shape diagram from corresponding BMD.



from $A \rightarrow B$, $\frac{dM}{dx} = \frac{\frac{9wa^2}{8} - 0}{a} = \frac{9}{8}wa$



from $B \rightarrow D$, $\frac{dM}{dx} = \frac{-\frac{5wa^2}{8} - \frac{9wa^2}{8}}{2a}$

from $D \rightarrow E$, $\Delta Bm = Bm_E - Bm_D = \text{Area of SFD}$

$$= -\frac{16wa^2}{8} - \left(-\frac{5wa^2}{8}\right) = -\frac{1}{2} \left(\frac{7wa}{8} + x\right)a$$

$$x = \frac{15wa}{8}$$

Since Slope of BMD at F is zero, SF at F is also zero.

from F to E (Right to left)

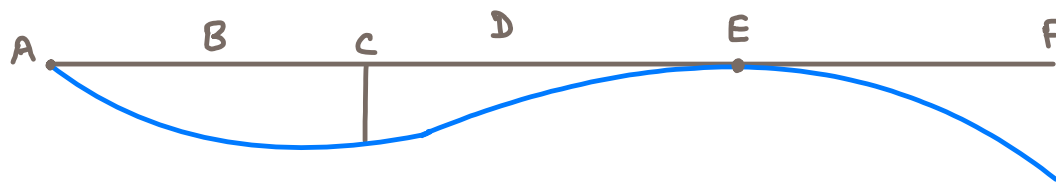
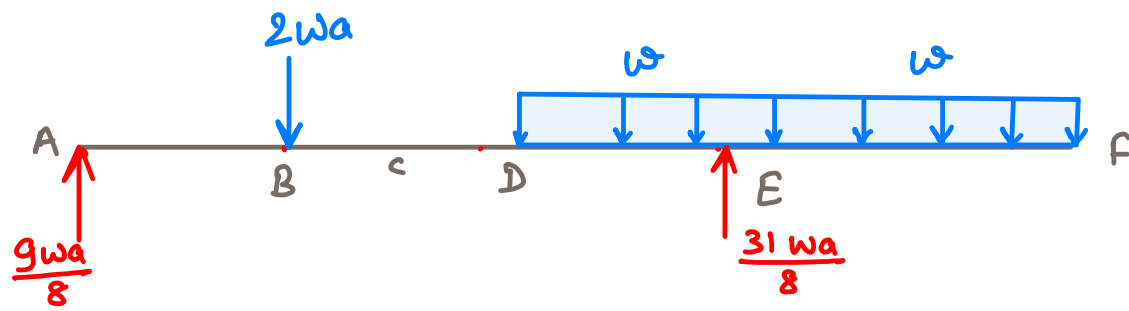
$$\Delta Bm = Bm_E - Bm_F = \text{Area of SFD}$$

$$-\left(-\frac{16a^2w}{8}\right) - 0 = \frac{1}{2} \times 2a \times y$$

$$y = 2wa$$

⑤

$$\Delta Bm_{E \rightarrow F} = Bm_F - Bm_E = \text{Area of SFD}$$



Deflected diagram.

TRANSFORMATION OF STRESS AND STRAIN

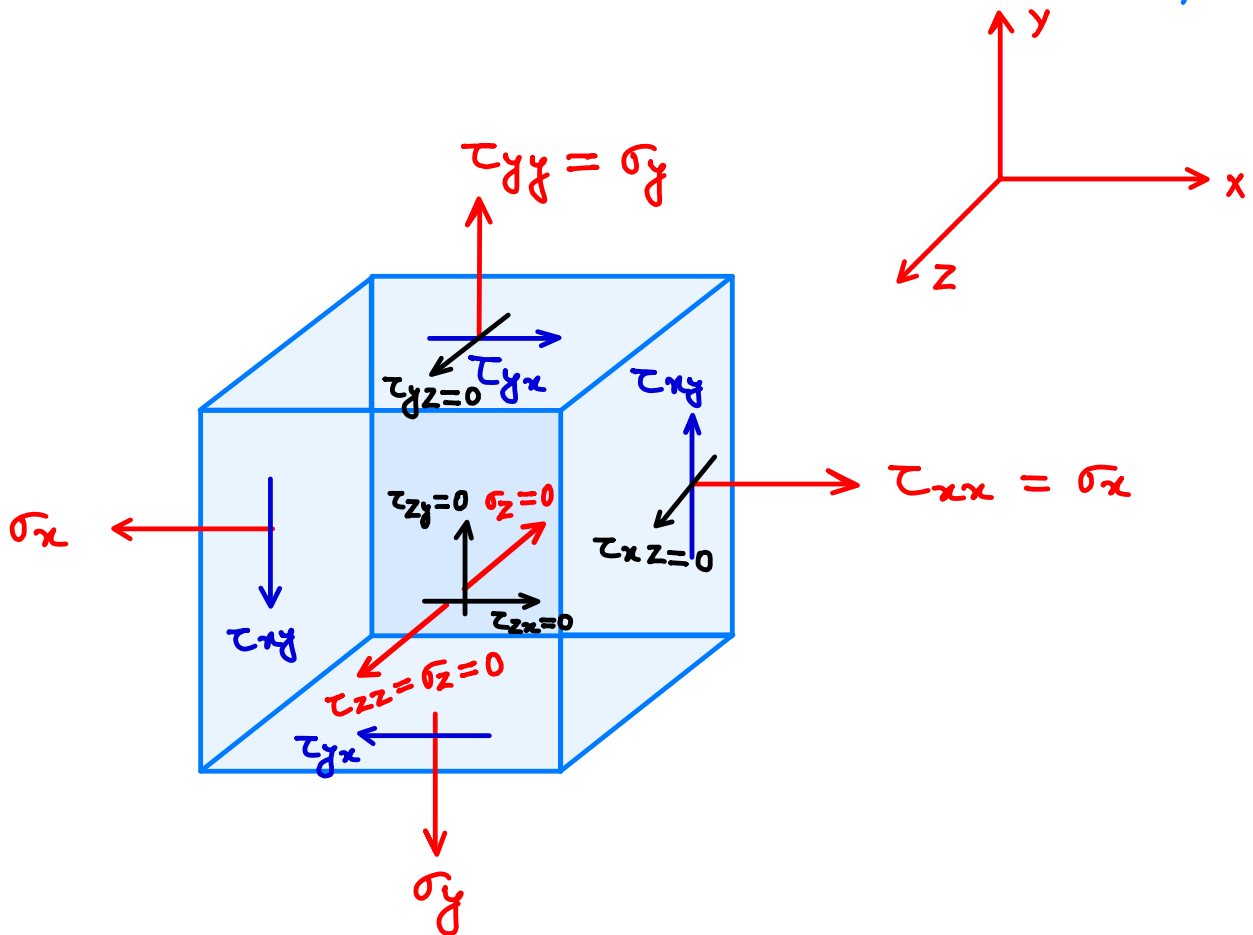
- from designing point of view Structural Component must not only be under equilibrium due to external forces but also it must be equilibrium under the impact of internal stresses at each and every point inside the volume.
- Hence we require the magnitude of maximum normal and shear stress and the plane on which these maximum stresses are acting
- Since the magnitude of these stresses depend on inclination of the plane they are acting on, they are computed from stress on the other known plane by the process termed as "TRANSFORMATION".
- In 3 dimension at a given point, 9 stress component exist, However, only 6 stress components are required to define the complete state of stress at that point. ($\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$)
- When 2 faces of Cubical element are free from any type of stress, the stress condition is termed as plane stress condition.
- for eg:- if no stress acts on Z plane. i.e.

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

154 it is plane stress condition for X-Y plane.

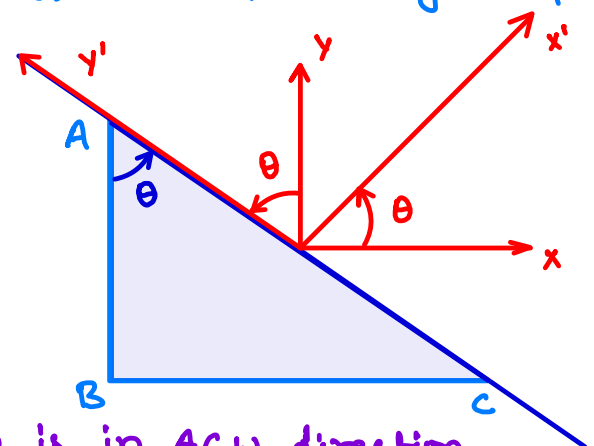
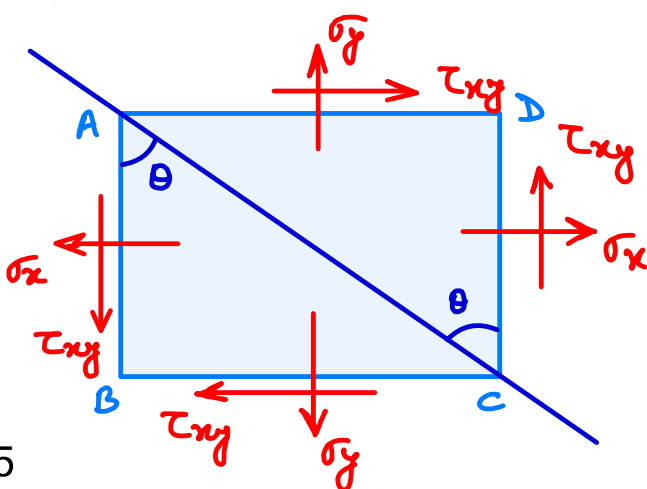
Stress components in this case would be $\sigma_x, \sigma_y, \tau_{xy}$ (Plane stress components).

- for eg:- Bar in tension/compression, Bending in beams, Torsion of shaft etc.



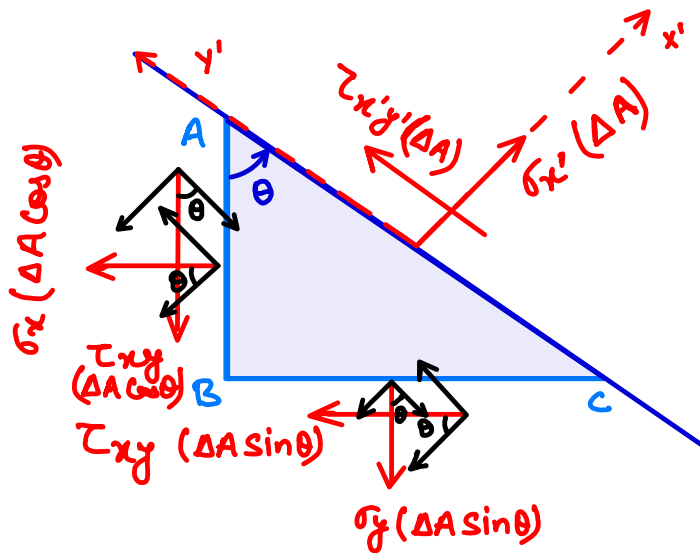
Transformation of Plane Stress

- Transformation is the process of finding the stresses on any plane with the help of the stress known on given plane.



Here θ is in ACW direction
So, Positive (+).

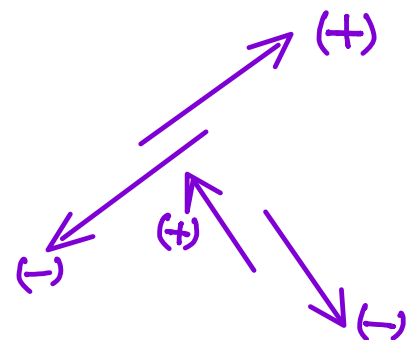
Let $\sigma_{x'}$ and $\tau_{x'y'}$ be the Normal & Shear stress acting on plane AC which is normal to positive (+) x' axis.



gf area of plane $AC = \Delta A$

$$\text{Area } AB = \Delta A \cos \theta$$

$$\text{Area } BC = \Delta A \sin \theta$$



$$\sum f_{x'} = 0$$

$$\sigma_{x'} \Delta A - \sigma_x (\Delta A \cos \theta) \cos \theta - \tau_{xy} (\Delta A \cos \theta) \sin \theta - \sigma_y (\Delta A \sin \theta) \sin \theta - \tau_{xy} (\Delta A \sin \theta) \cos \theta = 0$$

$$\sigma_{x'} = \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta$$

$$\sigma_{x'} = \sigma_x \frac{(1 + \cos 2\theta)}{2} + \sigma_y \frac{(1 - \cos 2\theta)}{2} + \tau_{xy} \sin 2\theta$$

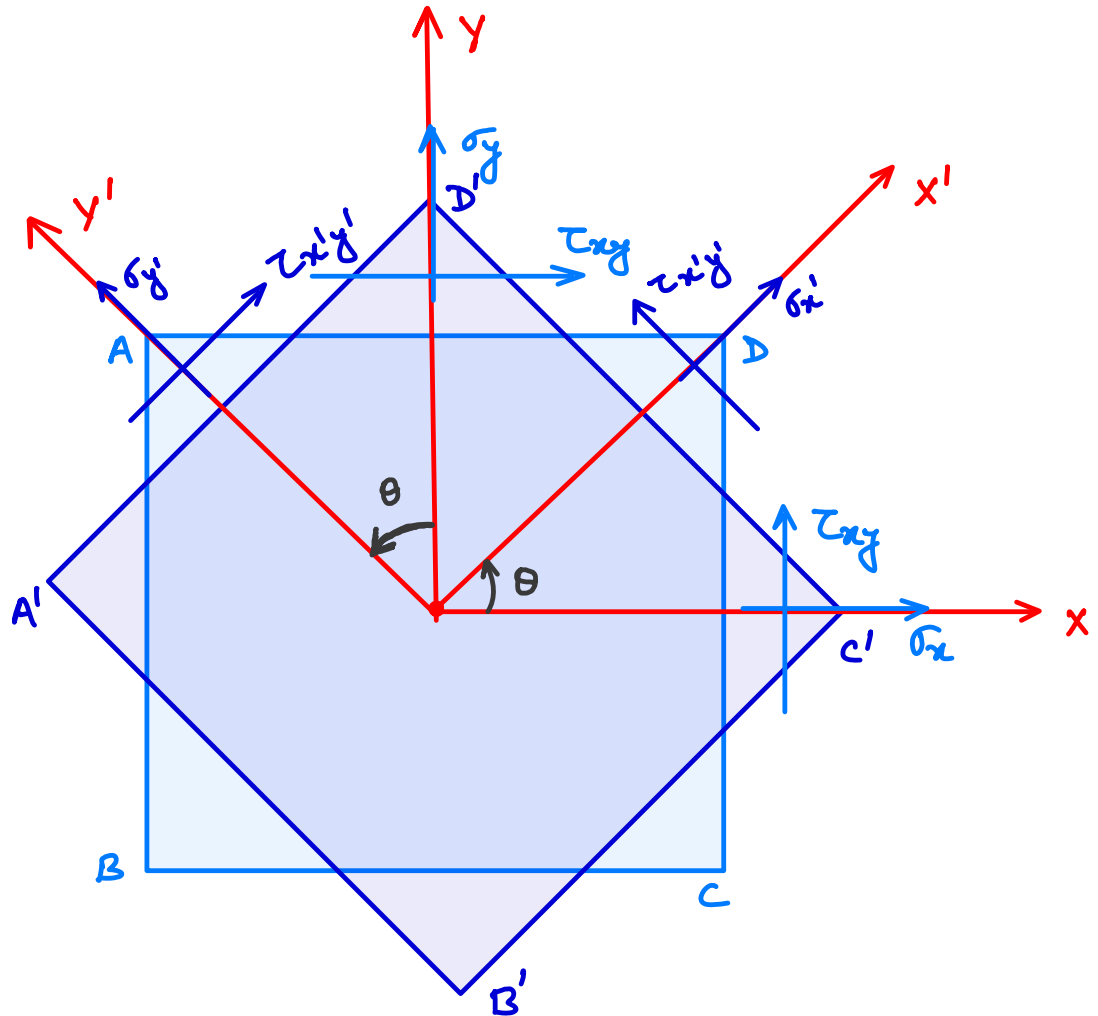
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \quad \text{--- I}$$

$$\sum f_{y'} = 0$$

$$\Rightarrow \tau_{x'y'} (\Delta A) + \sigma_x (\Delta A \cos \theta) \sin \theta - \tau_{xy} (\Delta A \cos \theta) \cos \theta - \sigma_y (\Delta A \sin \theta) \cos \theta + \tau_{xy} (\Delta A \sin \theta) \sin \theta = 0$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y) \sin \theta \cdot \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)$$

$$\tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \quad \text{--- II}$$



To obtain $\sigma_{y'}$, replace θ in $\sigma_{x'}$ by $\theta + 90^\circ$.

$$\text{Hence, } \cos 2(\theta + 90^\circ) = \cos (2\theta + 180^\circ) = -\cos 2\theta$$

$$\sin 2(\theta + 90^\circ) = \sin (2\theta + 180^\circ) = -\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \quad \text{--- III}$$

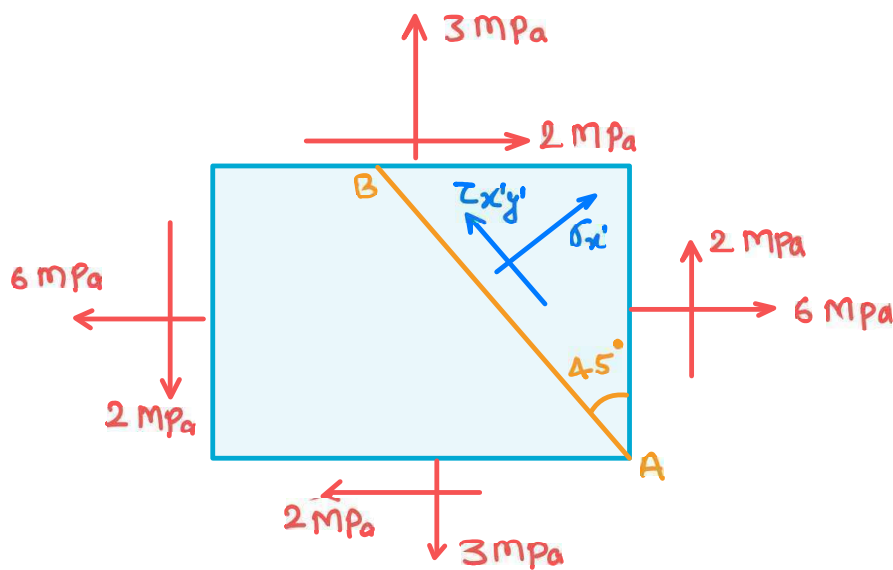
from I & III

$$\sigma_{x'} + \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta + \frac{\sigma_x + \sigma_y}{2}$$

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$$

- Hence in case of plane stress summation of Normal stress on any two mutually perpendicular plane is constant i.e. it is independent of the orientation of element.

Q. > A stress element diagram is shown in figure. Compute the normal & shear stress on plane AB.



Solution

$$\sigma_x = 6 \text{ MPa}, \quad \sigma_y = 3 \text{ MPa}, \quad \tau_{xy} = 2 \text{ MPa}$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$= \frac{6+3}{2} + \frac{6-3}{2} \cos 90^\circ + 2 \times \sin 90^\circ$$

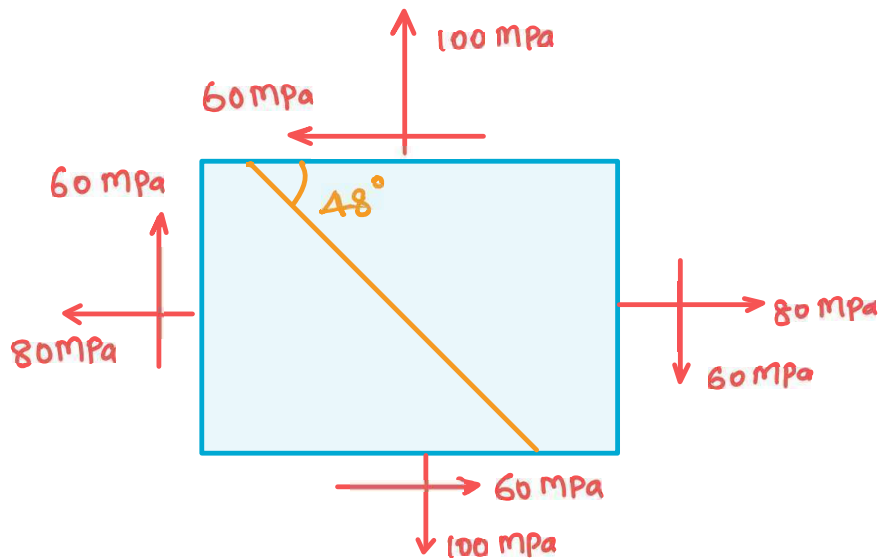
$$= 4.5 + 1.5 \times 0 + 2 = 6.5 \text{ MPa.}$$

$$\tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$= -\frac{(6-3)}{2} \sin 90^\circ + 2 \cos 90^\circ$$

$$= -1.5 \text{ MPa.}$$

b. >



$$\sigma_x = 80 \text{ MPa}, \quad \sigma_y = 100 \text{ MPa}, \quad \tau_{xy} = -60 \text{ MPa},$$

$$\theta = 90^\circ - 48^\circ$$

$$\theta = 42^\circ$$

$$\begin{aligned} \sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta + \tau_{xy} \sin 2\theta \\ &= \frac{80 + 100}{2} + \left(\frac{80 - 100}{2} \right) \cos 84^\circ + (-60) \sin 84^\circ \\ &= 29.28 \text{ MPa} \end{aligned}$$

$$\begin{aligned} \tau_{x'y'} &= -\frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \\ &= -\frac{(80 - 100)}{2} \sin 84^\circ + (-60) \cos 84^\circ \\ &= 3.67 \text{ MPa.} \end{aligned}$$

Principal Stress & maximum Shear Stress.

- for a stressed point in structural member infinite number of plane passes and every plane have different Normal and Shear Stress.
- Out of the infinite planes, 3 planes which are mutually perpendicular to each other exist, where normal stresses are either maximum or minimum.
- These maximum/minimum Normal Stresses are termed as "**principal Stresses**" i.e. major, intermediate and minor "principal Stresses".
- In plane stress condition, intermediate principal stress is zero.
- The planes over which principal stresses acts are termed as "**principal planes**".
- Over these principal planes Shear Stress is zero.
- for principal Stresses

$$\frac{d\sigma_{x'}}{d\theta} = 0 \quad \text{or} \quad \tau_{xy'} = 0$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\frac{d\sigma_{x'}}{d\theta} = 0 + \frac{\sigma_x - \sigma_y}{2} \sin 2\theta \cdot (-2) + \tau_{xy} \cos 2\theta \cdot (2) = 0$$

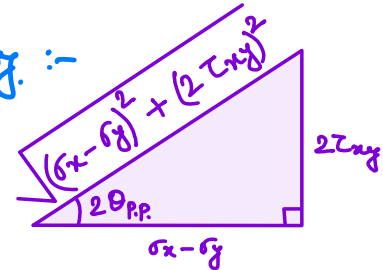
$$\sigma_x - \sigma_y (\sin 2\theta) = 2 \tau_{xy} \cos 2\theta$$

$$\tan 2\theta_{p.p.} = \frac{2 \tau_{xy}}{\sigma_x - \sigma_y}$$

- from here 2 values of $2\theta_{p,p}$ will be observed separated by 180° hence both values of $\theta_{p,p}$ will be separated by angle of 90° as $\tan(180^\circ + \alpha) = \tan \alpha$

— Now Values of principal stresses are given by :-

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta_{p,p} + \tau_{xy} \sin 2\theta_{p,p}$$



$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cdot \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}} + \frac{\tau_{xy} \times 2\tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2} \frac{[(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2]}{\sqrt{(\sigma_x - \sigma_y)^2 + (2\tau_{xy})^2}}$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Similarly,

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Hence,

$$\begin{array}{l} \sigma_{\max} / \sigma_{\min} \\ \text{or} \\ \sigma_{p_1} / \sigma_{p_2} \end{array} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- Now it can also be shown that Shear stress on principal plane is zero as follows:-

$$\tau_{x'y'} = \frac{-(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$\tau_{x'y'} = \frac{-(\sigma_x - \sigma_y)}{2} \frac{2 \tau_{xy}}{\sqrt{(\sigma_x - \sigma_y)^2 + (2 \tau_{xy})^2}} + \frac{\tau_{xy} (\sigma_x - \sigma_y)}{\sqrt{(\sigma_x - \sigma_y)^2 + (2 \tau_{xy})^2}}$$

$$\boxed{\tau_{x'y'} = 0}$$

- Out of infinite planes passing through a point, 3 mutually perpendicular plane exists on which magnitude of shear stress is maximum these planes are termed as **Maximum Shear Stress plane**.
- For plane stress condition only 2 mutually perpendicular planes exists on which shear stress is maximum.
- Maximum shear stress plane is **perpendicular bisector** of 2 principal planes.

- for shear stress to be maximum, $\frac{d \tau_{x'y'}}{d\theta} = 0$

$$\tau_{x'y'} = \frac{-(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$\frac{d \tau_{x'y'}}{d\theta} = -\frac{(\sigma_x - \sigma_y)}{2} \cos 2\theta \times 2 + \tau_{xy} \sin 2\theta \cdot (-2) = 0$$

$$2 \tau_{xy} \sin 2\theta = -(\sigma_x - \sigma_y) \cos 2\theta$$

$$\frac{\sin 2\theta}{\cos 2\theta} = -\frac{(\sigma_x - \sigma_y)}{2 \tau_{xy}}$$

$$\boxed{\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)}{2 \tau_{xy}}}$$

- from here 2 values of " $2\theta_s$ " will be observed separated by 180° hence two values of " θ_s " will be separated by angle of 90° .

— Now the value of Maximum Shear Stress is given by:-

$$\tau_{x'y'}_{\text{Max.}} = -\frac{(\sigma_x - \sigma_y)}{2} \sin 2\theta_s + \tau_{xy} \cos 2\theta_s$$

$$\tau_{x'y'}_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{P_1} - \sigma_{P_2} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} - \frac{\sigma_x - \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{P_1} - \sigma_{P_2} = 2 \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{P_1} - \sigma_{P_2} = 2 \tau_{x'y'}_{\text{max}}$$

$$\tau_{x'y'} = \frac{\sigma_{P_1} - \sigma_{P_2}}{2}$$

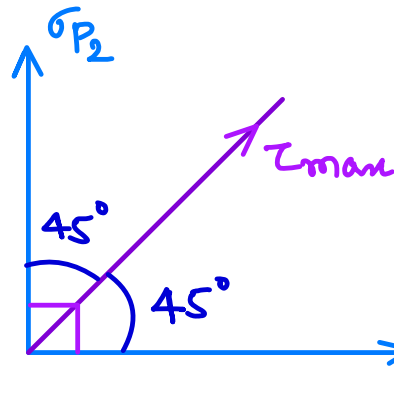
$$\Rightarrow \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}, \quad \tan 2\theta_s = \frac{(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

$$\tan 2\theta_p \cdot \tan 2\theta_s = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \cdot \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} = -1$$

Hence, $2\theta_p$ and $2\theta_s$ are separated by an angle of 90° .

$\therefore \theta_p$ and θ_s are separated by angle of 45° .

$\therefore$ Plane of maximum shear is perpendicular bisector of Principal Planes.



Now normal Stress at point where Shear Stress is maximum is given by:-

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta_s + \tau_{xy} \sin 2\theta_s$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cdot \frac{2\tau_{xy}}{\sqrt{(\sigma_y - \sigma_x)^2 + (2\tau_{xy})^2}} + \frac{\tau_{xy} \cdot (\sigma_y - \sigma_x)}{\sqrt{(\sigma_y - \sigma_x)^2 + (2\tau_{xy})^2}}$$

$$\boxed{\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2}}$$

Note:-

- Shear Stress is maximum on three mutually perpendicular planes out of infinite planes, hence absolute maximum Shear Stress is the maximum of Shear Stresses on these three planes.

$$\boxed{(\tau_{\max})_{\text{Absolute}} = \max \left| \frac{\sigma_{p1} - \sigma_{p2}}{2}, \frac{\sigma_{p2} - \sigma_{p3}}{2}, \frac{\sigma_{p1} - \sigma_{p3}}{2} \right|}$$

for safe designing, $\boxed{(\tau_{\max})_{\text{Absolute}} < \tau_y}$

— for plane stress condition (and not considering stresses on z-plane)

$$(\tau_{\max})_{\text{Absolute}} = \max \left| \frac{\sigma_{P_1} - \sigma_{P_2}}{2}, \frac{\sigma_{P_1}}{2}, \frac{\sigma_{P_2}}{2} \right|$$

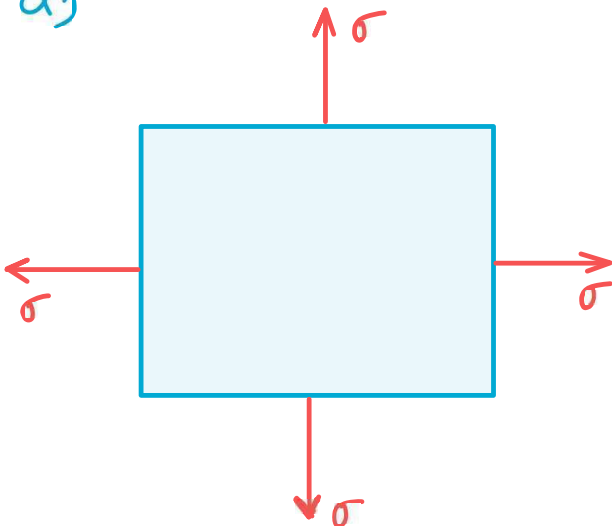
Q. > $\sigma_x = 60 \text{ MPa}$, $\sigma_y = 80 \text{ MPa}$, find principal stresses.
 $\tau_{xy} = 56 \text{ MPa}$.

$$\begin{aligned} \sigma_{P_1}/\sigma_{P_2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \\ &= \frac{60 + 80}{2} \pm \sqrt{\left(\frac{60 - 80}{2}\right)^2 + (56)^2} \end{aligned}$$

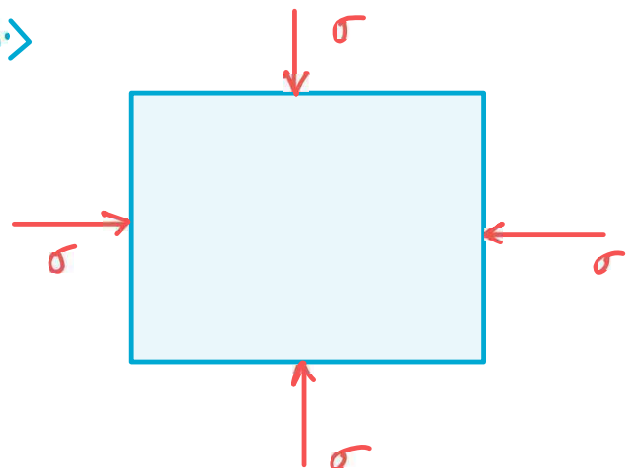
$$\sigma_{P_1} = 126.88 \text{ MPa}, \quad \sigma_{P_2} = 13.11 \text{ MPa}.$$

Q. > Stress element diagram as shown in figure is subjected to given stress condition. Compute the total number of principle planes.

a.)



b.)



Solution

$$\begin{aligned}
 \text{a) } \tau_{x'y'} &= \frac{-(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \\
 &= \frac{-(\sigma - \sigma)}{2} \sin 2\theta + 0 \cdot \cos 2\theta \\
 &= 0
 \end{aligned}$$

- Since for any value of θ , Shear stress is zero there are infinite principal planes.

$$\begin{aligned}
 \text{b) } \tau_{x'y'} &= \frac{-[-\sigma - (-\sigma)]}{2} \sin 2\theta + 0 \cdot \cos 2\theta \\
 &= 0.
 \end{aligned}$$

Q. > At a point in structural member triaxial stresses are experienced as follows:-

- Determine the magnitude of principal stresses & max. shear stress
- Determine the location of principal plane & plane of maximum shear
- Draw all the stresses by complete sketch.

Solution:-

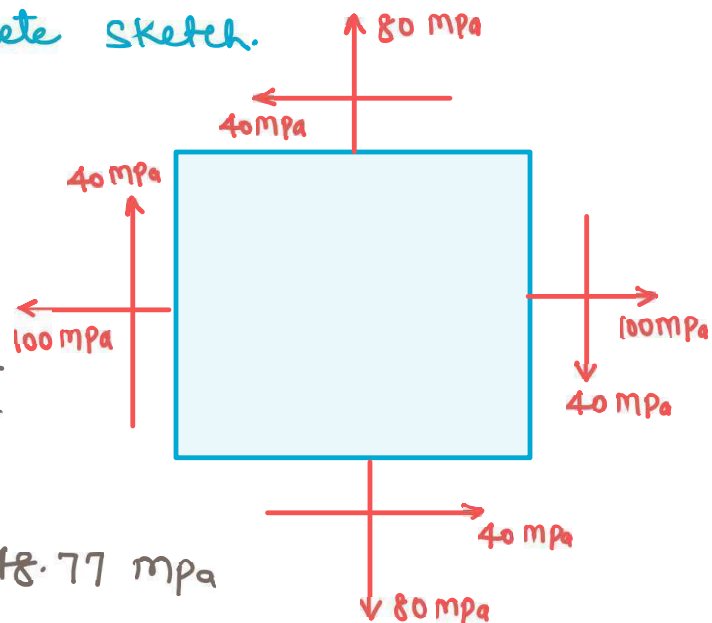
$$\begin{aligned}
 \text{a) } \sigma_{P_1} / \sigma_{P_2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \\
 &= \frac{100 + 80}{2} \pm \sqrt{\left(\frac{100 - 80}{2}\right)^2 + (-40)^2}
 \end{aligned}$$

$$\sigma_{P_1} = 131.23 \text{ MPa} \quad \sigma_{P_2} = 48.77 \text{ MPa}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-40)}{100 - 80} = \frac{-2 \times 40}{20} = -4$$

$$\theta_{P_1} = -37.98^\circ$$

$$\theta_{P_2} = 90 - 37.98^\circ = 52.02^\circ$$



check, $\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta_p + \tau_{xy} \sin 2\theta_p$

$$= \frac{180}{2} + \frac{20}{2} \cos(2 \times -37.98) + (-40) \sin(2 \times -37.98)$$

$$= 131.23 \text{ MPa.}$$

a) ii) $\tau_{max} = \frac{\sigma_{p_1} - \sigma_{p_2}}{2} = \frac{131.23 - 48.77}{2} = 41.23 \text{ MPa.}$

b) ii) $\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} = \frac{1}{4}$

$$\theta_{s_1} = 7.02^\circ$$

$$\theta_{s_2} = 90^\circ + 7.02^\circ = 97.02^\circ$$

check:-

$$\tau_{x'y'} = \frac{-(\sigma_x - \sigma_y)}{2} \sin 2\theta_{s_1} + \tau_{xy} \cos 2\theta_{s_1}$$

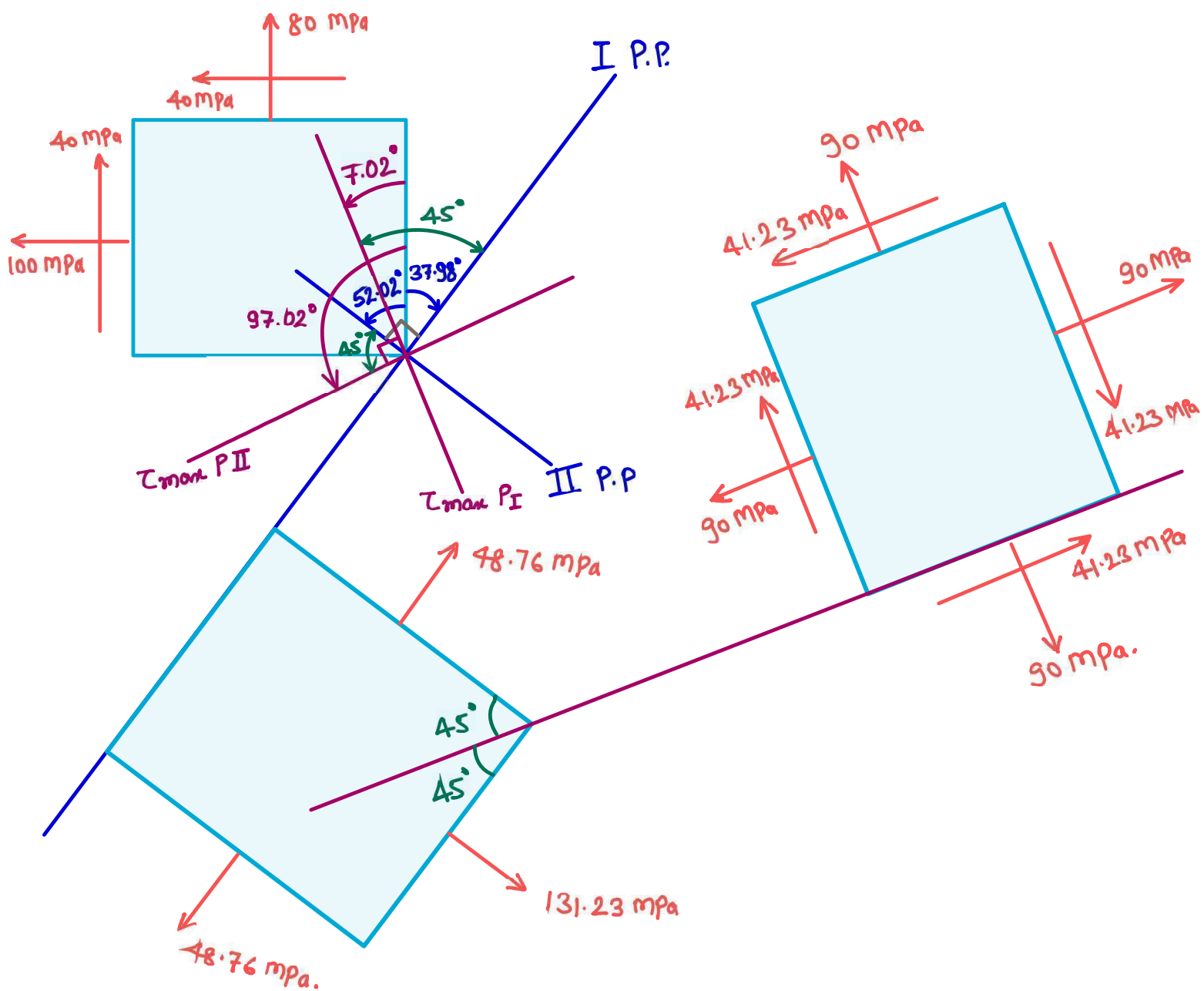
$$= \frac{-(100 - 80)}{2} \sin(2 \times 7.02) + (-40) \cos(2 \times 7.02)$$

$$\tau_{x'y'} = -41.23$$

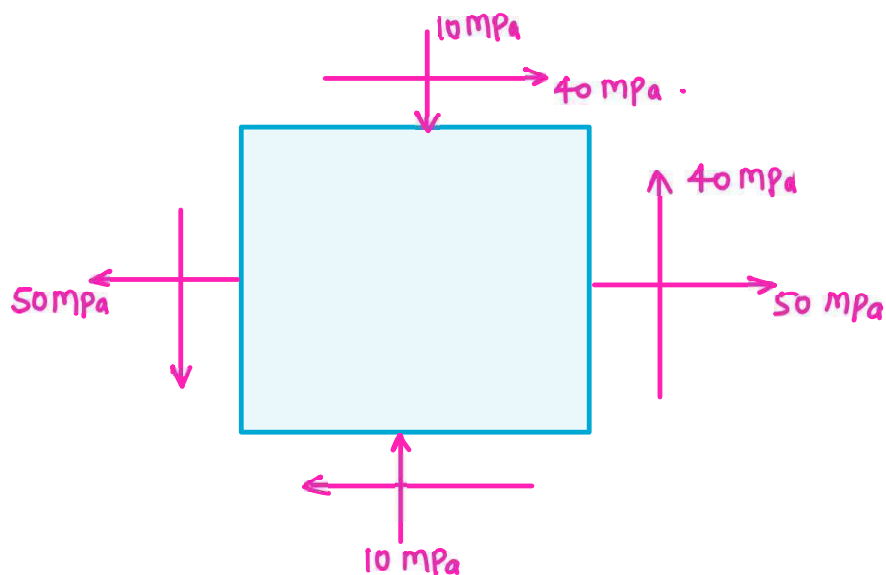
Also, $\sigma_{x'} \tau_{max} = \frac{\sigma_x + \sigma_y}{2}$

$$= \frac{100 + 80}{2} = 90 \text{ MPa.}$$

Stress invariants = 2 1r plane sum = constant (Applicable)



b. >



Solution.

$$\sigma_x = 50 \text{ mpa}$$

$$\sigma_y = -10 \text{ mpa}$$

$$\tau_{xy} = 40 \text{ mpa}$$

$$\begin{aligned}\sigma_{p_1}/\sigma_{p_2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= \frac{50 - 10}{2} \pm \sqrt{\left(\frac{50 - (-10)}{2}\right)^2 + (40)^2}\end{aligned}$$

$$\sigma_{p_1} = 70 \text{ mpa}$$

$$\sigma_{p_2} = -30 \text{ mpa}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{4}{3}$$

$$\theta_{p_1} = 26.56^\circ$$

$$\theta_{p_2} = 90 + 26.56^\circ = 116.56^\circ$$

$$\begin{aligned}\text{Check, } \underline{\sigma_x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta_{p_1} + \tau_{xy} \sin 2\theta_{p_1} \\ &= \frac{50 - 10}{2} + \frac{50 - (-10)}{2} \cos(2 \times 26.56^\circ) + 40 \cdot \sin(2 \times 26.56^\circ) \\ &= \underline{70 \text{ mpa}}\end{aligned}$$

$$\tau_{\max} = \frac{\sigma_{p_1} - \sigma_{p_2}}{2} = \frac{70 + 30}{2} = 50 \text{ mpa}$$

$$\sigma_x \tau_{\max} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2} = 20 \text{ mpa}$$

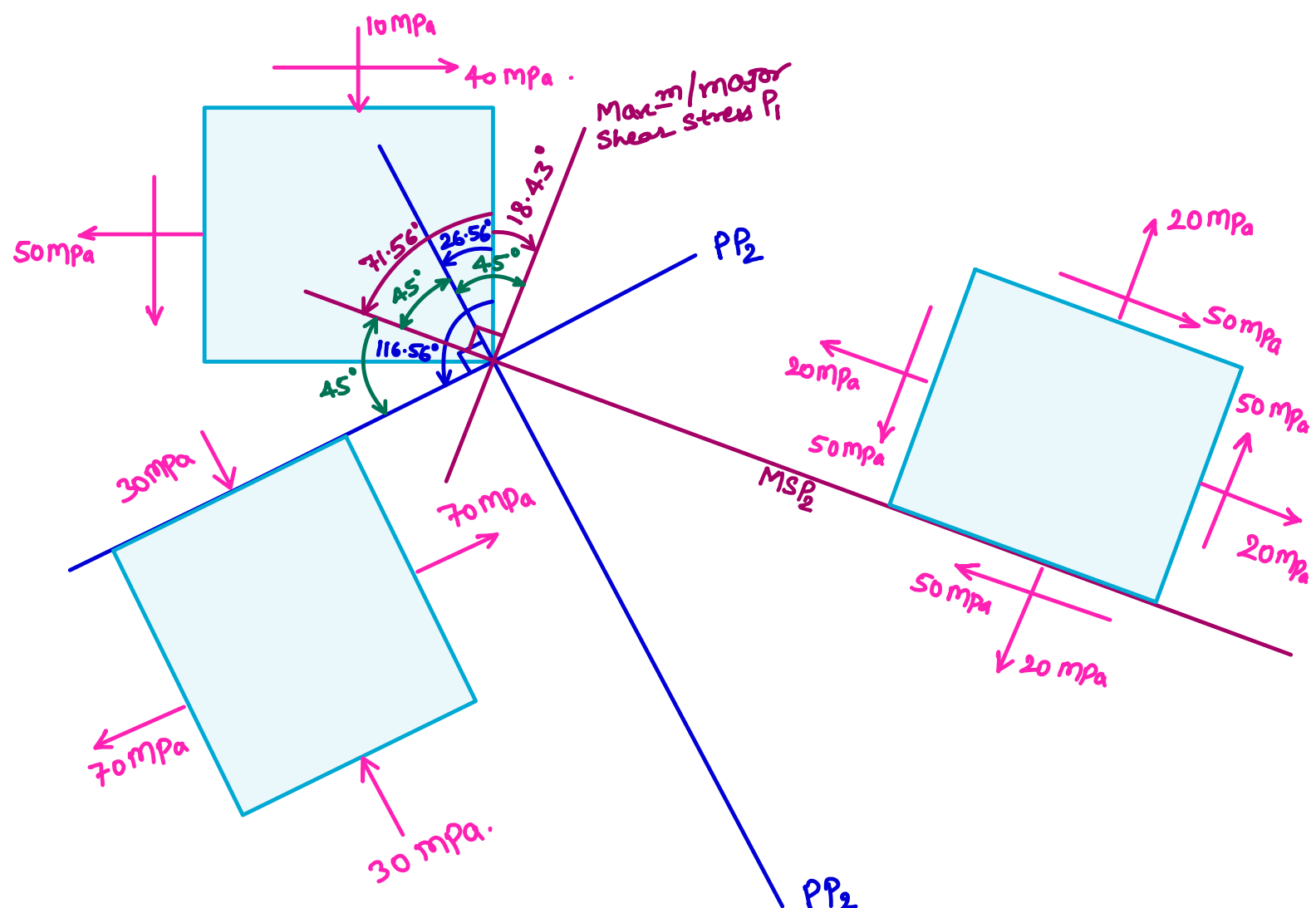
$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} = \frac{-(50 + 10)}{2 \times 40} = -\frac{3}{4}$$

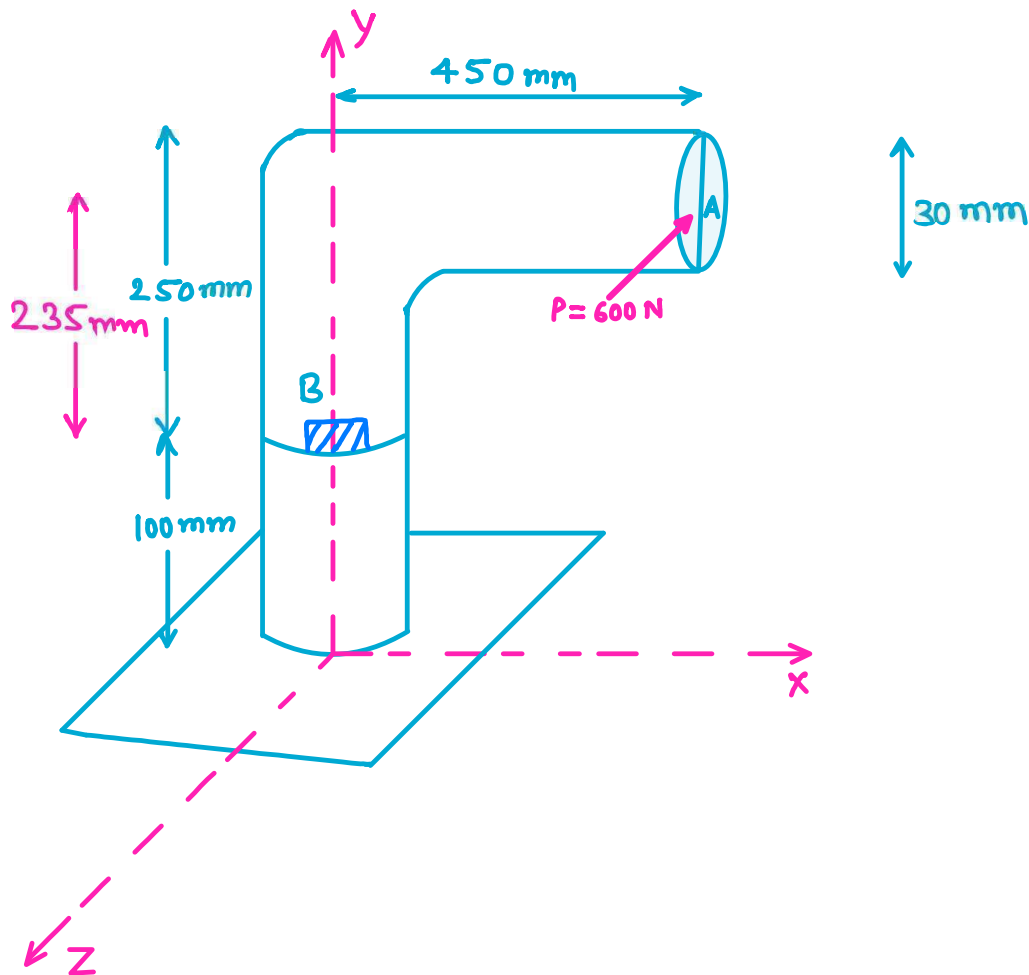
$$\theta_{s_1} = \tan^{-1}\left(-\frac{3}{4}\right) = -18.44^\circ$$

$$\theta_{s_2} = 90 - 18.44 = 71.56^\circ$$

$$\begin{aligned}
 \tau_{x'y'} &= \frac{-(\sigma_x - \sigma_y)}{2} \sin 2\theta_{s_1} + \tau_{xy} \cos 2\theta_{s_1} \\
 &= \frac{-(50 + 10)}{2} \sin(2 \times (-18.44)) + 40 \cdot \cos(2 \times (-18.44)) \\
 &= 50 \text{ MPa}.
 \end{aligned}$$

$$\begin{aligned}
 \tau_{x'y'} &= \frac{-(\sigma_x - \sigma_y)}{2} \sin 2\theta_{s_2} + \tau_{xy} \cos 2\theta_{s_2} \\
 &= -50 \text{ MPa}
 \end{aligned}$$

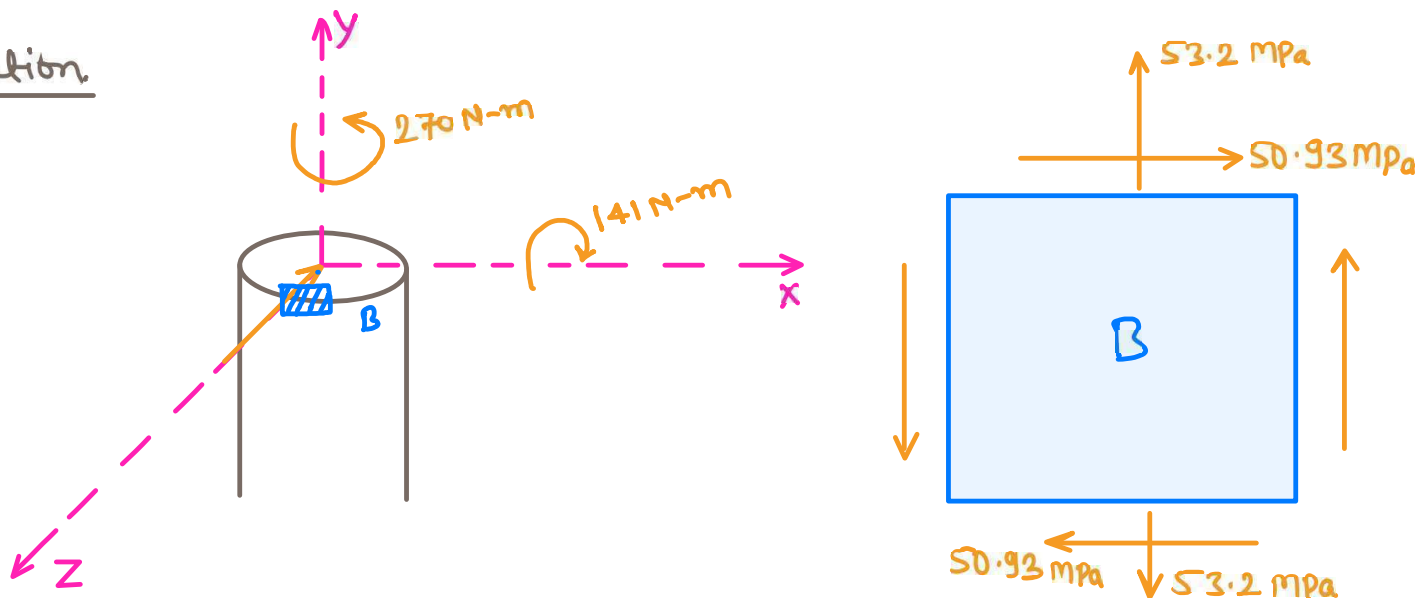




A single force of magnitude 600 N is applied at the end of lever at A. Determine

- The Normal & Shearing Stresses on a element located at point B are having sides parallel to the x & y axis.
- The principal plane and principal stresses at point B.

Solution



$$M_x = 600 \times (250 - 15) \times 10^{-3} = 141 \text{ Nm}$$

$$T_y = 600 \times (450) \times 10^{-3} = 270 \text{ Nm}$$

using Bending formula :-

$$\frac{M}{I} = \frac{\sigma}{y}$$

$$\sigma = \frac{M}{I} y = \frac{M \cdot \frac{d}{2}}{\pi \frac{d^4}{64}} = \frac{141 \cdot 15 \times 10^3}{\pi \cdot \frac{15^4}{64}} = 53.2 \text{ MPa.}$$

Using Torsion formula :-

$$\frac{T}{J} = \frac{\tau}{r} \Rightarrow \tau = \frac{T r}{J} = \frac{T \cdot d/2}{\pi d^4/32} = \frac{270 \times 15}{\pi \times 30^4} \times 32 \times 10^3$$

$$= 50.93 \text{ MPa.}$$

$$\sigma_{p_1}/\sigma_{p_2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

$$\sigma_x = 0, \quad \sigma_y = 53.2 \text{ MPa}, \quad \tau_{xy} = 50.93 \text{ MPa}$$

$$= \frac{0 + 53.2}{2} \pm \sqrt{\left(\frac{0 - 53.2}{2}\right)^2 + (50.93)^2}$$

$$\sigma_{p_1} = 84.05 \text{ MPa}$$

$$\sigma_{p_2} = -30.86 \text{ MPa}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times 50.93}{0 - 53.2} = -1.91$$

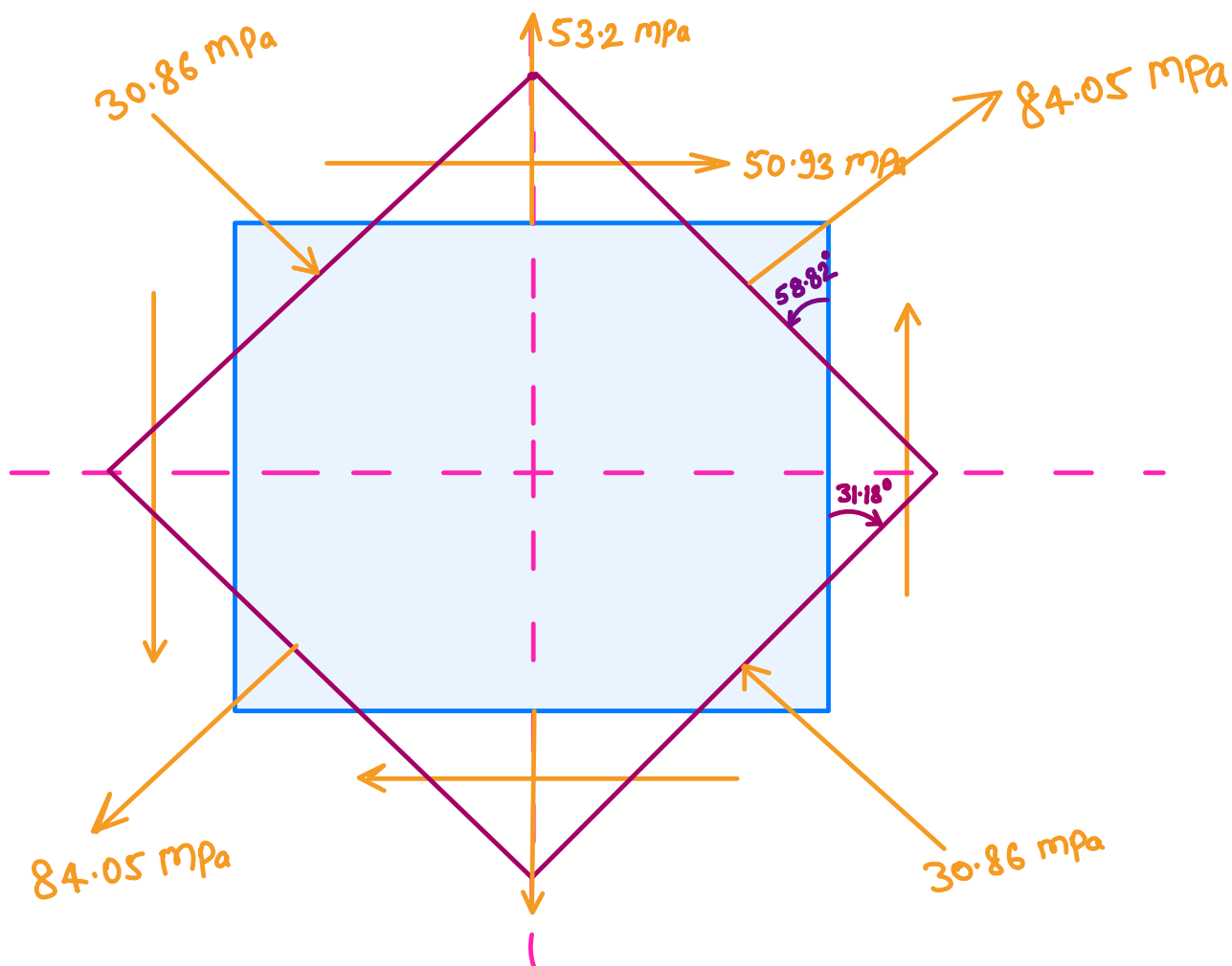
$$\theta_{p_1} = -31.18^\circ$$

$$\theta_{p_2} = 90 - 31.18^\circ = 58.82^\circ$$

check:- $\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta_p + \tau_{xy} \sin 2\theta_p$

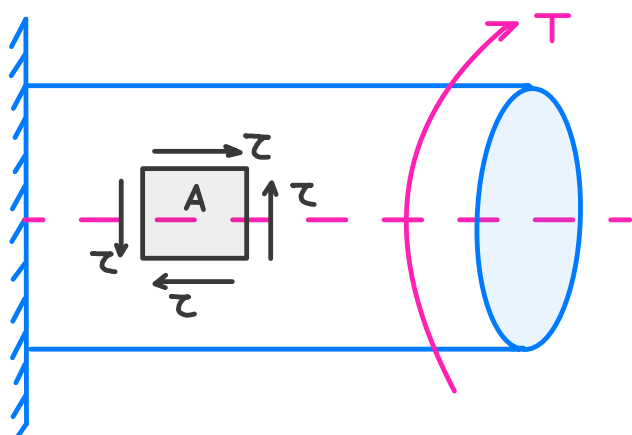
$$= \frac{0 + 53.2}{2} + \frac{0 - 53.2}{2} \cos(-2 \times 31.18) + 50.93 \cdot \sin(-2 \times 31.18)$$

$$\sigma_{x'} = -30.86 \text{ MPa.}$$

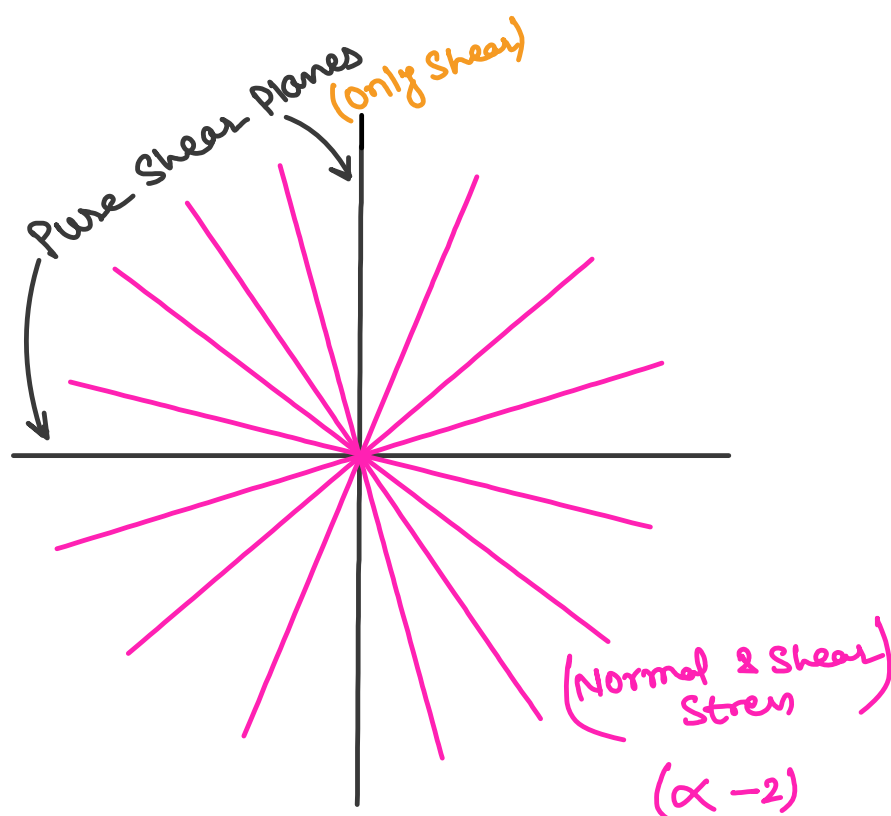


★ Principal Stress and Principal Plane in Pure Shear Condition:-

- Pure shear condition represents the state in which element is subjected to only shear stress & normal stress is absent.
- The planes over which only shear stress exists is termed as Pure Shear plane.



- In pure shear case, normal stress on two mutually perpendicular planes is zero and the remaining planes ($\alpha - 2$) are subjected to both normal & shear stress.



$$\sigma_x = 0, \quad \sigma_y = 0, \quad \tau_{xy} = \tau$$

$$\sigma_{p_1}/\sigma_{p_2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

$$= 0 \pm \sqrt{0 + 2\tau}$$

$$\sigma_{p_1} = +\tau$$

$$\sigma_{p_2} = -\tau$$

Normal Stress = Shear Stress
magnitude on PP

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \infty$$

$$\theta_{p_1} = 45^\circ$$

$$\theta_{p_2} = 90 + 45^\circ = 135^\circ$$

$$\tau_{max} = \frac{\sigma_{p_1} - \sigma_{p_2}}{2} = \frac{\tau - (-\tau)}{2} = \tau$$

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}} = 0$$

$$\theta_{s_1} = 0^\circ$$

$$\theta_{s_2} = 90^\circ$$

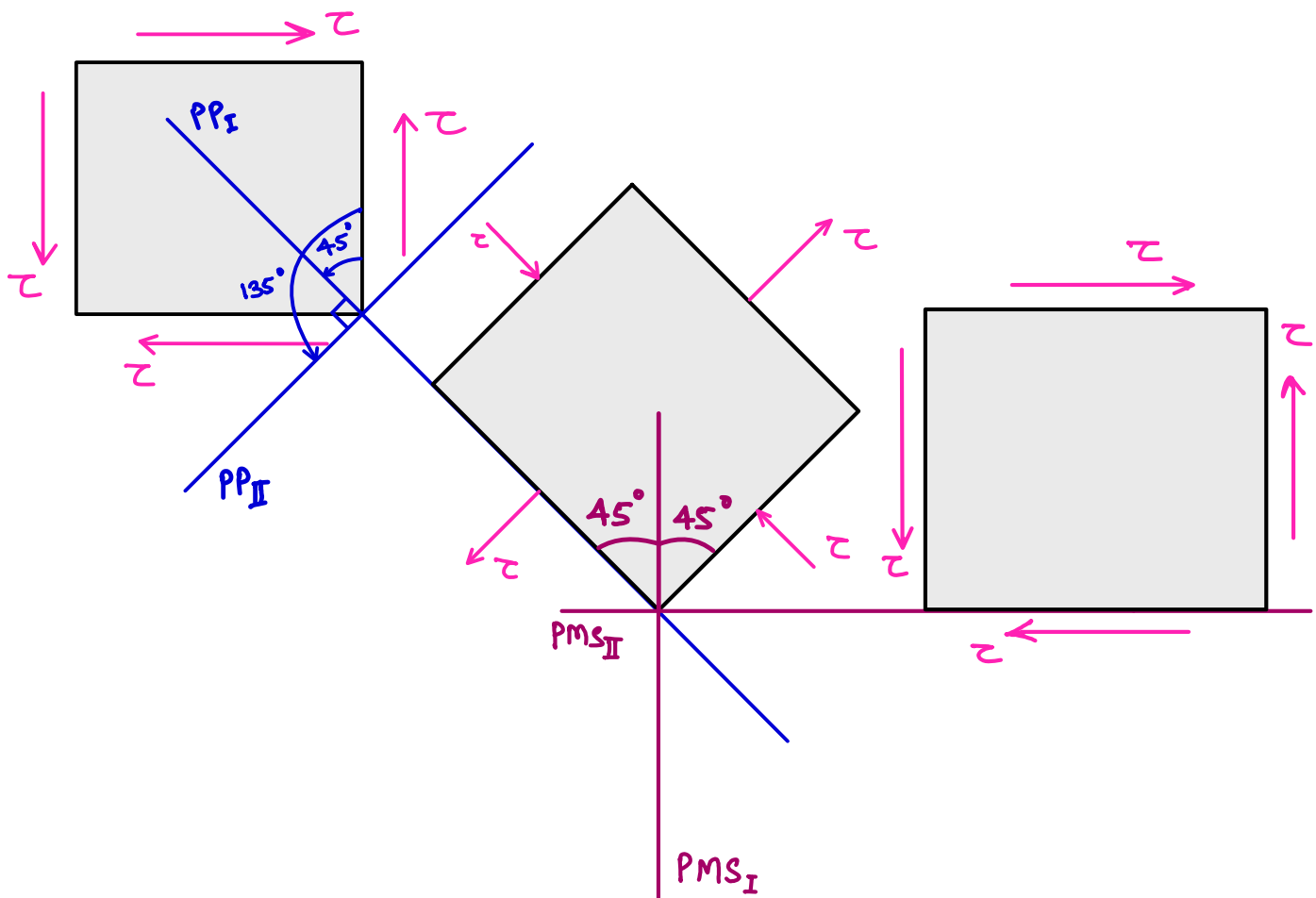
*

$$\sigma_{\tau_{max}} = \frac{\sigma_x + \sigma_y}{2} = 0$$

check:- $\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta_{p_1} + \tau_{xy} \sin 2\theta_{p_1}$

$$\sigma_{x'} = \tau$$

$$\begin{aligned}\tau_{n'y'} &= \frac{-(\sigma_x - \sigma_y)}{2} \sin 2\theta_s + \tau_{xy} \cos 2\theta_s \\ &= \tau \cos(2 \times 0) = \tau\end{aligned}$$

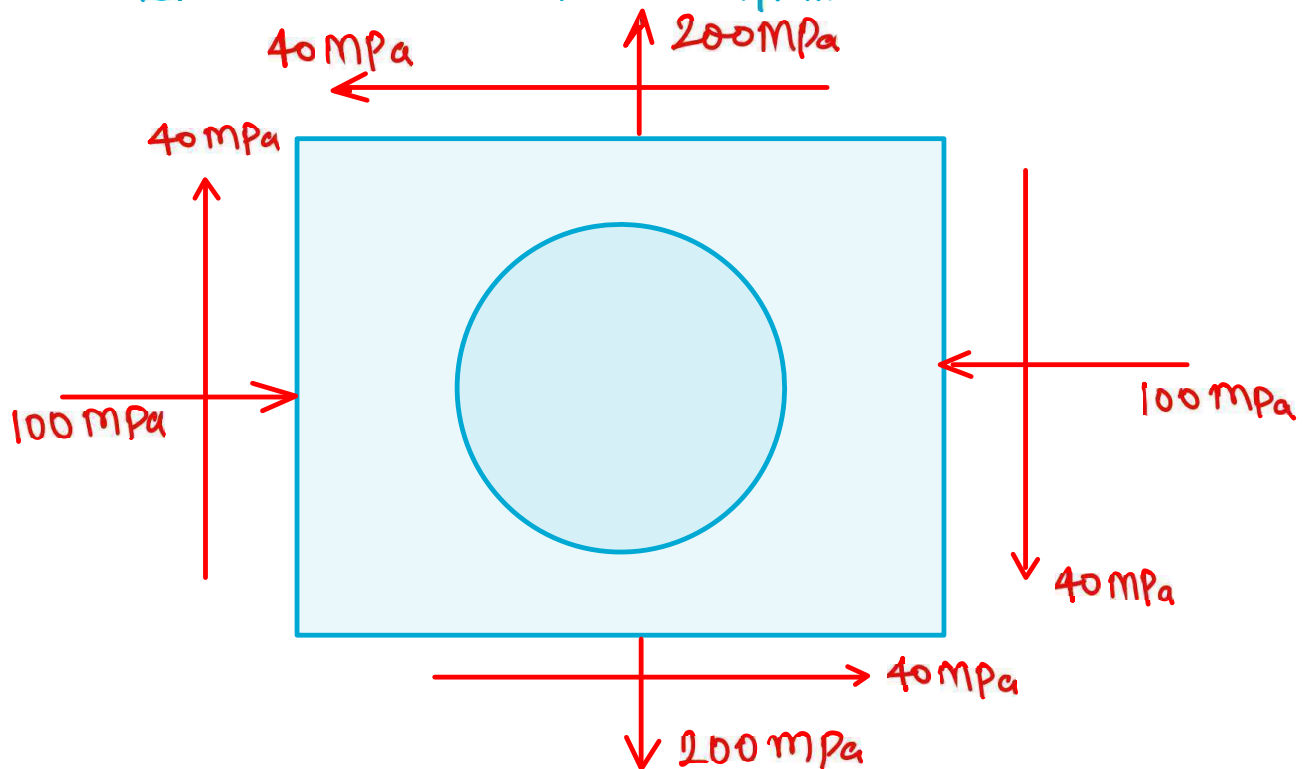


Conclusion :-

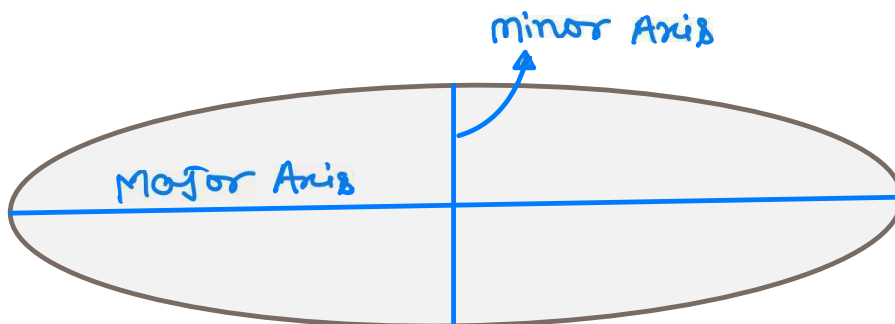
- In pure shear condition, magnitude of maximum normal stress / Major principal stress, minor principal stress and maximum shear stress is same but have different direction.
- On pure shear plane, Normal stress is zero, on plane of maximum shear Normal stress may assume any value $(\frac{\sigma_x + \sigma_y}{2})$, but in case of pure shear normal stress on plane of maximum shear is zero, hence it coincides with pure shear plane.



A circle of 100mm dia is drawn on Steel plate before application of load, loading is shown in the figure and due to load circle deforms into an ellipse. Determine the length of major & minor axis of ellipse. Take $\mu = 0.3$ & $E = 2.1 \times 10^5 \text{ N/mm}^2$



Solution



$$\sigma_x = -100 \text{ MPa}, \quad \sigma_y = 200 \text{ MPa}, \quad \tau_{xy} = -40 \text{ MPa}$$

$$\sigma_{P_1}/\sigma_{P_2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{-100 + 200}{2} \pm \sqrt{\left(\frac{-100 + 200}{2}\right)^2 + (-40)^2}$$

$$\sigma_{P_1} = 205.24 \text{ mPa}$$

$$\sigma_{P_2} = -105.24 \text{ mPa.}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times (-40)}{-100 + 200} = \frac{4}{15}$$

$$\theta_{P_1} = 7.46^\circ$$

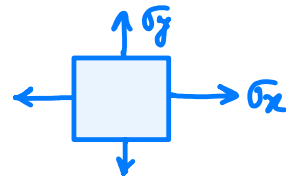
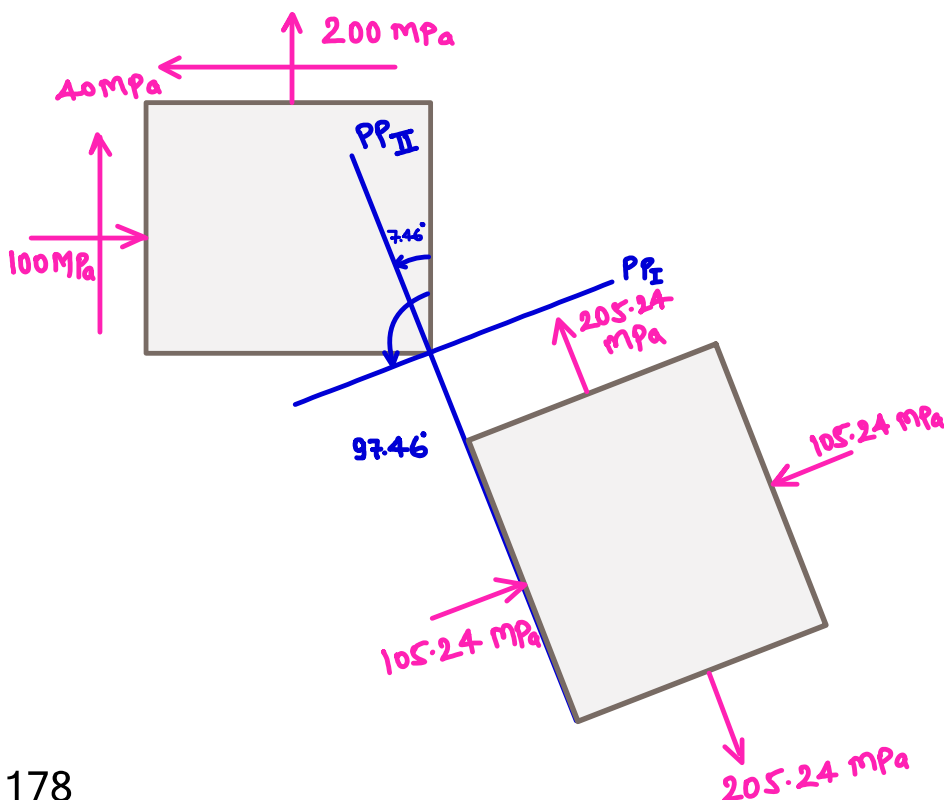
$$\theta_{P_2} = 90 + 7.46^\circ = 97.46^\circ$$

check:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta_{P_1} + \tau_{xy} \sin 2\theta_{P_1}$$

$$= \frac{-100 + 200}{2} + \frac{-100 - 200}{2} \cos 2 \times 7.46 + (-40) \sin(2 \times 7.46)$$

$$\sigma_{x'} = -105.24 \text{ mPa.}$$



$$\epsilon_x = \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E}$$

Along major Axis

$$\epsilon = \frac{\sigma_{P1}}{E} - \mu \frac{\sigma_{P2}}{E}$$

$$\frac{\Delta L}{L} = \frac{\sigma_{P1}}{E} - \mu \frac{\sigma_{P2}}{E}$$

$$\Delta L = 205.24 - 0.3 \left(-105.24 \right) \left[\frac{1}{2.1 \times 10^5} \right] \times 100$$

$$\Delta L = 0.1128 \text{ mm}$$

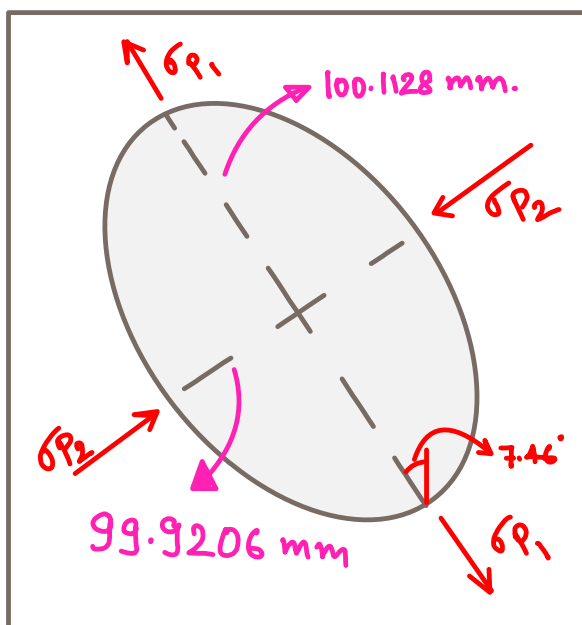
length of major Axis = $100 + 0.1128 = 100.1128 \text{ mm}$.

Along Minor Axis

$$\epsilon = \frac{\sigma_{P2}}{E} - \mu \frac{\sigma_{P1}}{E}$$

$$\Delta L = -0.0794 \text{ mm}$$

Length of minor Axis = $100 - 0.0794 = 99.9206 \text{ mm}$.



Analysis of Plane Stress

- Plane stress is used by the designer for designing of thin plane ex:- Skin of aircraft, ships, thin vessels (Boiler), thin spherical shell.
- In plane stress condition, stress on one plane is either zero or has very small magnitude, but strain on this plane is not zero.
- Hooke's law for plane stress condition. $(\sigma_x, \sigma_y, \tau_{xy})$

$$\begin{aligned}\epsilon_x &= \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} \\ \epsilon_y &= \frac{\sigma_y}{E} - \mu \frac{\sigma_x}{E} \\ \epsilon_z &= -\frac{\mu \sigma_x}{E} - \frac{\mu \sigma_y}{E} \\ \gamma_{xy} &= \frac{\tau_{xy}}{G}\end{aligned}$$

$$\begin{aligned}\sigma_x &= \frac{E}{1-\mu^2} (\epsilon_x + \mu \epsilon_y) \\ \sigma_y &= \frac{E}{1-\mu^2} (\epsilon_y + \mu \epsilon_x) \\ \tau_{xy} &= G \cdot \gamma_{xy}\end{aligned}$$

- for biaxial stress, $\tau_{xy}=0$, $\sigma_x \neq 0$, $\sigma_y \neq 0$
- for uniaxial stress, $\tau_{xy} = \sigma_y = 0$, $\sigma_x \neq 0$
- for pure shear, $\sigma_x = \sigma_y = 0$, $\tau_{xy} \neq 0$

$$\epsilon_v = \frac{\Delta V}{V} = \left[\frac{\sigma_x + \sigma_y + \sigma_z}{E} \right] [1-2\mu] = \left[\frac{\sigma_x + \sigma_y}{E} \right] [1-2\mu]$$

— Strain Energy per Unit Volume (u)

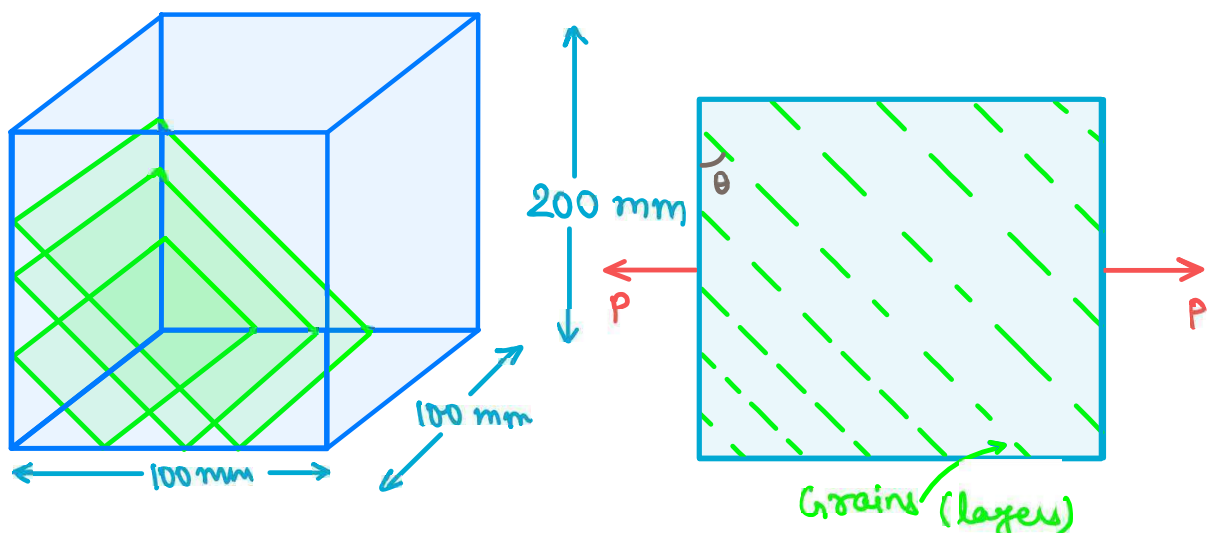
$$u = \frac{1}{2} \left[\underbrace{\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z}_{u_1} + \underbrace{\tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{zy} \gamma_{zy}}_{u_2} \right]$$

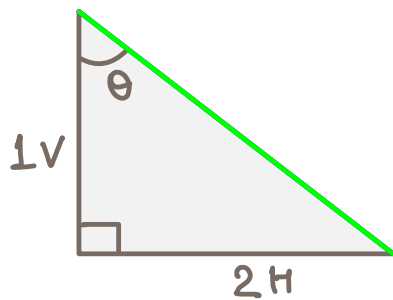
u_1 (SE due to Normal Stress) u_2 (SE due to Shear Stress)

$$u = \frac{1}{2} [\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy}]$$

$$u = \frac{1}{2} [\sigma_{P_1} \epsilon_1 + \sigma_{P_2} \epsilon_2 + 0] = \frac{1}{2E} [\sigma_{P_1}^2 + \sigma_{P_2}^2 - 2\mu \sigma_{P_1} \sigma_{P_2}]$$

Q. A rectangular block of dimension shown in figure is subjected to an axial force. The grains of the wood are oriented in a pattern as shown in figure and the layers of the grain is inclined at one vertical & 2 horizontal. If permissible tensile stress for the plane of the grain is 8 MPa and permissible shear stress parallel to the grain is 6 MPa then find the maximum axial force that can be applied without failure. (Design for magnitude only).



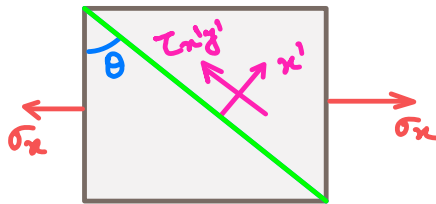
Solution

$$\tan \theta = \frac{2}{1}$$

$$\Rightarrow \theta = 63.43^\circ$$

$$\sigma_x = ?$$

$$\sigma_y = 0, \quad \tau_{xy} = 0$$



$$\sigma_{x'} = \left(\frac{\sigma_x + \sigma_y}{2} \right) + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$8 = \frac{\sigma_x}{2} + \frac{\sigma_x}{2} \cos(2 \times 63.43) + 0$$

$$\sigma_x = 40 \text{ MPa}$$

$$\begin{aligned} \text{permissible load} &= \sigma_x \times \text{Area} \\ &= 40 \times (200 \times 100) \\ &= 800000 \text{ N} \\ &= 800 \text{ kN} \end{aligned}$$

$$\tau_{x'y'} = \frac{-(\sigma_x - \sigma_y)}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$6 = -\frac{\sigma_x}{2} \sin(2 \times 63.43) + 0$$

$$\sigma_x = -15 \text{ MPa}$$

$$\begin{aligned} \text{permissible value} &= 15 \times 200 \times 100 \times 10^{-3} \\ &= 300 \text{ kN} \end{aligned}$$

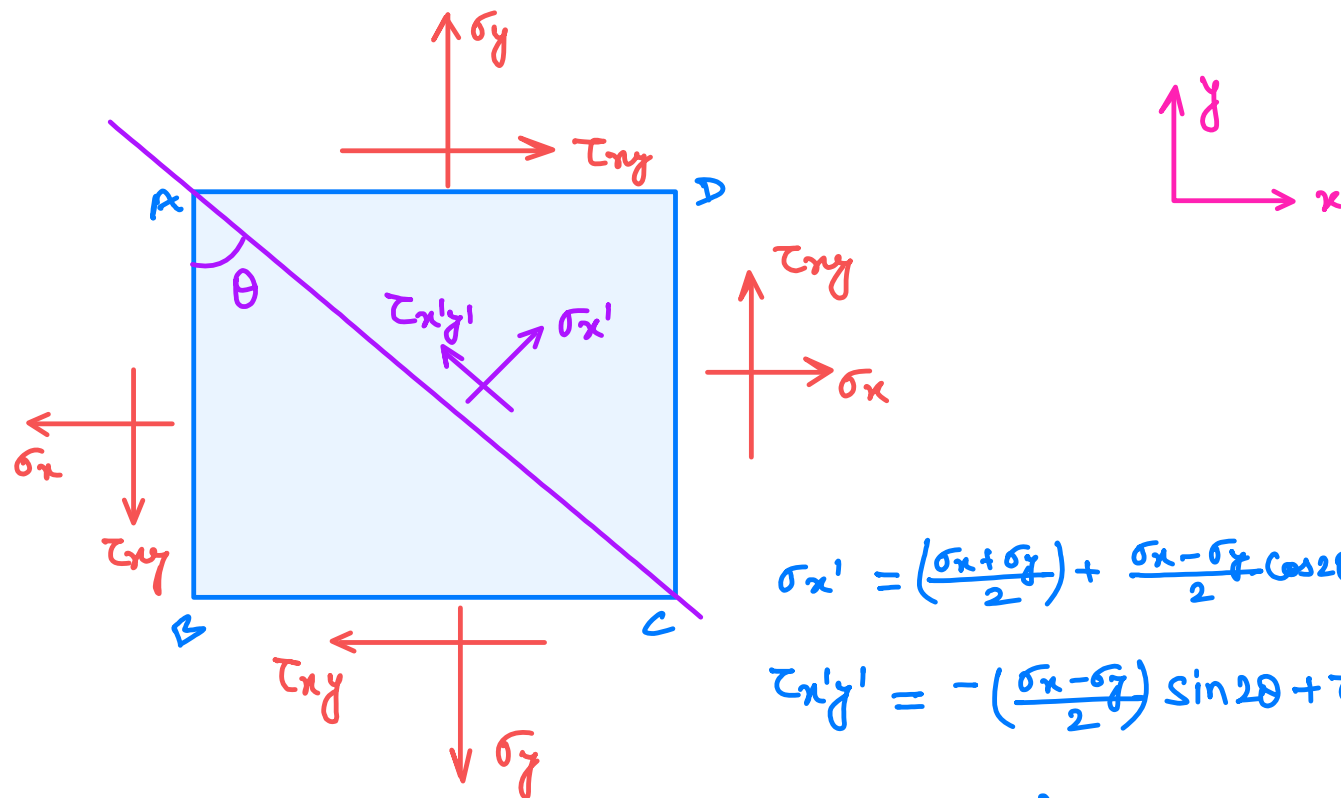
Hence, minimum of (800, 300).

$$P_{\max} = 300 \text{ kN.}$$

#

Mohr circle for Plane Stress.

- Mohr circle is used for transformation of stresses.
- State of stress at a point in the body can be expressed as point on the circumference of this circle
- Hence this Mohr circle can be used to find the stresses on any plane passing through a point using the known stresses on 2 perpendicular planes.



$$\sigma_{x'} = \left(\frac{\sigma_x + \sigma_y}{2} \right) + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = - \left(\frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta$$

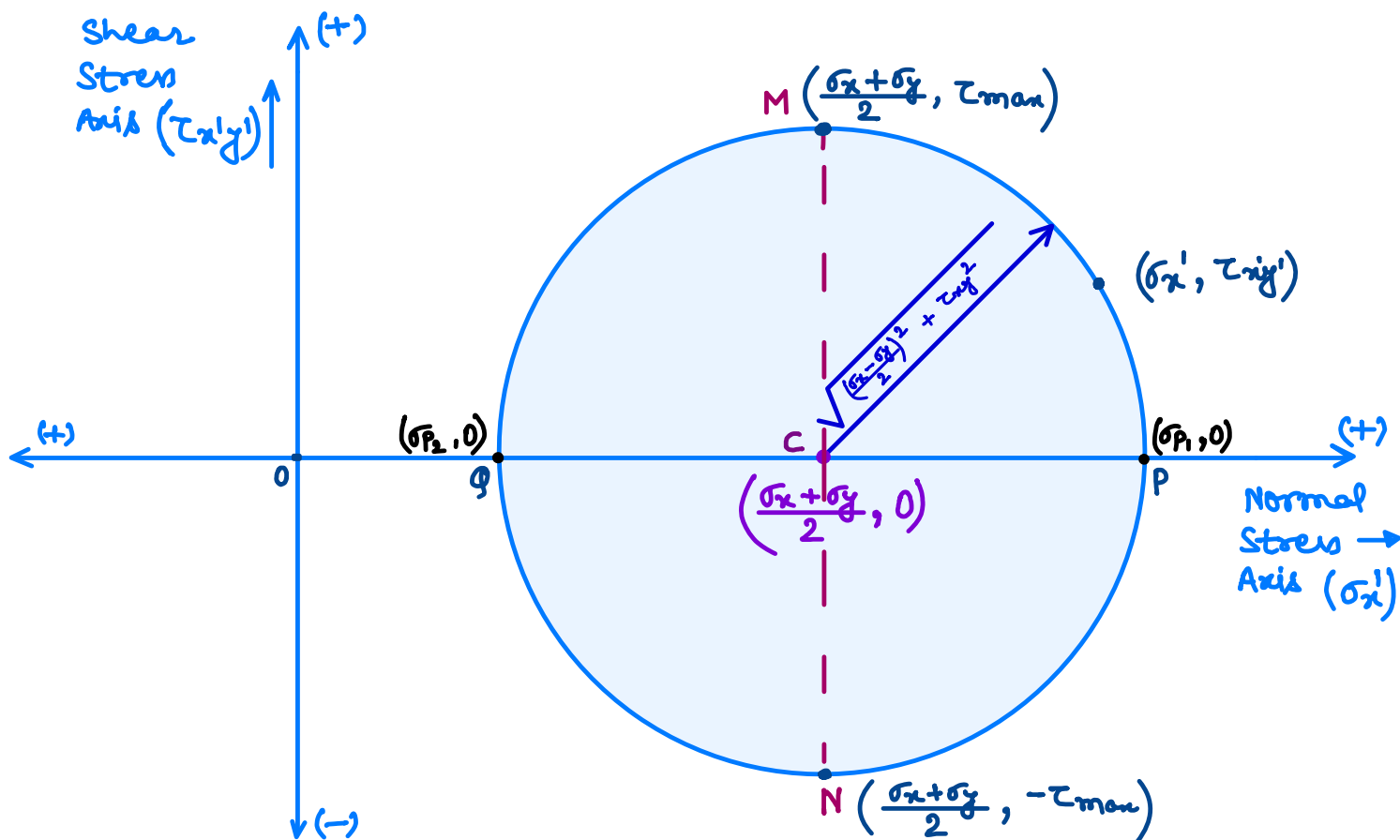
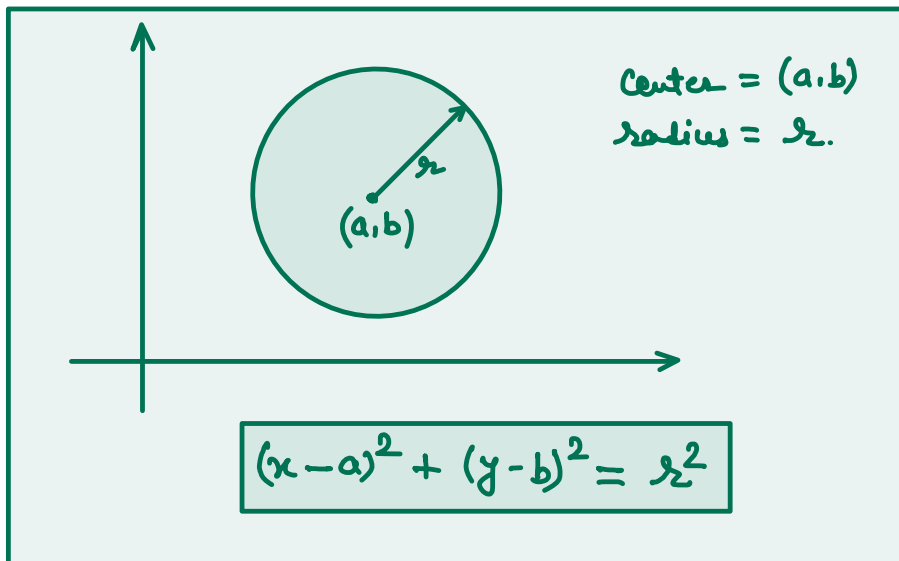
$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} \right)^2 + (\tau_{x'y'} - 0)^2 = \left(\frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \right)^2 + \left(- \left(\frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta \right)^2$$

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} \right)^2 + (\tau_{x'y'} - 0)^2 = \left(\frac{\sigma_x - \sigma_y}{2} \cos 2\theta \right)^2 + (\tau_{xy} \sin 2\theta)^2 + \left(\frac{\sigma_x - \sigma_y}{2} \sin 2\theta \right)^2 + (\tau_{xy} \cos 2\theta)^2$$

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} \right)^2 + (\tau_{x'y'} - 0)^2 = \left(\frac{\sigma_x - \sigma_y}{2} \right)^2 [\sin^2 2\theta + \cos^2 2\theta] + \tau_{xy}^2 [\sin^2 2\theta + \cos^2 2\theta]$$

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} \right)^2 + (\tau_{x'y'} - 0)^2 = \left(\sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} \right)^2$$

Note:-



$$\text{Diameter of circle} = \sigma_{P1} - \sigma_{P2}$$

$$\text{Radius of circle} = \frac{\sigma_{P1} - \sigma_{P2}}{2}$$

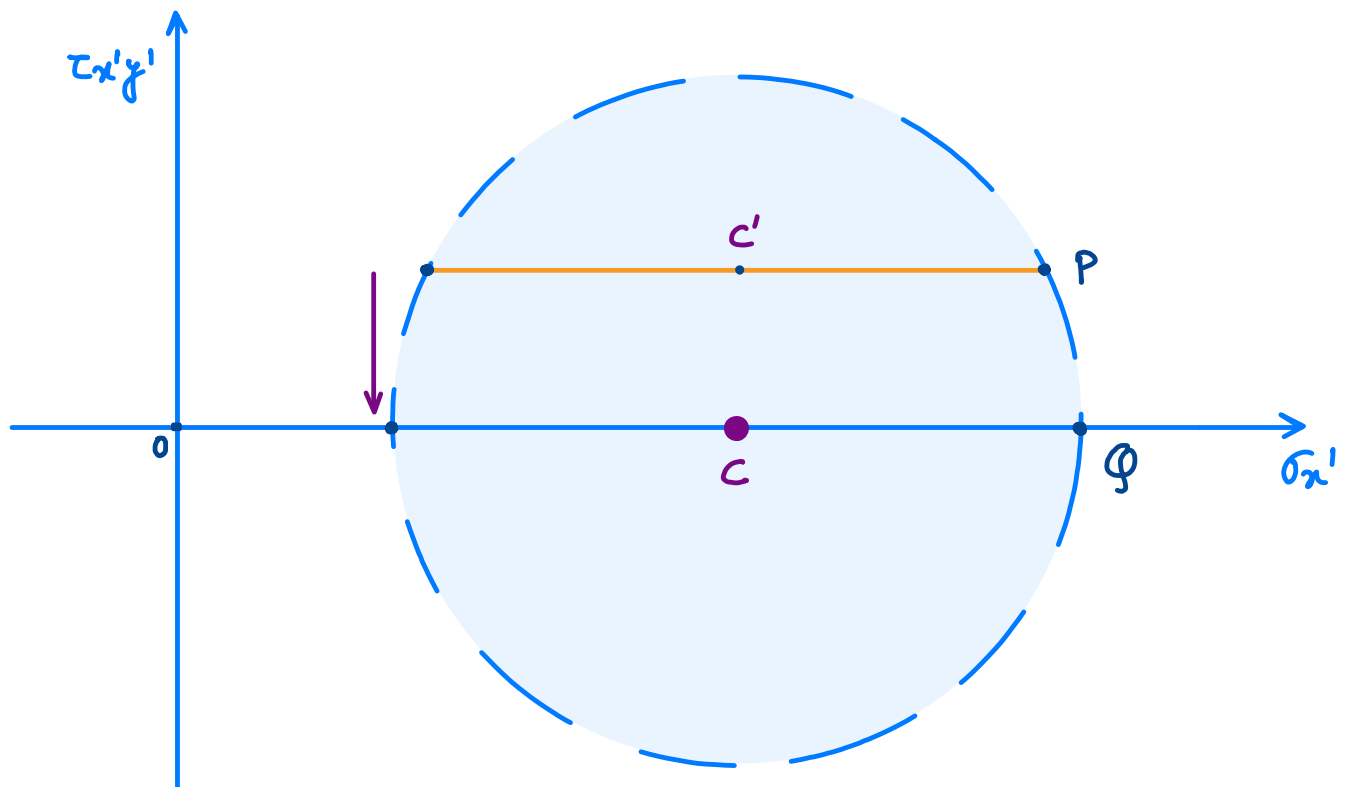
$$\text{Radius of the circle} = \frac{\sigma_{P1} - \sigma_{P2}}{2} = \text{maximum shear stress.}$$

$$OC = OQ + QC = \sigma_{p_2} + \frac{\sigma_{p_1} - \sigma_{p_2}}{2}$$

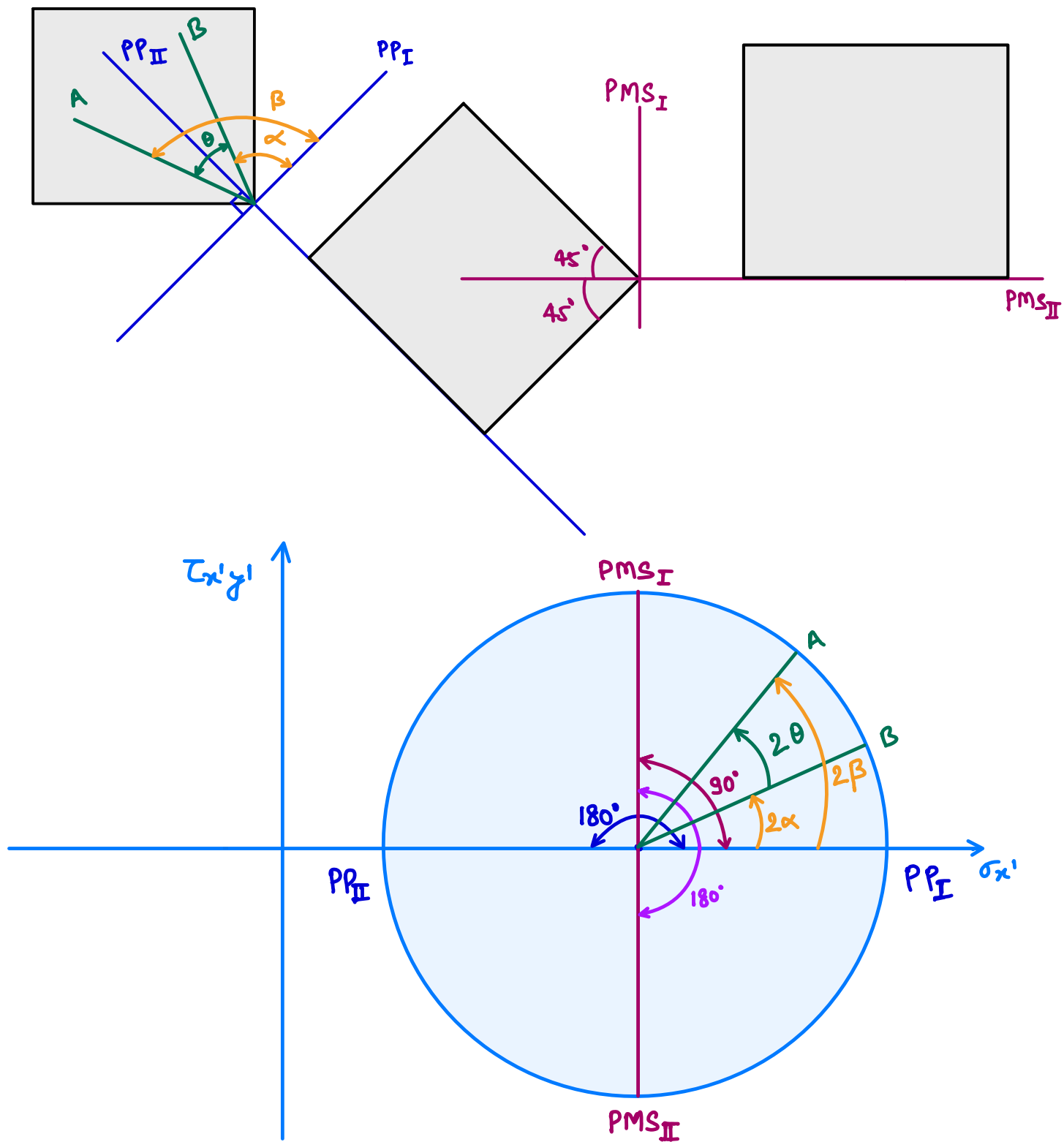
$$OC = \frac{\sigma_{p_1} + \sigma_{p_2}}{2} \quad (\text{This Justifies Stress invariant Concept}).$$

★ properties of Mohr circle:-

i) centre of Mohr circle always lies over normal stress axis.



ii) If two planes make an angle of " θ " in stress element then angle between these two planes when represented on Mohr circle (as a point on circumference) would be " 2θ ".

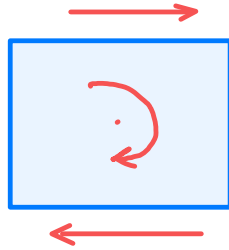


iii.) Sign convention to be used for Mohr circle:

a.) Tension (+ve)

b.) Compression (-ve)

c.)

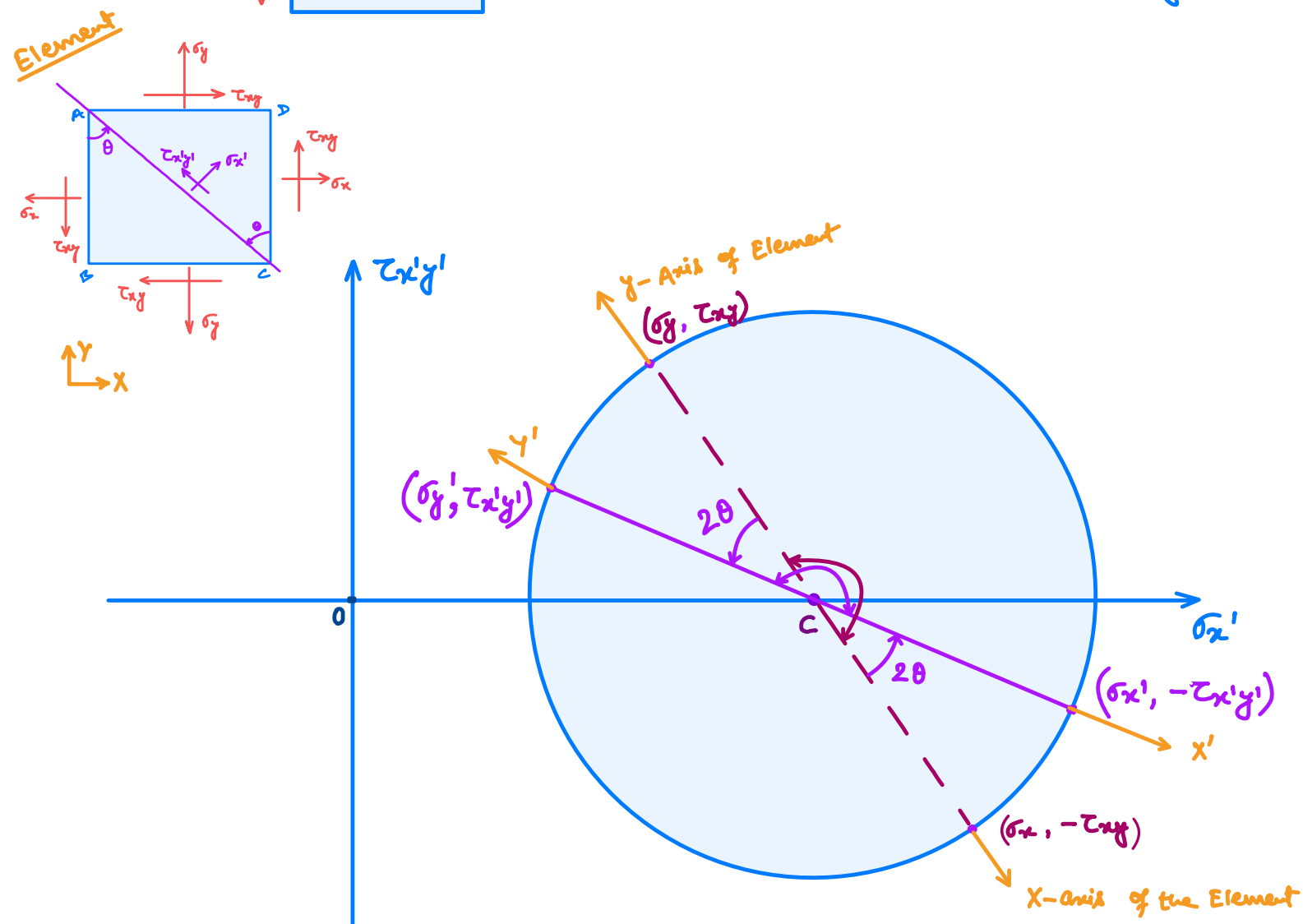


If Shear produces clockwise moment about center, it is positive.

d.)

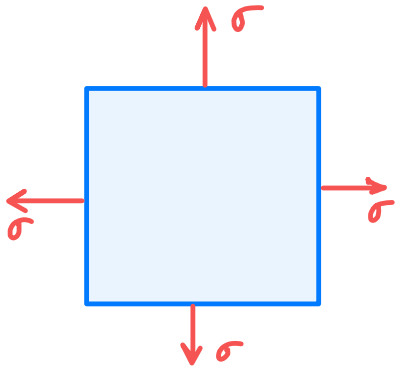


If Shear produces anti-clockwise moment about center it is Negative.



Special Cases of Mohr's Circle

a) point Mohr circle



$$\sigma_x = \sigma, \sigma_y = \sigma, \tau_{xy} = 0$$

$$c = \left(\frac{\sigma_x + \sigma_y}{2}, 0 \right) = (\sigma, 0)$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$= \sqrt{\left(\frac{\sigma - \sigma}{2} \right)^2 + 0^2}$$

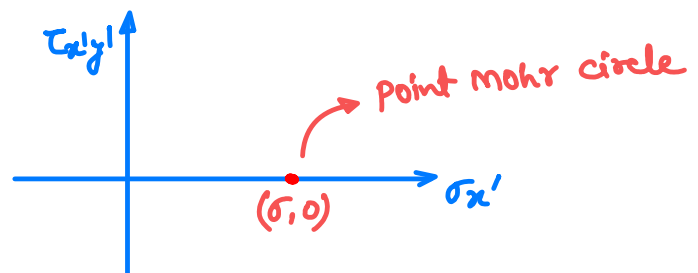
$$R = 0$$

$$\tau_{max} = R = 0 \Rightarrow \tau_{max} = \frac{\sigma_{P1} - \sigma_{P2}}{2} = 0$$

$$\sigma_{P1} = \sigma_{P2}$$

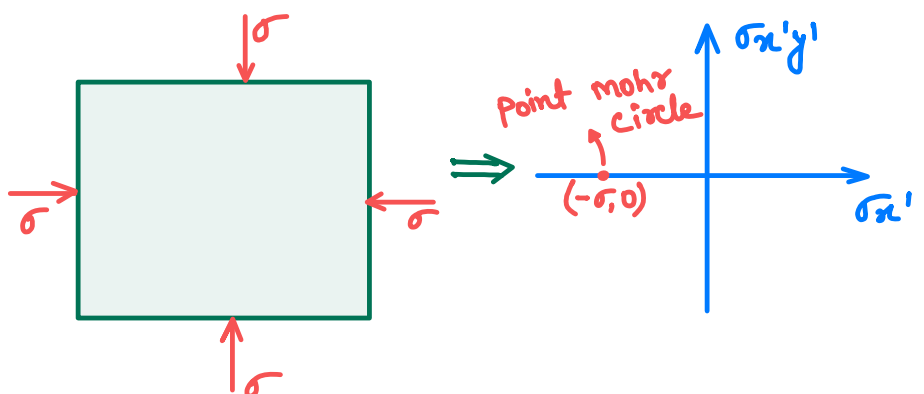
$\Rightarrow$ Since it is a point on Normal Stress axis, $\tau_{x'y'} = 0$
hence in this case there are infinite principal plane.

eg:- This is found in case of hydrostatic tension.

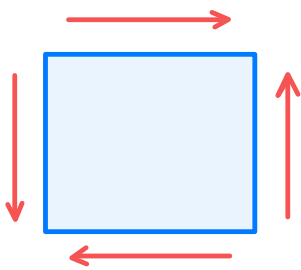


Note:-

if instead of tensile, compressive stresses are acting on element Mohr circle would have centre on Negative Normal Stress axis.



b> Mohr circle for pure shear condition :-



$$\sigma_x = 0, \quad \sigma_y = 0, \quad \tau_{xy} = \tau$$

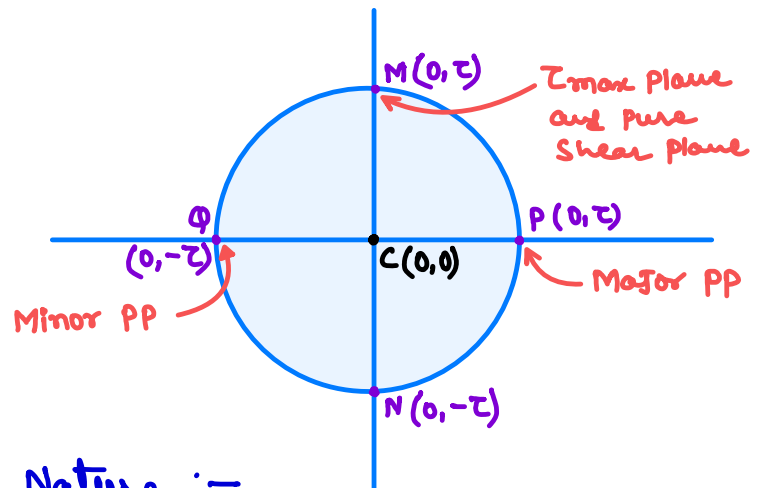
$$C = \left(\frac{\sigma_x + \sigma_y}{2}, 0 \right) = (0, 0)$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{0-0}{2} \right)^2 + \tau^2}$$

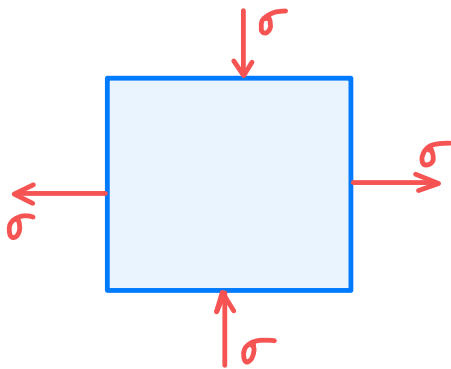
$$R = \tau$$

$$\tau_{\max} = R = \tau$$

$$\sigma_{p_1}/\sigma_{p_2} = C \pm R = \pm \tau$$



(c) Biaxial stress of opposite nature :-



$$\sigma_x = \sigma, \quad \sigma_y = -\sigma, \quad \tau_{xy} = 0$$

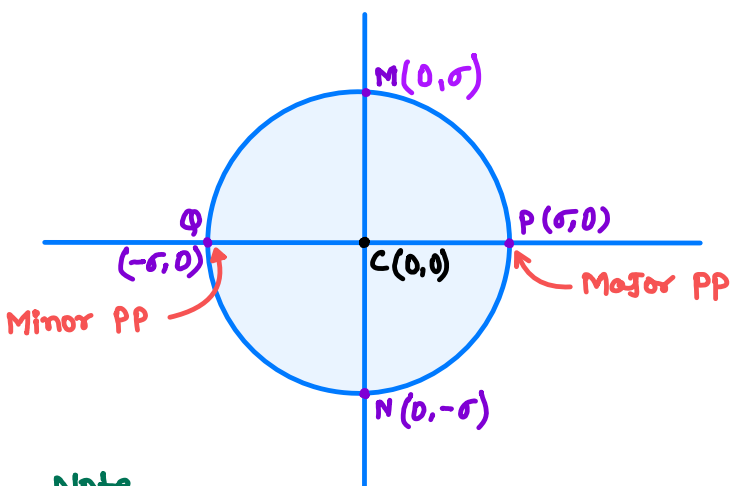
$$C = \left(\frac{\sigma_x + \sigma_y}{2}, 0 \right) = (0, 0)$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{\sigma - (-\sigma)}{2} \right)^2 + 0^2} = \sigma$$

$$R = \sigma$$

$$\tau_{\max} = R = \sigma$$

$$\sigma_{p_1}/\sigma_{p_2} = C \pm R = \pm \sigma$$

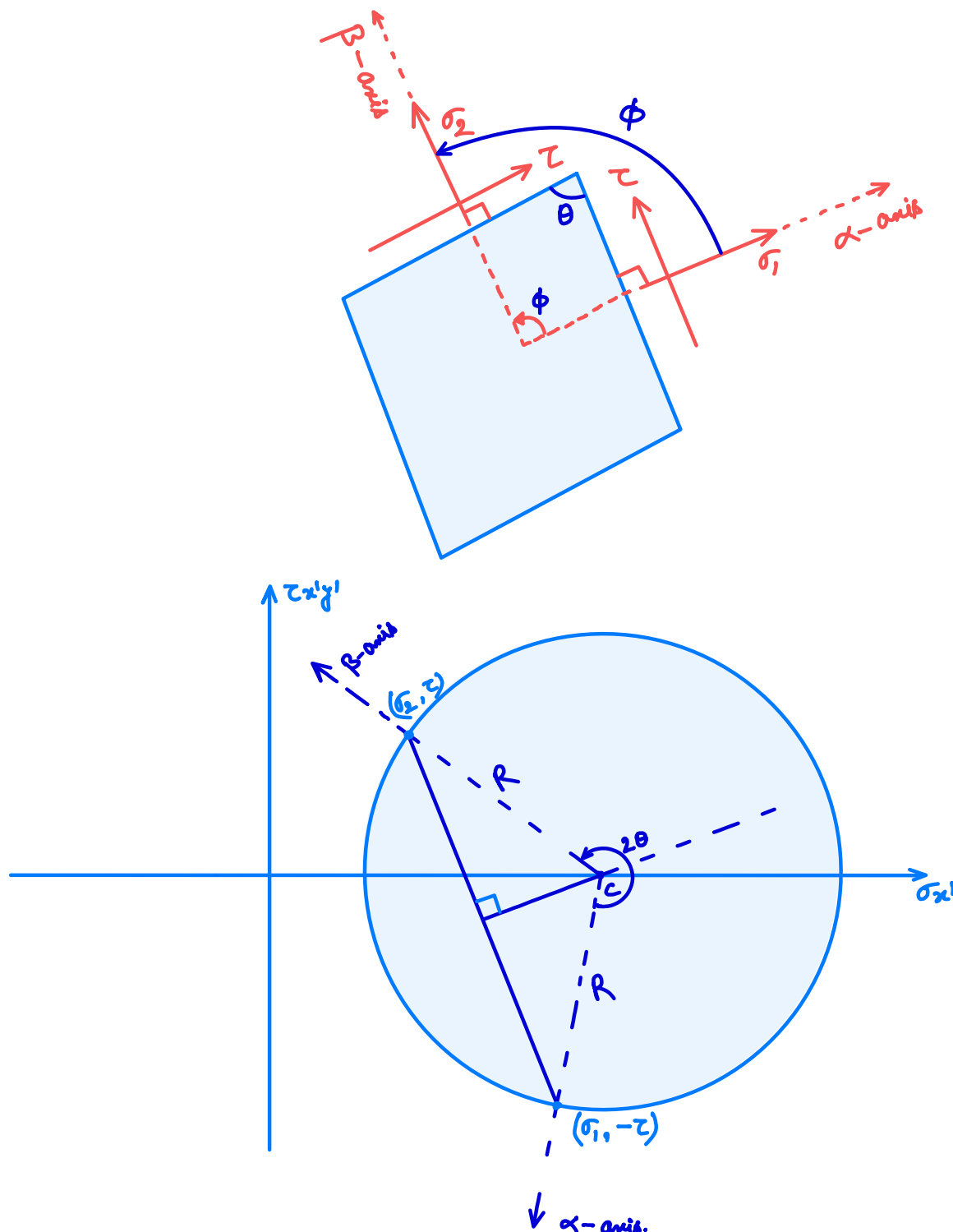


Note

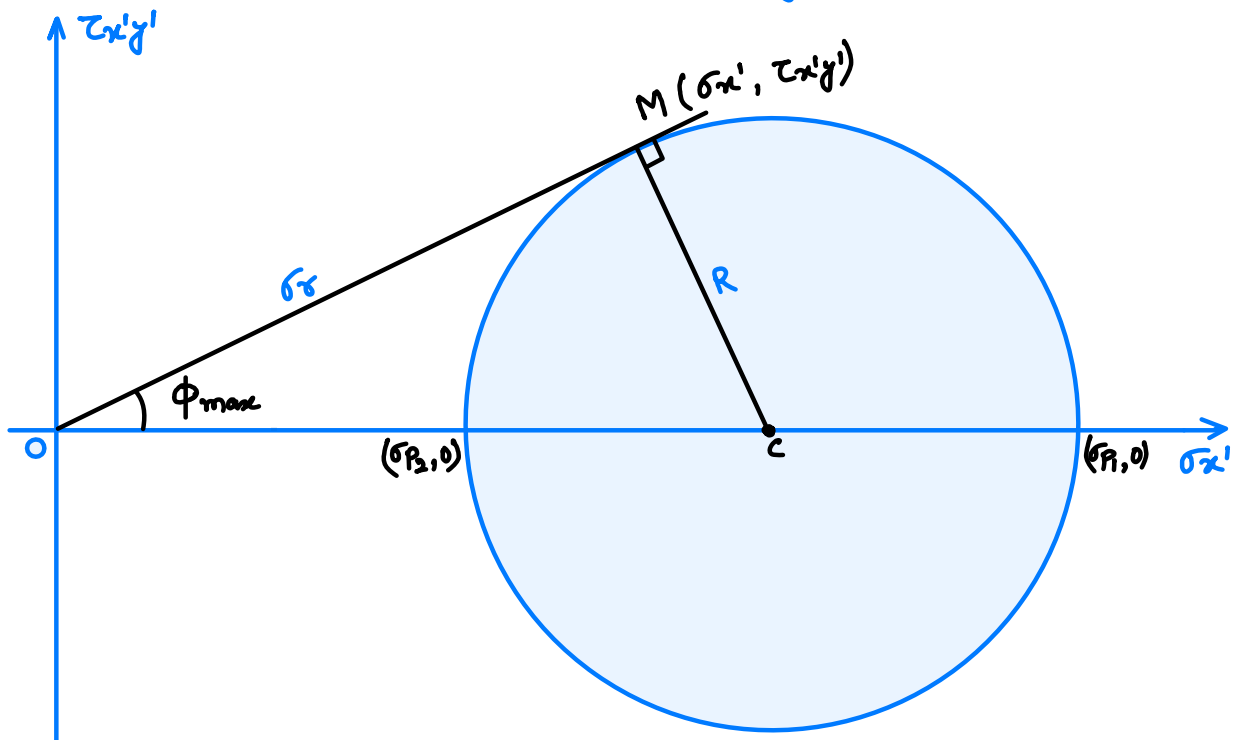
This case is equivalent to that of pure shear case.

Property

iv. If the two plane, the stresses of which have been plotted are not perpendicular to each other, then to get the centre of the Mohr circle, a perpendicular bisector is drawn on the line joining the two points and the intersection of this perpendicular bisector with the Normal axis is the centre of the circle.



★ Angle of obliquity (ϕ) would be maximum when resultant stress on a plane would be tangential to Mohr circle.



$$\sin \phi_{\max} = \frac{CM}{OC} = \frac{R}{c} = \frac{\frac{\sigma_{P1} - \sigma_{P2}}{2}}{\frac{\sigma_{P1} + \sigma_{P2}}{2}} = \frac{\sigma_{P1} - \sigma_{P2}}{\sigma_{P1} + \sigma_{P2}}$$

$$\frac{1 + \sin \phi_{\max}}{1 - \sin \phi_{\max}} = \frac{\sigma_{P1}}{\sigma_{P2}}$$

Stress & Strain tensor for Plane Stress Condition.

Stress tensor

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

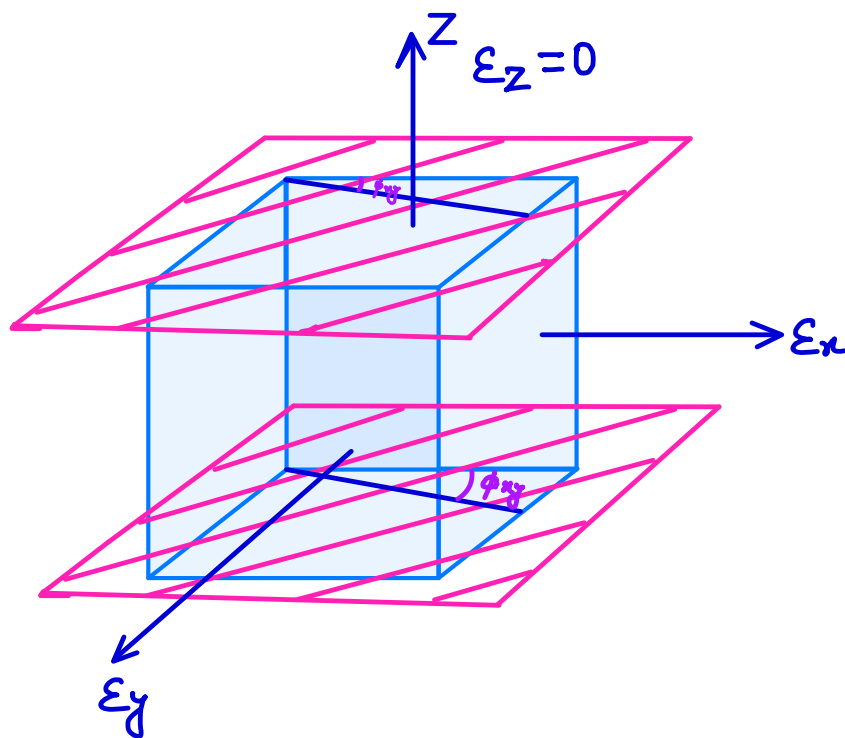
Strain tensor

$$\begin{bmatrix} \epsilon_x & \frac{\phi_{xy}}{2} & 0 \\ \frac{\phi_{yx}}{2} & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}$$

Shear strain
2

Plain Strain State :-

- If an element of the material is subjected to only strain in 2 direction (where the strain/deformation in third direction is zero) it is said to be in the state of Plain Strain.
- In this case where strain in third direction is zero it does not mean stress in this direction is zero infact stress will be required in this direction to make the deformation zero.



$$\epsilon_z = \frac{\sigma_z}{E} - \mu \frac{\sigma_x}{E} - \mu \frac{\sigma_y}{E} = 0$$

$$\sigma_z = \mu (\sigma_x + \sigma_y) \neq 0$$

Note:-

- if $\sigma_z = 0$, it does not mean $\epsilon_z = 0$
[Except in case when $\mu = 0$, i.e. ideal material
(or) when $\sigma_x = -\sigma_y$]
- if $\epsilon_z = 0$, it does not means, $\sigma_z = 0$
[except in case when $\mu = 0$, i.e. ideal material (or) when $\sigma_x = -\sigma_y$]

- Hence Plane Stress & Plain Strain Component are not Same.

Stress Tensor

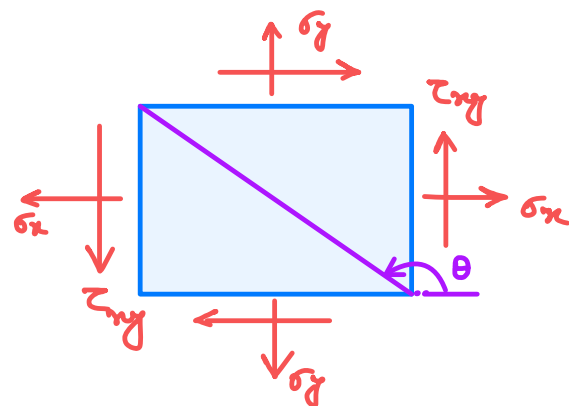
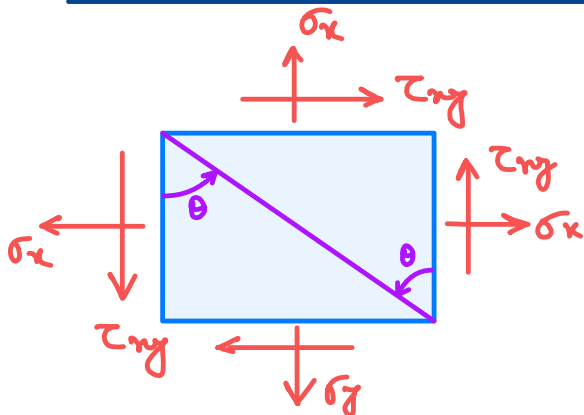
$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

Strain Tensor

$$\begin{bmatrix} \epsilon_x & \frac{\phi_{xy}}{2} & 0 \\ \frac{\phi_{yx}}{2} & \epsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Plain Strain Condition is used by designer for designing of thick plates eg:- Gun Barrel, Cannon, The pressure vessel, Dams, Tunnels.

Transformation of plane strain



- θ is measured from vertical plane. θ is measured from horizontal plane.
- θ is taken as positive for ACW rotation & (-ve) for CW.
- Plane Strain transformation can be obtained by replacing normal stress by corresponding normal strain and shear stress by $\frac{1}{2}$ the corresponding shear strain in stress transformation equation.

- Normal tensile Strain is taken positive
- Normal compressive Strain is taken Negative.
- ϕ_{xy} is taken Positive if angle between positive faces reduces otherwise it is Negative.

Stress transformation

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\sigma_x + \sigma_y = \sigma_x' + \sigma_y'$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$\sigma_{p_1}/\sigma_{p_2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tau_{max} = \frac{\sigma_{p_1} - \sigma_{p_2}}{2}$$

or

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_s = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

Strain transformation.

$$\epsilon_x' = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\phi_{xy}}{2} \sin 2\theta$$

$$\epsilon_y' = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta - \frac{\phi_{xy}}{2} \sin 2\theta$$

$$\epsilon_x + \epsilon_y = \epsilon_x' + \epsilon_y'$$

$$\frac{\phi_{x'y'}}{2} = -\frac{(\epsilon_x - \epsilon_y)}{2} \sin 2\theta + \frac{\phi_{xy}}{2} \cos 2\theta$$

$$\epsilon_{p_1}/\epsilon_{p_2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\phi_{xy}}{2}\right)^2}$$

$$\tan 2\theta_p = \frac{\phi_{xy}}{\epsilon_x - \epsilon_y}$$

$$\frac{\phi_{max}}{2} = \frac{\epsilon_{p_1} - \epsilon_{p_2}}{2}$$

or

$$\frac{\phi_{max}}{2} = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\phi_{xy}}{2}\right)^2}$$

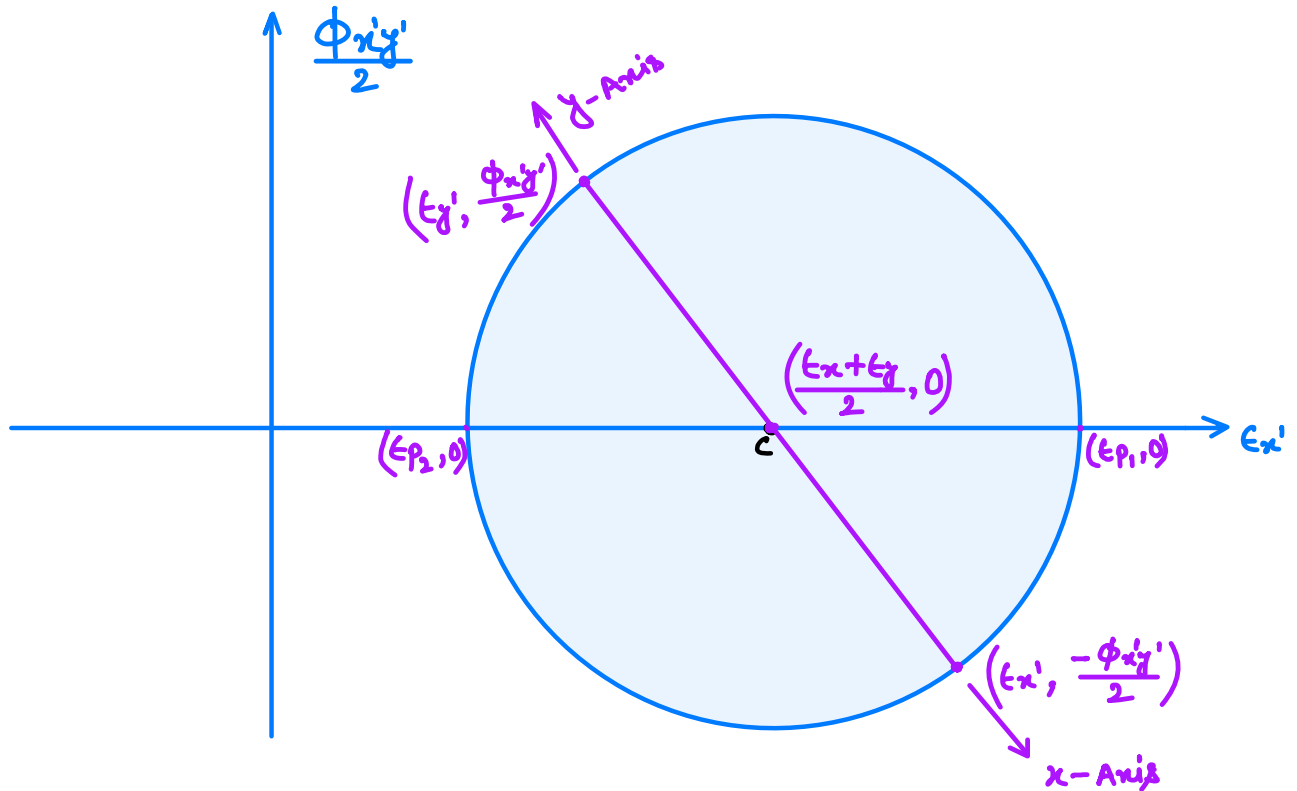
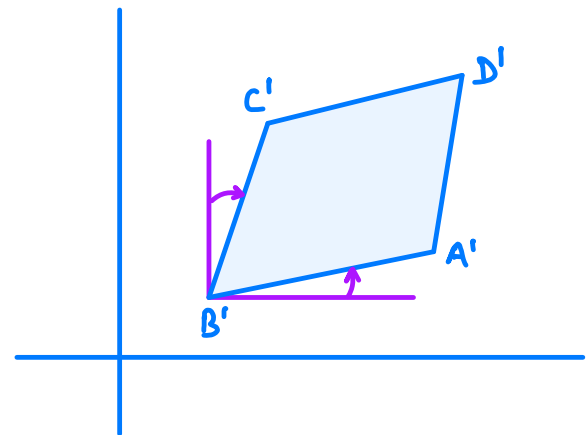
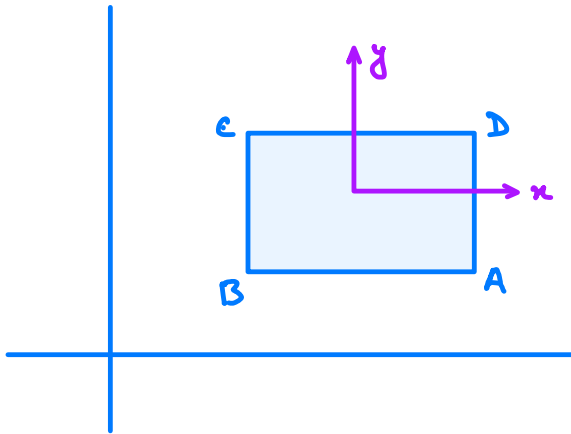
$$\tan 2\theta_s = \frac{-(\epsilon_x - \epsilon_y)}{\phi_{xy}}$$

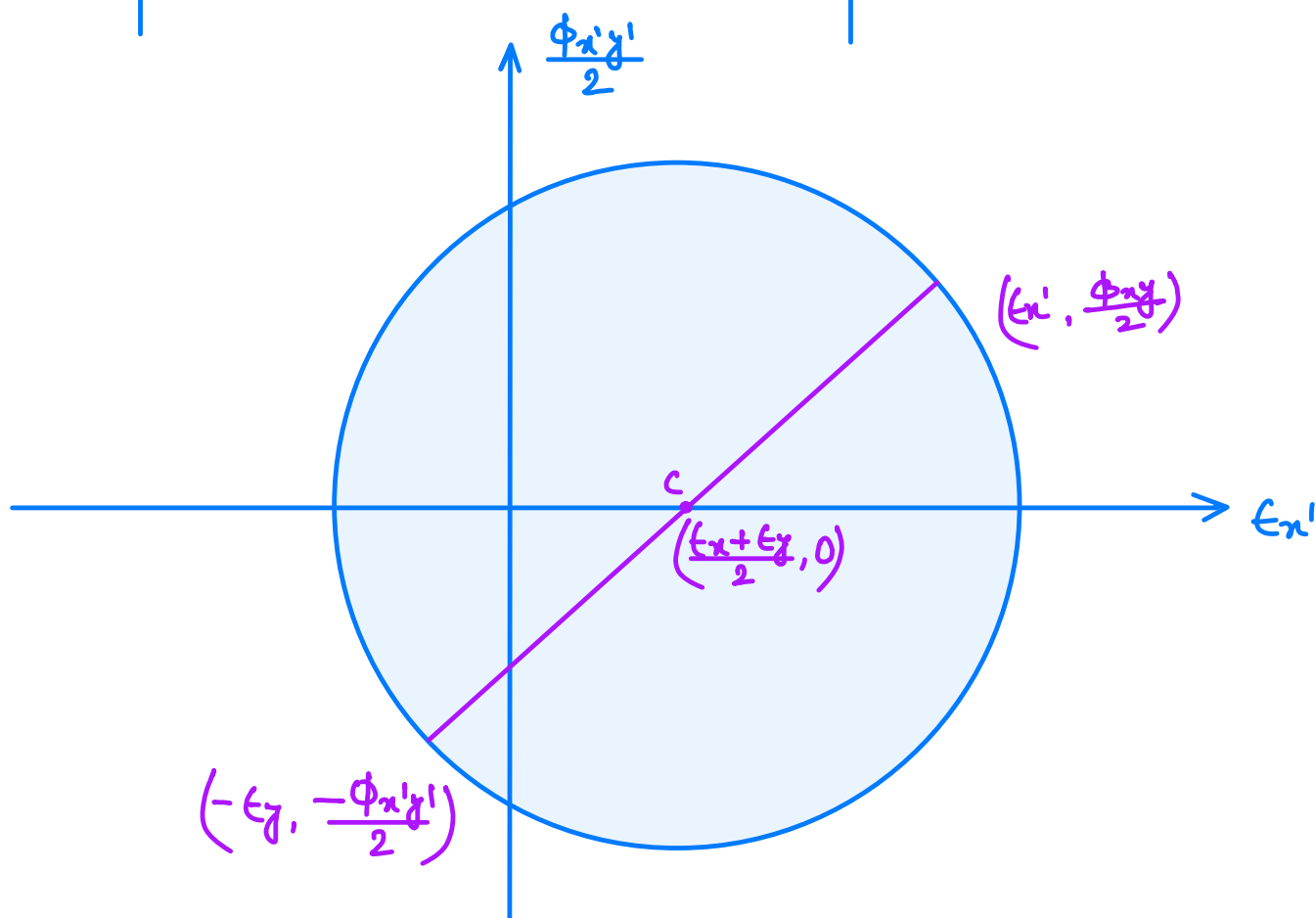
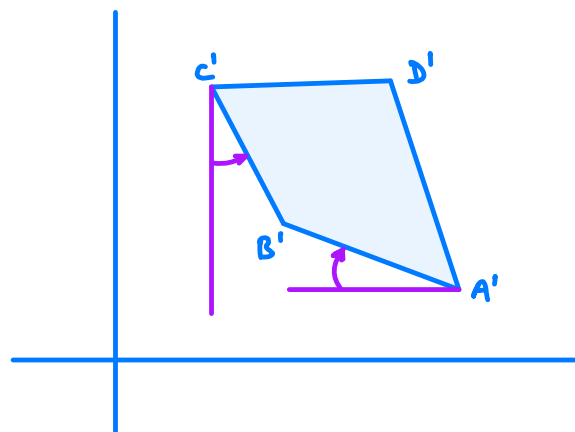
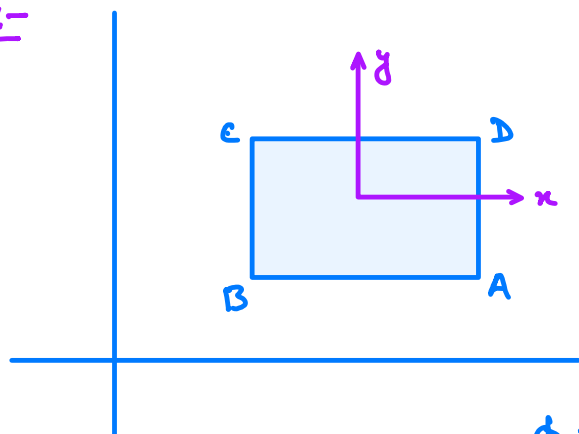
Mohr circle for plane strain

- tensile Normal strain is taken as positive
- compressive Normal strain is taken as Negative.

- if shear deformation causes given side to rotate in clockwise direction, corresponding point on Mohr circle for plane strain is taken positive.
- if shear deformation causes given side to rotate in anticlockwise direction, corresponding point on Mohr circle is taken Negative.

Ex:-



Eg:-

$$C = \left(\frac{\sigma_x + \sigma_y}{2}, 0 \right)$$

$$R = \frac{\sigma_{P1} - \sigma_{P2}}{2}$$

$$\textcircled{\sigma} R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2}$$

$$C = \left(\frac{\epsilon_x + \epsilon_y}{2}, 0 \right)$$

$$R = \frac{\epsilon_{P1} - \epsilon_{P2}}{2}$$

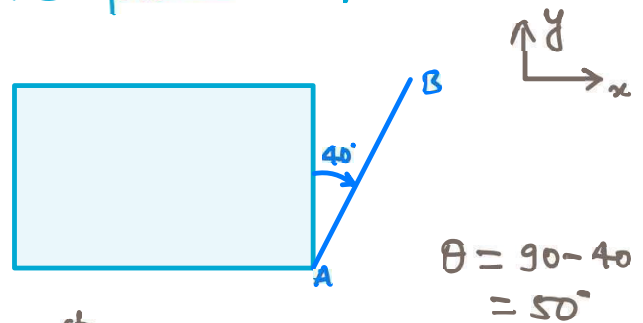
$$\textcircled{\sigma} R = \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2} \right)^2 + \left(\frac{\phi_{xy}}{2} \right)^2}$$

Q. A strain arrangement shown in the figure. Find out the normal and Shear Strain in the plane AB.

$$\epsilon_x = 800 \times 10^{-6}$$

$$\epsilon_y = -1200 \times 10^{-6}$$

$$\phi_{xy} = -600 \times 10^{-6}$$



Soln

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\phi_{xy}}{2} \sin 2\theta$$

$$= \frac{800 \times 10^{-6} - 1200 \times 10^{-6}}{2} + \frac{(800 + 1200) \times 10^{-6}}{2} \cos(2 \times 50^\circ) + \frac{-600 \times 10^{-6}}{2} \sin(2 \times 50^\circ)$$

$$= -6.69 \times 10^{-4} \quad (\text{Contraction})$$

$$\epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta - \frac{\phi_{xy}}{2} \sin 2\theta$$

$$= \frac{(800 - 1200) \times 10^{-6}}{2} - \frac{2000 \times 10^{-6}}{2} \cos(100^\circ) - \frac{-600 \times 10^{-6}}{2} \times \sin 100^\circ$$

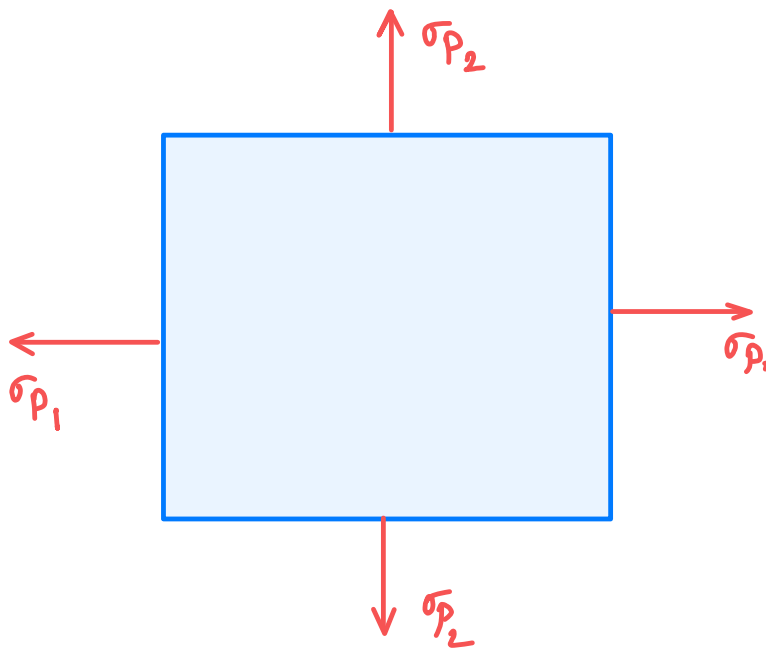
$$= 2.69 \times 10^{-4} \quad (\text{Extension})$$

$$\frac{\phi_{x'y'}}{2} = -\frac{(\epsilon_x - \epsilon_y)}{2} \sin 2\theta + \frac{\phi_{xy}}{2} \cos 2\theta$$

$$\phi_{x'y'} = \left(-\frac{2000 \times 10^{-6}}{2} \sin 100^\circ + \frac{-600 \times 10^{-6}}{2} \sin 100^\circ \right) \times 2$$

$$= -18.65 \times 10^{-4} \quad - \text{sign indicate Angle will be } > 90^\circ.$$

Analysis of principal strain



$$\epsilon_{P1}/\epsilon_{P2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\phi_{xy}}{2}\right)^2}$$

$$\epsilon_{P1} = \frac{\sigma_{P1}}{E} - \mu \frac{\sigma_{P2}}{E}$$

$$\epsilon_{P2} = \frac{\sigma_{P2}}{E} - \mu \frac{\sigma_{P1}}{E}$$

$$\sigma_{P1} = \frac{E}{1-\mu^2} [\epsilon_{P1} + \mu \epsilon_{P2}]$$

$$\sigma_{P2} = \frac{E}{1-\mu^2} [\epsilon_{P2} + \mu \epsilon_{P1}]$$

$$\epsilon_{P1} > \epsilon_{P2} \Rightarrow \sigma_{P1} > \sigma_{P2}$$

Note:-

In case of plane stress state

$$\sigma_{P3} = 0 \text{ , but } \epsilon_{P3} \neq 0$$

$$\epsilon_{P3} = -\frac{\mu}{E} (\sigma_{P1} + \sigma_{P2})$$

$$\begin{aligned} \epsilon_{P3} &= -\frac{\mu}{E} \left[\frac{E}{1-\mu^2} (\epsilon_{P1} + \mu \epsilon_{P2}) + \frac{E}{1-\mu^2} (\epsilon_{P2} + \mu \epsilon_{P1}) \right] \\ &= -\mu \left[\frac{(\epsilon_{P1} + \epsilon_{P2})}{1-\mu^2} (1+\mu) \right] \end{aligned}$$

$$\epsilon_{P3} = \frac{-\mu}{1-\mu} [\epsilon_{P1} + \epsilon_{P2}]$$

$$\frac{\phi_{max}}{2} = \frac{\epsilon_{Pmax} - \epsilon_{Pmin}}{2} = \frac{\epsilon_{P1} - \epsilon_{P3}}{2}$$

Q. The Strain arrangement on a body at a point under the state of plane strain condition is $\epsilon_x = 1200 \times 10^{-6}$, $\epsilon_y = -900 \times 10^{-6}$, $\phi_{xy} = 600 \times 10^{-6}$. Compute principal strain & maximum shear strain?

Soln

$$\epsilon_{P_1}/\epsilon_{P_2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_y - \epsilon_x}{2}\right)^2 + \left(\frac{\phi_{xy}}{2}\right)^2}$$

$$= \frac{(1200 - 900) \times 10^{-6}}{2} \pm \sqrt{\left(\frac{2100 \times 10^{-6}}{2}\right)^2 + \left(\frac{600 \times 10^{-6}}{2}\right)^2}$$

$$\epsilon_{P_1} = 1.242 \times 10^{-3}$$

$$\epsilon_{P_2} = -9.42 \times 10^{-4}$$

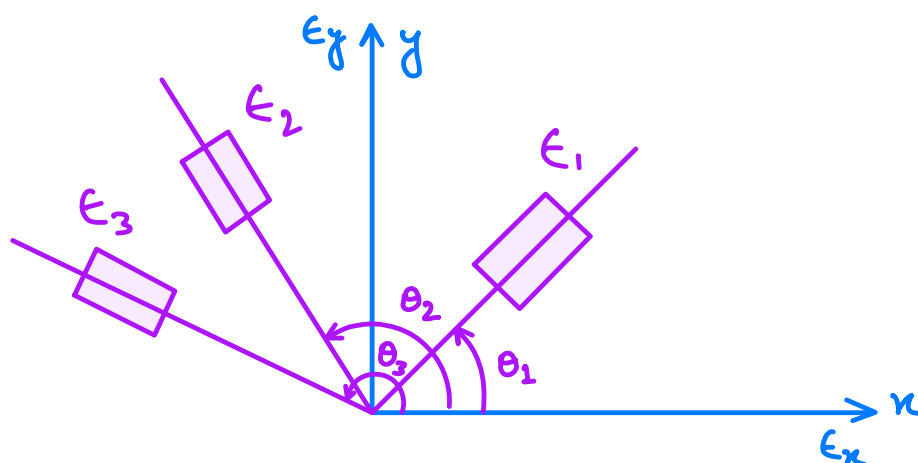
$$\frac{\phi_{\max}}{2} = \frac{\epsilon_{P_1} - \epsilon_{P_2}}{2} = \frac{1.242 \times 10^{-3} + 9.42 \times 10^{-4}}{2}$$

$$\phi_{\max} = 2.184 \times 10^{-3}$$

Strain Rosette

- For the designing of any component (beam, column, etc) the stresses developed in the body must be less than the strength of the material of the body.
- Hence we should know the principal stresses developed in the body but there is no method available to find the principal stresses practically.
- Hence the strain developed in the body is measured with the help of Extensometer which is connected with wheatstone Bridge which indicates the resistance against flow of current which is further related to strain/extension.

- It is capable of measuring the strain of the order of 10^{-6} .
- So, with the help of Extensometer Normal and Shear Strain are measured, which is used to compute Normal & Shear Stress and finally principal stress.
- But it involves error in measurement of Shear Strain hence principal stress calculated is not accurate by this method.
- Hence in order to find the Exact value of principal stresses **Strain Rosette** is used.
- A Strain Rosette is a group of 3 gauges arranged in a particular pattern such that it can measure normal strain in three different direction on the body subjected to Plane Stress.
- With the help of these three Normal Strains ϵ_y , ϵ_x & ϕ_{xy} can be computed to give principal stresses.



$$\epsilon_1 = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta_1 + \frac{\phi_{xy}}{2} \sin 2\theta_1$$

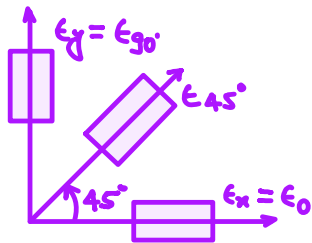
$$\epsilon_2 = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta_2 + \frac{\phi_{xy}}{2} \sin 2\theta_2$$

$$\epsilon_3 = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta_3 + \frac{\phi_{xy}}{2} \sin 2\theta_3$$

By solving above equations ϵ_x , ϵ_y , ϕ_{xy} can be computed.

— Generally following types of Strain Rosettes are used :-

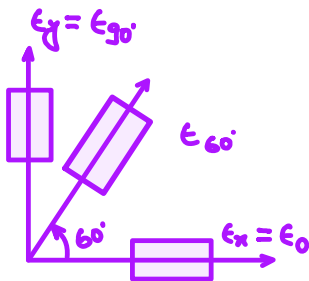
a> 45° Strain Rosette.



$$\epsilon_{45^\circ} = \frac{\epsilon_0 + \epsilon_{90}}{2} + \frac{\epsilon_0 - \epsilon_{90}}{2} \cos 90^\circ + \frac{\phi_{xy}}{2} \sin 90^\circ$$

$$\phi_{xy} = ?$$

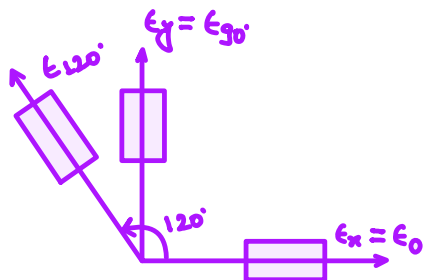
b> 60° Strain Rosette



$$\epsilon_{60^\circ} = \frac{\epsilon_0 + \epsilon_{90}}{2} + \frac{\epsilon_0 - \epsilon_{90}}{2} \cos 120^\circ + \frac{\phi_{xy}}{2} \sin 120^\circ$$

$$\phi_{xy} = ?$$

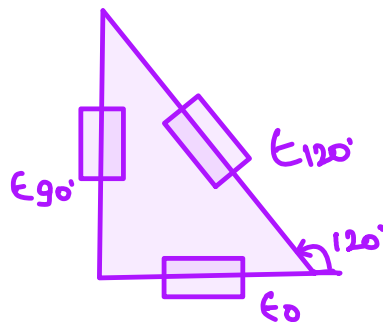
c> 120° Strain Rosette



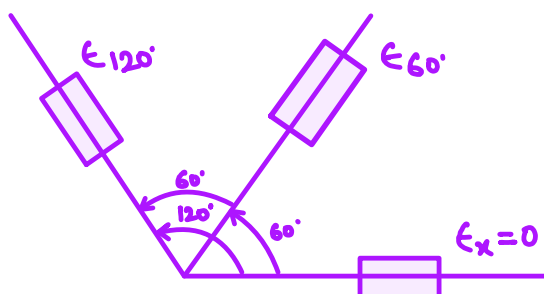
$$\epsilon_{120^\circ} = \frac{\epsilon_0 + \epsilon_{90}}{2} + \frac{\epsilon_0 - \epsilon_{90}}{2} \cos 240^\circ + \frac{\phi_{xy}}{2} \sin 240^\circ$$

$$\phi_{xy} = ?$$

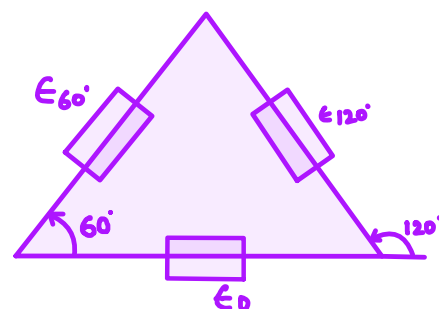
≡



d> Delta Strain Rosette



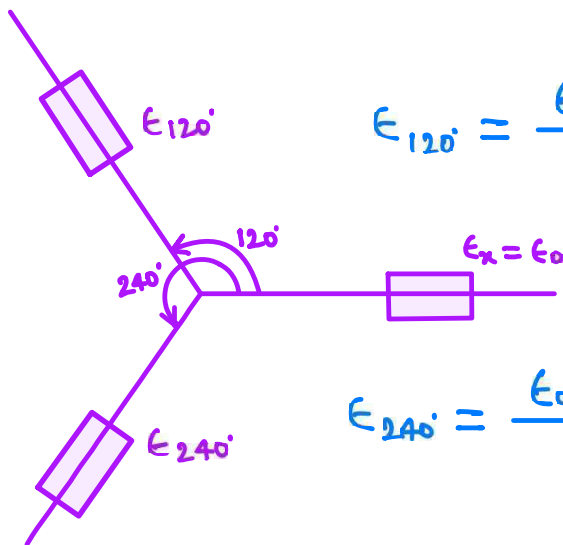
≡



$$\epsilon_{60^\circ} = \frac{\epsilon_0 + \epsilon_y}{2} + \frac{\epsilon_0 - \epsilon_y}{2} \cos 120^\circ + \frac{\phi_{xy}}{2} \sin 120^\circ$$

$$\epsilon_{120^\circ} = \frac{\epsilon_0 + \epsilon_y}{2} + \frac{\epsilon_0 - \epsilon_y}{2} \cos 240^\circ + \frac{\phi_{xy}}{2} \sin 240^\circ$$

e. > Star Strain Rosette



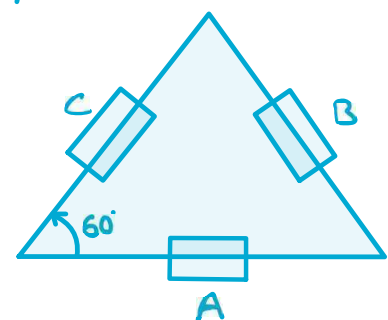
$$\epsilon_{120^\circ} = \frac{\epsilon_0 + \epsilon_y}{2} + \frac{\epsilon_0 - \epsilon_y}{2} \cos 240^\circ + \frac{\phi_{xy}}{2} \sin 240^\circ$$

$$\epsilon_{240^\circ} = \frac{\epsilon_0 + \epsilon_y}{2} + \frac{\epsilon_0 - \epsilon_y}{2} \cos 480^\circ + \frac{\phi_{xy}}{2} \sin 480^\circ$$

Q. > A delta Strain Rosette of 3 electrical resistance strain gauge making an angle of 0° , 60° , 120° is shown in figure. If Poisson's ratio is $1/3$ under plane stress condition and the following strain is measured.

$$\epsilon_A = 1600 \times 10^{-6}, \quad \epsilon_B = -130 \times 10^{-6}, \quad \epsilon_C = -650 \times 10^{-6}$$

Determine the magnitude of principal strain with their direction & maximum shear strain.



Soln

$$\epsilon_{60^\circ}/\epsilon_c = \frac{\epsilon_0 + \epsilon_x}{2} + \frac{\epsilon_0 - \epsilon_x}{2} \cos 120^\circ + \frac{\phi_{xy}}{2} \sin 120^\circ$$

$$-650 \times 10^{-6} = \frac{1000 \times 10^{-6} + \epsilon_x}{2} + \frac{1000 \times 10^{-6} - \epsilon_x}{2} \cos 120^\circ + \frac{\phi_{xy}}{2} \cos 120^\circ \dots (i)$$

$$\epsilon_{120^\circ}/\epsilon_B = \frac{\epsilon_0 + \epsilon_x}{2} + \frac{\epsilon_0 - \epsilon_x}{2} \cos 240^\circ + \frac{\phi_{xy}}{2} \sin 240^\circ$$

$$-130 \times 10^{-6} = \frac{1000 \times 10^{-6} + \epsilon_x}{2} + \frac{1000 \times 10^{-6} - \epsilon_x}{2} \cos 240^\circ + \frac{\phi_{xy}}{2} \sin 240^\circ \dots (ii)$$

on Solving equation (i) & (ii)

$$\epsilon_x = -8.53 \times 10^{-4}$$

$$\phi_{xy} = 6 \times 10^{-4}$$

$$\begin{aligned} \epsilon_{P_1}/\epsilon_{P_2} &= \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\phi_{xy}}{2}\right)^2} \\ &= \frac{1000 \times 10^{-6} + (-8.53 \times 10^{-4})}{2} \pm \sqrt{\left(\frac{1000 \times 10^{-6} + 8.53 \times 10^{-4}}{2}\right)^2 + \left(\frac{6 \times 10^{-4}}{2}\right)^2} \end{aligned}$$

$$\epsilon_{P_1} = 1.047 \times 10^{-3}$$

$$\epsilon_{P_2} = -9 \times 10^{-4}$$

$$\epsilon_{P_3} = \frac{-1}{1-1} (\epsilon_{P_1} + \epsilon_{P_2})$$

$$= \frac{-1/2}{1-1/2} (1.047 \times 10^{-3} - 9 \times 10^{-4})$$

$$\epsilon_{P_3} = -7.35 \times 10^{-5}$$

$$\begin{aligned} \frac{\phi_{max}}{2} &= \frac{\epsilon_{P_{max}} - \epsilon_{P_{min}}}{2} = \frac{\epsilon_{P_1} - \epsilon_{P_3}}{2} = \frac{1.047 \times 10^{-3} + 7.35 \times 10^{-5}}{2} \\ &= 1.12 \times 10^{-3} \end{aligned}$$

$$\tan 2\theta_p = \frac{\phi_{xy}}{\epsilon_x - \epsilon_y} = \frac{1.12 \times 10^{-3}}{(1000 \times 10^{-6} + 8.53 \times 10^{-4})} = 17.94$$

$$\theta_{p_1} = 8.97^\circ$$

$$\theta_{p_2} = 90^\circ + 8.97^\circ = 98.97^\circ$$

θ_{p_2} is perpendicular to both θ_{p_1} & θ_{p_2}

Theories of failure

- Under uniaxial tension or compression practically yielding begins at yield strength but when several components of the stresses exist, yielding depends upon some combination of these components.
- No theoretical method is available that correlates yielding in the uniaxial tensile test with yielding in complex loading condition.
- For this purpose several theories of failure are evolved from the behaviour of material subjected to uniaxial loading.
- By failure here it signifies **failure of material** only however in actual failure may take place due to local buckling & elastic instability.
- Each theory of failure is applicable for particular material, situations and loading conditions.

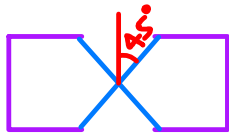
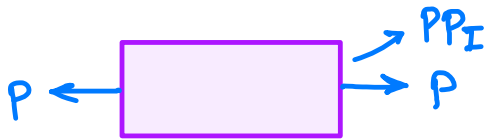
Note:-

Here failure is considered under static loading.

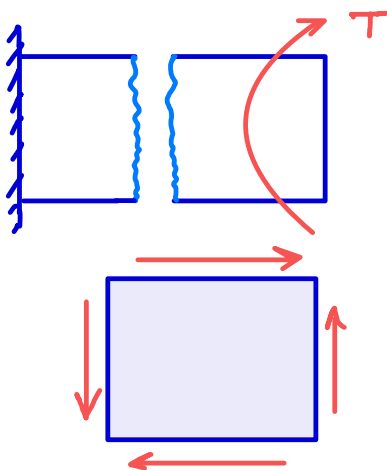
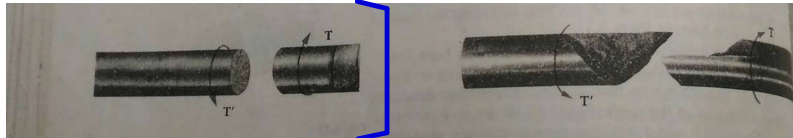
for eg:- failure due to uniaxial loading in ductile & brittle material.

Ductile

$$\sigma_{yc} > \sigma_{yt} > \tau$$



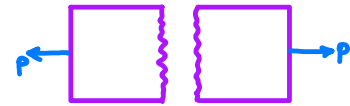
- Hence it shows cup-cone failure.
- failure plane 45° to the longitudinal axis.



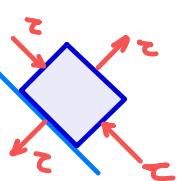
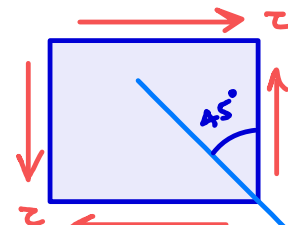
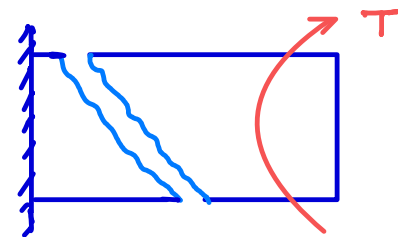
- In pure shear, failure plane is at 90° to longitudinal axis.

Brittle

$$\sigma_{uc} > \tau > \sigma_{ut}$$



- failure plane at 90° to longitudinal axis.



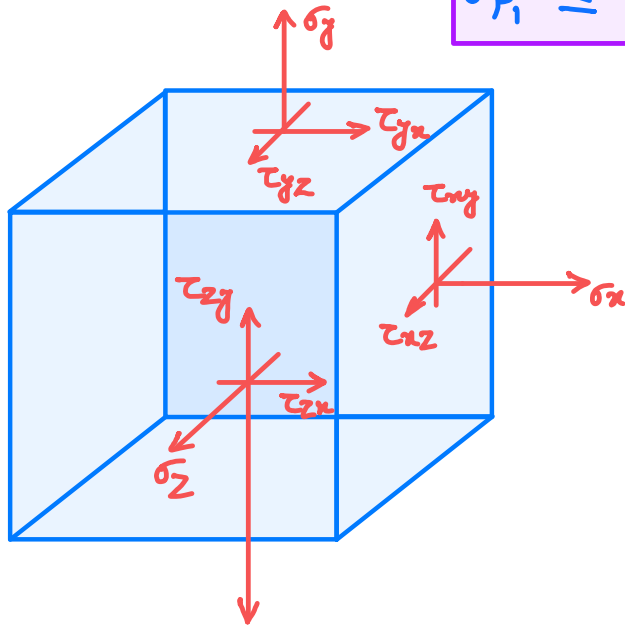
- Here failure plane is at 45° to longitudinal axis.

★ The various theories of failure are as follows :-

(A) Maximum Stren/Maximum principal Stren/Maximum Normal Stren/
Rankines / Lamme's Theory.

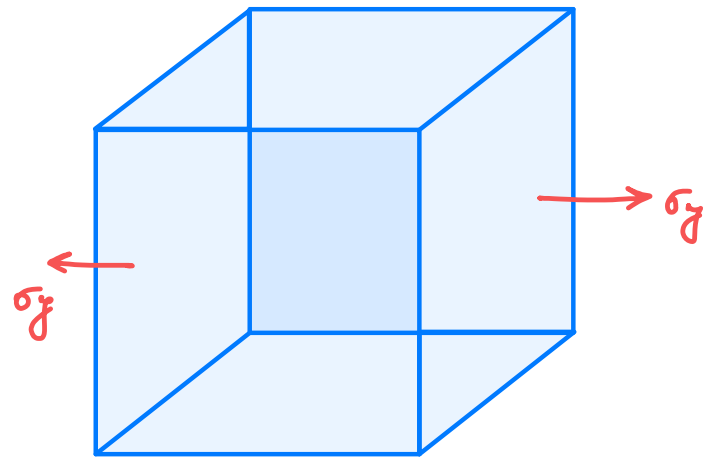
— for No failure, maximum principal stress should be less than yield stress under uniaxial loading.

$$\sigma_{p_1} \leq f_y$$



$$\sigma_{p_1} > \sigma_{p_2} > \sigma_{p_3}$$

principal stresses



$$\sigma_y$$

yield stress.

— for design purpose,

$$\sigma_{p_1} \leq \frac{\sigma_y}{FOS}$$

- This theory is applicable for brittle material as they fail under tension
- This theory is not suitable for ductile material in which failure is governed by shear.
- It is not suitable for pure shear case as under pure shear maximum principal stress is equal to shear stress

Hence according to this theory,

$$\sigma_{p_1} = \tau \leq \sigma_y$$

- It is not preferred for Hydrostatic loading as in this case failure takes place due to Normal Stress and not Shear.

B. Maximum principal strain / St. Venant theory

- for No failure, maximum principal strain should be less than or equal to yield strain.

$$\epsilon_{p_1} \leq \epsilon_y$$

$$\frac{\sigma_{p_1} - \mu(\sigma_{p_2} + \sigma_{p_3})}{E} \leq \frac{\sigma_y}{E}$$

$$\sigma_{p_1} - \mu(\sigma_{p_2} + \sigma_{p_3}) \leq \sigma_y$$

- for design purpose:

$$\sigma_{p_1} - \mu(\sigma_{p_2} + \sigma_{p_3}) \leq \frac{\sigma_y}{FOS}$$

- This theory is also suitable for Brittle material.
- It overestimates the elastic strength of ductile material.
- Not suitable for Hydrostatic loading as

$$\sigma_{p_1} = \sigma_{p_2} = \sigma_{p_3} = \sigma$$

$$\text{as per this theory, } \sigma - \mu(\sigma + \sigma) \leq \sigma_y$$

$$\sigma - 2\mu\sigma \leq \sigma_y$$

$$\sigma(1 - 2\mu) \leq \sigma_y$$

$$\sigma \leq \frac{\sigma_y}{(1 - 2\mu)}$$

$$\text{if } \mu = 0.25 \text{ (for eg:- Steel)}$$

$$\sigma \leq 2\sigma_y \rightarrow \text{(Not acceptable)}$$

208 It is not suitable for pure shear condition as

$$\sigma_{p_1} = \tau, \quad \sigma_{p_2} = -\tau$$

according to this theory,

$$\tau - \mu(-\tau + 0) \leq \sigma_y$$

$$\tau(1+\mu) \leq \sigma_y$$

$$\tau \leq \frac{\sigma_y}{1+\mu}$$

if $\mu = 0.25$ (for steel)

$$\tau \leq 0.8 \sigma_y \text{ (Not suitable).}$$

C. > Maximum Shear Stress / Tresca / Guest / Coulomb Theory.

- for no failure maximum shear stress should be less than or equal to maximum shear stress under uniaxial loading.

$$\tau_{\max} \leq \tau_{\max} \text{ Under uniaxial loading.}$$

$$\text{Max.} \left[\frac{\sigma_{p_1} - \sigma_{p_2}}{2}, \frac{\sigma_{p_2} - \sigma_{p_3}}{2}, \frac{\sigma_{p_1} - \sigma_{p_3}}{2} \right] \leq \frac{\sigma_y}{2}$$

- for design purpose :

$$\text{Max.} \left[\frac{\sigma_{p_1} - \sigma_{p_2}}{2}, \frac{\sigma_{p_2} - \sigma_{p_3}}{2}, \frac{\sigma_{p_1} - \sigma_{p_3}}{2} \right] \leq \frac{\sigma_y}{2 \text{ FOS}}$$

- This theory gives unsafe result for brittle material hence, not suitable to be used.
- This theory gives oversafe result for ductile material in case of pure shear hence, also termed as conservative theory (thereby is uneconomical).
- In case of Hydrostatic loading, $[\sigma_{p_1} = \sigma_{p_2} = \sigma_{p_3} = \sigma]$ Since shear stress on all the planes is zero, thereby it would not be able to predict the failure.

hence is not suitable.

$$\tau_{max} = 0 \leq \frac{\sigma_y}{2}$$

No conclusion can be drawn.

D> Maximum Strain Energy theory/ Beltrami - Heigh's Theory

- According to this theory for No failure, maximum Strain energy per unit volume should be less than or equal to Strain energy per unit volume under uniaxial loading at yielding.

Total Strain energy per unit volume (u) is given by

$$u = \frac{1}{2} [\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{xz} \gamma_{xz} + \tau_{yz} \gamma_{yz}]$$

$$\textcircled{\text{or}} u = \frac{1}{2} [\sigma_{p1} \epsilon_1 + \sigma_{p2} \epsilon_2 + \sigma_{p3} \epsilon_3]$$

$$u = \frac{1}{2E} [\sigma_{p1}^2 + \sigma_{p2}^2 + \sigma_{p3}^2 - 2\mu(\sigma_{p1}\sigma_{p2} + \sigma_{p2}\sigma_{p3} + \sigma_{p1}\sigma_{p3})]$$

According to this theory,

$$\frac{1}{2E} [\sigma_{p1}^2 + \sigma_{p2}^2 + \sigma_{p3}^2 - 2\mu(\sigma_{p1}\sigma_{p2} + \sigma_{p2}\sigma_{p3} + \sigma_{p1}\sigma_{p3})] \leq \frac{\sigma_y^2}{2E}$$

$$\boxed{\sigma_{p1}^2 + \sigma_{p2}^2 + \sigma_{p3}^2 - 2\mu(\sigma_{p1}\sigma_{p2} + \sigma_{p2}\sigma_{p3} + \sigma_{p1}\sigma_{p3}) \leq \sigma_y^2}$$

- for design purpose

$$\sigma_{p1}^2 + \sigma_{p2}^2 + \sigma_{p3}^2 - 2\mu(\sigma_{p1}\sigma_{p2} + \sigma_{p2}\sigma_{p3} + \sigma_{p1}\sigma_{p3}) \leq \left[\frac{\sigma_y}{FOS} \right]^2$$

- for plane stress state, $\sigma_{p3} = 0$

$$\sigma_{p1}^2 + \sigma_{p2}^2 - 2\mu(\sigma_{p1}\sigma_{p2}) \leq \sigma_y^2$$

- for pure shear case, $\sigma_{p1} = \tau$, $\sigma_{p2} = -\tau$

$$\tau^2 - (-\tau)^2 - 2\mu(\tau)(-\tau) \leq \sigma_y^2$$

$$2\tau^2 + 2\mu\tau^2 \leq \sigma_y^2$$

$$2\tau^2(1+\mu) \leq \sigma_y^2$$

$$\tau \leq \frac{\sigma_y}{\sqrt{2(1+\mu)}}$$

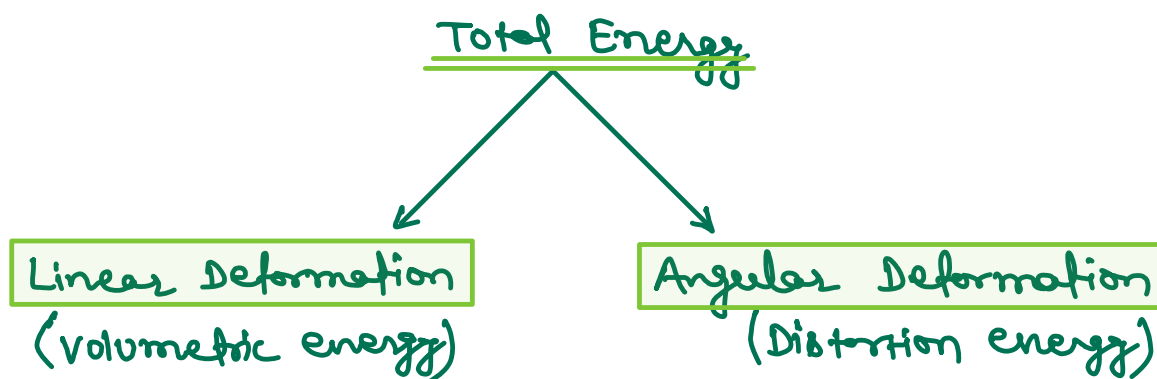
for $\mu = 0.25$, $\tau \leq 0.632$

- This theory also gives little unsafe result for ductile material in case of pure shear but it is very close to experimental result (of $0.577 \sigma_y$)
- This theory gives unsafe result for brittle material hence not suitable to be used.

e.g. Maximum Shear Strain Energy theory / Distortion Energy theory
HUBER-HENKY-VON MISES theory

- According to this theory, maximum shear strain energy must be less than or equal to maximum shear strain energy under uniaxial loading.

NOTE:-



$$\begin{aligned} \text{volumetric strain energy} &= \frac{1}{2} \times \text{Volumetric Stress} \times \text{Volumetric Strain} \\ &= \frac{1}{2} \left(\frac{\sigma_{P1} + \sigma_{P2} + \sigma_{P3}}{3} \right) \times \frac{\sigma_{P1} + \sigma_{P2} + \sigma_{P3}}{E} (1-2\mu) \end{aligned}$$

$$= \frac{(\sigma_{P_1} + \sigma_{P_2} + \sigma_{P_3})^2 (1-2\mu)}{6E}$$

$$= \frac{(\sigma_{P_1}^2 + \sigma_{P_2}^2 + \sigma_{P_3}^2 + 2\sigma_{P_1}\sigma_{P_2} + 2\sigma_{P_2}\sigma_{P_3} + 2\sigma_{P_3}\sigma_{P_1})(1-2\mu)}{6E}$$

Now,

$$\text{Total Strain Energy} = \text{Volumetric Strain Energy} + \text{Distortion Energy}$$

$$\text{Distortion Energy} = \text{Total Strain Energy} - \text{Volumetric Strain Energy}$$

$$= \frac{1}{2E} [\sigma_{P_1}^2 + \sigma_{P_2}^2 + \sigma_{P_3}^2 - 2\mu(\sigma_{P_1}\sigma_{P_2} + \sigma_{P_2}\sigma_{P_3} + \sigma_{P_3}\sigma_{P_1})] -$$

$$\frac{(1-2\mu)}{6E} [\sigma_{P_1}^2 + \sigma_{P_2}^2 + \sigma_{P_3}^2 + 2(\sigma_{P_1}\sigma_{P_2} + \sigma_{P_2}\sigma_{P_3} + \sigma_{P_3}\sigma_{P_1})]$$

$$= \frac{1+\mu}{3E} [\sigma_{P_1}^2 + \sigma_{P_2}^2 + \sigma_{P_3}^2 - \sigma_{P_1}\sigma_{P_2} - \sigma_{P_2}\sigma_{P_3} - \sigma_{P_3}\sigma_{P_1}]$$

$$= \frac{1+\mu}{6E} [(\sigma_{P_1} - \sigma_{P_2})^2 + (\sigma_{P_2} - \sigma_{P_3})^2 + (\sigma_{P_3} - \sigma_{P_1})^2]$$

$$= \frac{(1+\mu)}{6 \cdot 2G(1+\mu)} [(\sigma_{P_1} - \sigma_{P_2})^2 + (\sigma_{P_2} - \sigma_{P_3})^2 + (\sigma_{P_3} - \sigma_{P_1})^2]$$

$$\text{Distortion Energy} = \frac{1}{12G} [(\sigma_{P_1} - \sigma_{P_2})^2 + (\sigma_{P_2} - \sigma_{P_3})^2 + (\sigma_{P_3} - \sigma_{P_1})^2]$$

for uniaxial loading, $\sigma_{P_1} = \sigma_y$, $\sigma_{P_2} = \sigma_{P_3} = 0$

$$\text{Distortion Energy} = \frac{\sigma_y^2}{6G}$$

- As per this theory,

$$\frac{1}{12G} [(\sigma_{P_1} - \sigma_{P_2})^2 + (\sigma_{P_2} - \sigma_{P_3})^2 + (\sigma_{P_3} - \sigma_{P_1})^2] \leq \frac{\sigma_y^2}{6G}$$

$$\frac{1}{2} [(\sigma_{P_1} - \sigma_{P_2})^2 + (\sigma_{P_2} - \sigma_{P_3})^2 + (\sigma_{P_3} - \sigma_{P_1})^2] \leq \sigma_y^2$$

— for design purpose

$$\frac{1}{2} \left[(\sigma_{p1} - \sigma_{p2})^2 + (\sigma_{p2} - \sigma_{p3})^2 + (\sigma_{p3} - \sigma_{p1})^2 \right] \leq \left[\frac{\sigma_y}{FOS} \right]^2$$

— for plane stress state, $\sigma_{p3} = 0$

$$\sigma_{p1}^2 + \sigma_{p2}^2 - \sigma_{p1} \sigma_{p2} \leq \sigma_y^2$$

— for pure shear case: $\sigma_{p1} = \tau$, $\sigma_{p2} = -\tau$

$$\tau^2 + (-\tau)^2 - \tau(+\tau) \leq \sigma_y^2$$

$$3\tau^2 \leq \sigma_y^2$$

$$\tau \leq \frac{\sigma_y}{\sqrt{3}}$$

$$\tau \leq 0.577 \sigma_y$$

— This theory gives best suitable result for ductile material in case of pure shear hence it preferred for these.

— This theory gives unsafe results for brittle material hence is not suitable to be used.

F. > Octahedral Shear Stress theory

— According to this theory for no failure Octahedral Shear stress must be less than equal Octahedral Shear stress under uniaxial loading.

NOTE :-

— A plane which have equal inclination with all the three principal plane is termed as Octahedral plane.

— out of infinite planes passing through a point there exist 3 octahedral planes.

$$\sigma_{\text{octahedral}} = \frac{\sigma_{P_1} + \sigma_{P_2} + \sigma_{P_3}}{3}$$

$$\tau_{\text{octahedral}} = \frac{1}{3} \sqrt{(\sigma_{P_1} - \sigma_{P_2})^2 + (\sigma_{P_2} - \sigma_{P_3})^2 + (\sigma_{P_3} - \sigma_{P_1})^2}$$

- for uniaxial loading, $\sigma_{P_1} = \sigma_y$, $\sigma_{P_2} = \sigma_{P_3} = 0$

$$\tau_{\text{octahedral}} = \frac{\sqrt{2}}{3} \sigma_y$$

- According to this theory :-

$$\frac{1}{3} \sqrt{(\sigma_{P_1} - \sigma_{P_2})^2 + (\sigma_{P_2} - \sigma_{P_3})^2 + (\sigma_{P_3} - \sigma_{P_1})^2} \leq \frac{\sqrt{2}}{3} \sigma_y$$

- for design purpose

$$\frac{1}{3} \sqrt{(\sigma_{P_1} - \sigma_{P_2})^2 + (\sigma_{P_2} - \sigma_{P_3})^2 + (\sigma_{P_3} - \sigma_{P_1})^2} \leq \frac{\sqrt{2}}{3} \frac{\sigma_y}{\text{FOS}}$$

- for plane stress state, $\sigma_{P_3} = 0$

$$\frac{\sqrt{2}}{3} \sqrt{\sigma_{P_1}^2 + \sigma_{P_2}^2 - \sigma_{P_1} \sigma_{P_2}} \leq \frac{\sqrt{2}}{3} \sigma_y$$

$$\sigma_{P_1}^2 + \sigma_{P_2}^2 - \sigma_{P_1} \sigma_{P_2} \leq \sigma_y^2$$

- Thus, this theory is same as maximum distortion energy theory in 2D condition.

- Hence it gives best result for ductile material in pure shear case.

NOTE:-

- The Experimental results for ductile material shows that its shear strength is approximately 57% of its yield strength.

- Hence material will fail if $\tau > 0.577 \sigma_y$

- Hence if all the theories are compared to be used for ductile material, following conclusion is obtained:-

Type of theory	$\frac{\tau_{max}}{\sigma_y}$	Remarks.
I.) Rankine	1	Unsafe
II.) St. Venant	0.8 ($\mu = 0.25$)	Unsafe.
III.) Tresca	0.5	Extra safe
IV.) Beltrami-Heigelt's	0.62 ($\mu = 0.25$)	Unsafe
V.) Huber-Henky	0.577	Accurate.
VI.) Octahedral	0.577	Accurate.

- Maximum Shear stress theory is most conservative for ductile material.
- Max. distortion energy & Octahedral Shear Stress theory is the most accurate for ductile material.
- Maximum principal stress theory is the most accurate for brittle material.
- All the theories compare the values under general state of stress with that under uniaxial loading. Hence, all the theories give similar result under the uniaxial loading as follows :-

a.) Rankine theory.

$$\sigma_p \leq \sigma_y$$

b.) St. Venant theory

$$\epsilon_1 \leq \epsilon_y$$

$$\epsilon_1 \leq \frac{\sigma_y}{E}$$

$$\frac{\sigma_{p1}}{E} \leq \frac{\sigma_y}{E}$$

$\Rightarrow$

$$\sigma_{p1} \leq \sigma_y$$

c) Tresca's theory

$$\tau_{\max} = \frac{\sigma_y}{2}$$

$$\frac{\sigma_{p1}}{2} \leq \frac{\sigma_y}{2}$$

$$\boxed{\sigma_{p1} \leq \sigma_y}$$

d) Beltrami - Heigat's theory

$$u \leq \frac{\sigma_y^2}{2E}$$

$$\frac{\sigma_{p1}^2}{2E} \leq \frac{\sigma_y^2}{2E}$$

$$\boxed{\sigma_{p1} \leq \sigma_y}$$

e) Huber - Henky - VON - mises theory

$$\frac{1}{2}(\sigma_{p1}^2 + \sigma_{p2}^2) \leq \sigma_y^2$$

$$\boxed{\sigma_{p1} \leq \sigma_y}$$

Graphical Representation of theories of failure

a) Max. principal stress theory

- According to this theory for no failure

$$\sigma_1 \leq \pm \sigma_y$$

$$\sigma_{p2} \leq \pm \sigma_y$$

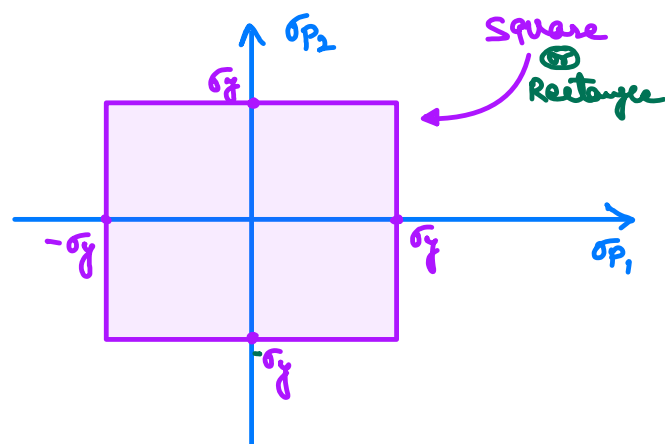
$$\sigma_{p3} \leq \pm \sigma_y$$

for 2D stress, $\sigma_{p3} = 0$

$$x = \pm c$$

$$y = \pm c$$

$\left\{ \begin{array}{l} (+) : \text{Tension} \\ (-) : \text{Compression} \end{array} \right\}$



Note:-

if yield stress (σ_y) in tension & compression is different then it would be converted to rectangle.

b) Maximum principal strain theory

— According to this theory, for No failure

$$\epsilon_1 \leq \pm \frac{\sigma_y}{E} \Rightarrow \frac{\sigma_{P1}}{E} - \mu \frac{(\sigma_{P2} + \sigma_{P3})}{E} \leq \pm \frac{\sigma_y}{E}$$

$$\epsilon_2 \leq \pm \frac{\sigma_y}{E} \Rightarrow \frac{\sigma_{P2}}{E} - \mu \frac{(\sigma_{P3} + \sigma_{P1})}{E} \leq \pm \frac{\sigma_y}{E}$$

$$\epsilon_3 \leq \pm \frac{\sigma_y}{E} \Rightarrow \frac{\sigma_{P3}}{E} - \mu \frac{(\sigma_{P1} + \sigma_{P2})}{E} \leq \pm \frac{\sigma_y}{E}$$

— For 2D stress :- $\sigma_{P3} = 0$ and $\epsilon_3 < \epsilon_2, \epsilon_1$

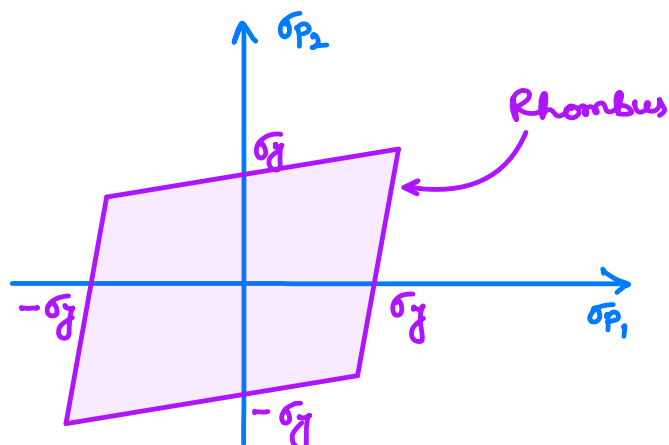
$$\sigma_{P1} - \mu \sigma_{P2} \leq \pm \sigma_y$$

$$\sigma_{P2} - \mu \sigma_{P1} \leq \pm \sigma_y$$

— $\mu (\sigma_{P1} + \sigma_{P2}) \leq \pm \sigma_y \Rightarrow$ Since $\epsilon_3 < \epsilon_2, \epsilon_1$
material can't fail in z-direction

In general:

$$x - ay = \pm c$$

$$y - ax = \pm c$$


c) Maximum Shear Stress theory

- According to this theory for no failure

$$\tau_{\max 1,2} \leq \pm \frac{\sigma_y}{2} \Rightarrow \frac{\sigma_{P1} - \sigma_{P2}}{2} \leq \pm \frac{\sigma_y}{2}$$

$$\tau_{\max 2,3} \leq \pm \frac{\sigma_y}{2} \Rightarrow \frac{\sigma_{P2} - \sigma_{P3}}{2} \leq \pm \frac{\sigma_y}{2}$$

$$\tau_{\max 1,3} \leq \pm \frac{\sigma_y}{2} \Rightarrow \frac{\sigma_{P1} - \sigma_{P3}}{2} \leq \pm \frac{\sigma_y}{2}$$

- for 2D stress, $\sigma_{P3} = 0$

$$\sigma_{P1} - \sigma_{P2} \leq \pm \sigma_y$$

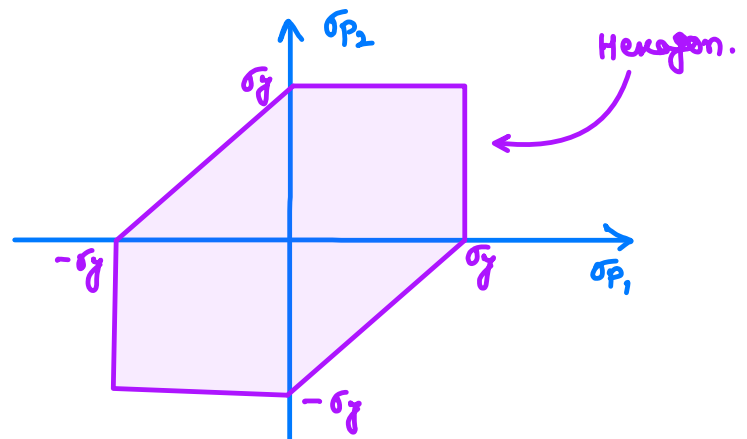
$$\sigma_{P2} \leq \pm \sigma_y$$

$$\sigma_{P1} \leq \pm \sigma_y$$

in general: $x - y = \pm c$

$$x = \pm c$$

$$y = \pm c$$



d) Maximum Strain Energy theory

- According to this theory for no failure

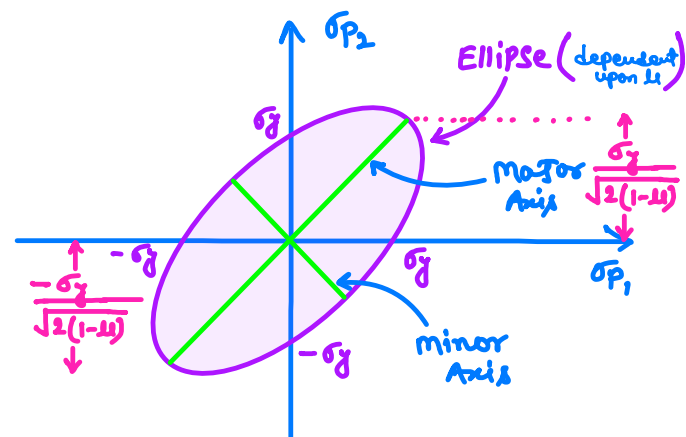
$$\frac{1}{2E} [\sigma_{P1}^2 + \sigma_{P2}^2 + \sigma_{P3}^2 - 2\mu(\sigma_{P1}\sigma_{P2} + \sigma_{P2}\sigma_{P3} + \sigma_{P3}\sigma_{P1})] \leq \frac{\sigma_y^2}{2E}$$

- for 2D stress, $\sigma_{P3} = 0$

$$\sigma_{P1}^2 + \sigma_{P2}^2 - 2\mu \sigma_{P1} \sigma_{P2} \leq \sigma_y^2$$

(Dependent upon μ)

Major axis is in
1st & 3rd quadrant.



e) Maximum Distortion Energy theory

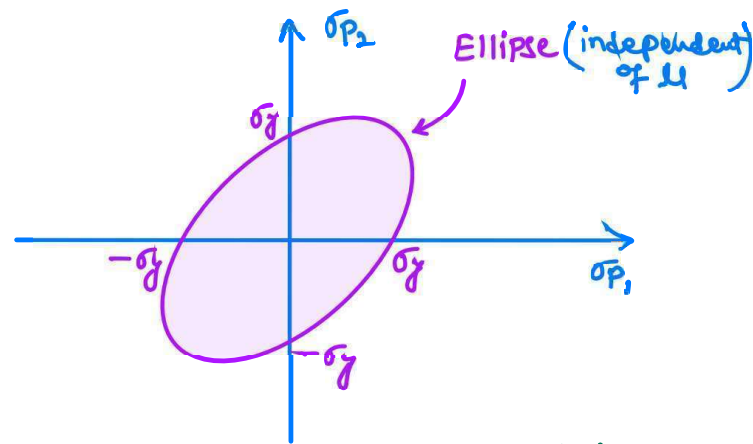
- According to this theory, for no failure

$$\frac{1}{12G} [(\sigma_{P1} - \sigma_{P2})^2 + (\sigma_{P2} - \sigma_{P3})^2 + (\sigma_{P1} - \sigma_{P3})^2] \leq \frac{\sigma_y^2}{6G}$$

- for 2D Stress, $\sigma_{p_3} = 0$

$$\sigma_{p_1}^2 + \sigma_{p_2}^2 - \sigma_{p_1} \sigma_{p_2} \leq \sigma_y^2$$

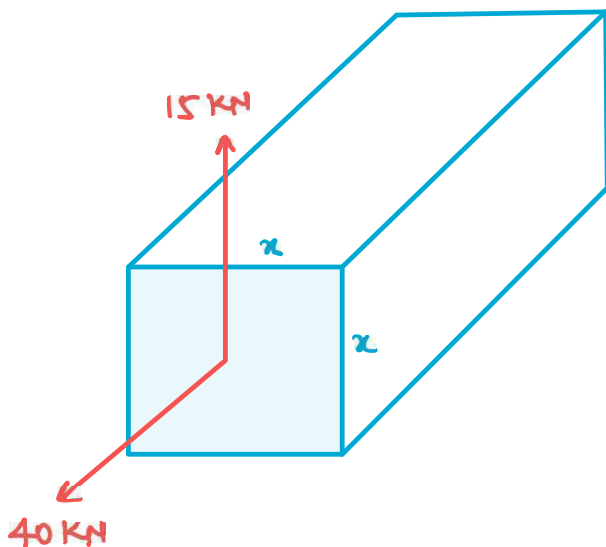
independent of μ .



Note:-

for octahedral shear stress theory, graphical representation is same as above.

Q. > A square pin is required to resist a pull of 40 kN and shear force of 15 kN. Derive a suitable section of the pin according to maximum strain energy theory. Given that $\sigma_y = 350 \text{ mpa}$, $\mu = 0.3$ & $\text{fos} = 2$, Assume uniform distribution of Normal & shear stress on section.



let the size of square section be $x \text{ mm}$.

$$\sigma = \frac{40}{x^2}, \quad \tau = \frac{15}{x^2}$$

$$\begin{aligned} \sigma_{p_1}/\sigma_{p_2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2} \\ &= \frac{\frac{40}{x^2} + 0}{2} \pm \sqrt{\left(\frac{\frac{40}{x^2} - 0}{2}\right)^2 + \left(\frac{15}{x^2}\right)^2} \end{aligned}$$

$$\sigma_{p_1} = \frac{45}{x^2} \text{ kN/mm}^2$$

$$\sigma_{p_2} = -\frac{5}{x^2} \text{ kN/mm}^2$$

According to max strain Energy theory, for NO failure

$$\frac{1}{2E} [\sigma_{p1}^2 + \sigma_{p2}^2 - 2\mu \sigma_{p1} \sigma_{p2}] \leq \frac{(\sigma_y / FOS)^2}{2E}$$

$$\left[\left(\frac{45}{x^2} \right)^2 + \left(\frac{-5}{x^2} \right)^2 - 2 \times 0.3 \times \frac{45}{x^2} \times \left(\frac{-5}{x^2} \right) \right] \leq \left(\frac{350}{2} \right)^2 \times \frac{1}{106}$$

$$x = 16.14 \text{ mm}$$

Q.7

The principal stress at a point in a elastic material are 1.5σ (T), σ (T), 0.5σ (C). The elastic limit in tension is 210 MPa and $\mu = 0.3$, what would be the value of σ at failure when computed by different theories of failure.

Solution: (a) maximum principal stress theory,

$$\sigma_{p1} \leq \sigma_y$$

$$1.5\sigma \leq 210$$

$$\sigma \leq 140 \text{ N/mm}^2$$

(b) maximum principal strain theory

$$\epsilon_1 \leq \epsilon_y$$

$$\frac{\sigma_{p1} - \mu(\sigma_{p2} + \sigma_{p3})}{E} \leq \frac{\sigma_y}{E}$$

$$1.5\sigma - 0.3(\sigma + (-0.5\sigma)) \leq 210$$

$$\sigma \leq 155.5 \text{ N/mm}^2$$

(c) maximum shear stress theory

$$\tau_{max} \leq \frac{\sigma_y}{2}$$

$$\max \left| \frac{\sigma_{p1} - \sigma_{p2}}{2}, \frac{\sigma_{p2} - \sigma_{p3}}{2}, \frac{\sigma_{p1} - \sigma_{p3}}{2} \right| \leq \frac{\sigma_y}{2}$$

$$\max. \left| \frac{1.5\sigma - \sigma}{2}, \frac{\sigma + 0.5\sigma}{2}, \frac{1.5\sigma + 0.5\sigma}{2} \right| \leq \frac{\sigma_y}{2}$$

$$2\sigma \leq 210$$

$$\sigma \leq 105 \text{ N/mm}^2$$

(d.) Maximum Strain Energy theory

$$\frac{1}{2E} [\sigma_{P1}^2 + \sigma_{P2}^2 + \sigma_{P3}^2 - 2\mu (\sigma_{P1}\sigma_{P2} + \sigma_{P2}\sigma_{P3} + \sigma_{P1}\sigma_{P3})] \leq \sigma_y$$

$$\frac{1}{2E} [(1.5\sigma)^2 + (\sigma)^2 + (-0.5\sigma)^2 - 2 \times 0.3 (1.5\sigma \cdot \sigma + \sigma \cdot (-0.5\sigma) + 1.5\sigma \cdot (-0.5\sigma))] \leq 210^2$$

$$\sigma \leq 114.73 \text{ MPa}$$

(e.) Max. distortion Energy theory

$$2 \cdot \frac{1}{12G} [(\sigma_{P1} - \sigma_{P2})^2 + (\sigma_{P2} - \sigma_{P3})^2 + (\sigma_{P3} - \sigma_{P1})^2] \leq \frac{\sigma_y^2}{6G}$$

$$\frac{1}{2} [(1.5\sigma - \sigma)^2 + (\sigma + 0.5\sigma)^2 + (-0.5\sigma - 1.5\sigma)^2] \leq 210^2$$

$$\sigma \leq 116.487 \text{ N/mm}^2$$

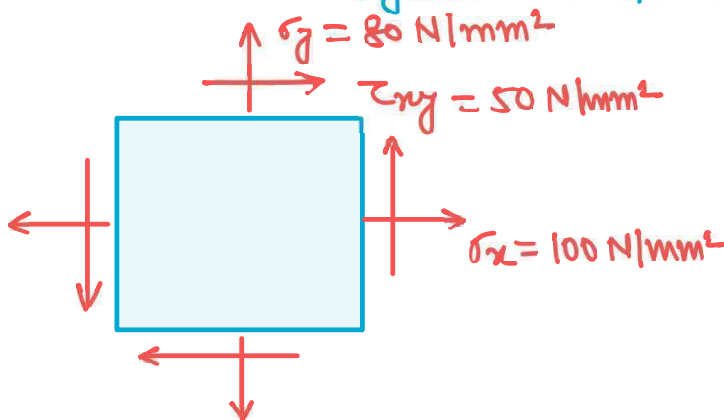
(f.) Octahedral Shear Stress theory

$$\frac{1}{3} \times \sqrt{(\sigma_{P1} - \sigma_{P2})^2 + (\sigma_{P2} - \sigma_{P3})^2 + (\sigma_{P3} - \sigma_{P1})^2} \leq \frac{\sqrt{2}}{3} \sigma_y$$

$$\frac{1}{3} \times \sqrt{(1.5\sigma - \sigma)^2 + (\sigma + 0.5\sigma)^2 + (-0.5\sigma - 1.5\sigma)^2} \leq \frac{\sqrt{2}}{3} \times 210$$

$$\sigma \leq 116.48 \text{ N/mm}^2$$

Q.7 A plane element of the body is subjected to stresses as shown in figure. Compute the FOS as per different theories of failure, if yield stress for the material is 200 N/mm^2 & $\mu = 0.3$.



Solution:-

$$\sigma_{p1}/\sigma_{p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{p1} = 141 \text{ N/mm}^2$$

$$\sigma_{p2} = 39 \text{ N/mm}^2$$

(a) max principal stress theory

$$\sigma_{p1} \leq \frac{\sigma_y}{\text{FOS}}$$

$$\text{FOS} \leq \frac{\sigma_y}{\sigma_{p1}} = \frac{200}{141}$$

$$\text{FOS} \leq 1.418.$$

(b) max principal strain theory

$$\epsilon_1 \leq \epsilon_y$$

$$\frac{\sigma_{p1} - \mu \sigma_{p2}}{E} \leq \frac{\sigma_y}{E}$$

$$\frac{141 - 0.3(39)}{E} \leq \frac{200}{\text{FOS}}$$

$$\text{FOS} \leq \frac{200}{129.3}$$

$$\text{FOS} \leq 1.546.$$

c.) maximum shear stress theory

$$\tau_{max} \leq \frac{\sigma_y}{2}$$

$$\max \left| \frac{\sigma_{p1} - \sigma_{p2}}{2}, \frac{\sigma_{p1}}{2}, \frac{\sigma_{p2}}{2} \right| \leq \frac{\sigma_y}{2 \cdot FOS}$$

$$\frac{\sigma_{p1}}{2} \leq \frac{\sigma_y}{2 \cdot FOS}$$

$$141 \leq \frac{200}{FOS}$$

$$FOS \leq 1.418$$

d.) maximum strain energy theory

$$FOS = 1.486.$$

e.) Max. distortion Energy theory & Octahedral S. Stress theory

$$FOS = 1.586.$$

mcq

Q.)

The Homogeneous State of stress for a metal part undergoing deformation is given by

- using VON-MISES yield criteria, Compute the value of shear yield stress.

$$\begin{bmatrix} 10 & 5 & 0 \\ 5 & 20 & 0 \\ 0 & 0 & -10 \end{bmatrix} \text{ mpa}$$

Solution

According to this theory : $\tau \leq 0.577 \sigma_y$

$$\sigma_x = 10 \text{ mpa}, \sigma_y = 20 \text{ mpa}, \sigma_z = -10 \text{ mpa}$$

$$\tau_{xy} = \tau_{yx} = 5 \text{ mpa}, \tau_{xz} = \tau_{zx} = 0 \text{ mpa}$$

$$\tau_{yz} = \tau_{zy} = 0 \text{ mpa}.$$

$$\sigma_{P1} / \sigma_{P2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{10+20}{2} \pm \sqrt{\left(\frac{10-20}{2}\right)^2 + 5^2}$$

$$\sigma_{P1} = 22 \text{ MPa.}$$

$$\sigma_{P2} = 7.93 \text{ MPa}$$

$$\sigma_{P3} = -10 \text{ MPa (comp)}$$

$$\frac{1}{12G} [(\sigma_{P1} - \sigma_{P2})^2 + (\sigma_{P2} - \sigma_{P3})^2 + (\sigma_{P3} - \sigma_{P1})^2] \leq \frac{\sigma_y^2}{6G}$$

$$\frac{1}{2} [(22 - 7.93)^2 + (7.93 + 10)^2 + (-10 - 22)^2] \leq \sigma_y^2$$

$$\sigma_y \geq 27.83 \text{ MPa}$$

$$\tau \leq 0.577 \sigma_y$$

$$\tau \leq 0.577 \times 27.83$$

$$\tau \leq 16 \text{ MPa}$$

Q: In mild steel shaft of 50 mm diameter subjected to Bending moment of 1 kNm and a Torque of "T" kNm. If the yield point of steel in tension is 210 MPa. then find the maximum value of torque without causing yielding according to the

- maximum principal stress theory
- max. shear stress theory.

Soln As per Bending formula: $\frac{M}{I} = \frac{\sigma}{y} \Rightarrow \sigma = \frac{M}{\frac{I}{y}} = \frac{M}{Z}$

$$\sigma = \frac{M}{Z} = \frac{M}{\frac{\pi d^3}{32}}$$

As per Torsion formula:-

$$\frac{T}{J} = \frac{\tau}{r} \Rightarrow T = \frac{\tau}{\frac{J}{r}} = \frac{\tau}{Z_p} = \frac{\tau}{\frac{\pi d^3}{16}}$$

a) maximum principal stress theory

$$\sigma_{p1} \leq f_y$$

$$\frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} \leq f_y$$

$$\frac{16M}{\pi d^3} + \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2} \leq 210$$

$$\frac{16 \times 1 \times 10^6}{\pi \times (50)^3} + \sqrt{\left(\frac{16 \times 1 \times 10^6}{\pi \times (50)^3}\right)^2 + \left(\frac{16 \times T}{\pi \times (50)^3}\right)^2} \leq 210$$

$$T \leq 5.056 \text{ kN-m}$$

b) max Shear Stress theory

$$\tau_{max} \leq \frac{\sigma_y}{2}$$

$$\frac{16}{\pi d^3} \sqrt{M^2 + T^2} \leq \frac{\sigma_y}{2}$$

$$\Rightarrow T \leq 2.375 \text{ kN-m.}$$

Q.7 A shaft of uniform circular section is subjected to pure torsion. Find the ratio of dia according to maximum shear stress theory & maximum principal stress theory.

Solution max. Shear stress theory

$$\tau_{max} \leq \frac{\sigma_y}{2}$$

$$\frac{16T}{\pi d_1^3} \leq \frac{\sigma_y}{2} \quad \text{--- (i)}$$

(ii) max. principal stress theory

$$\sigma_{p1} \leq \sigma_y$$

$$\frac{16T}{\pi d_2^3} \leq \sigma_y \quad \text{--- (ii)}$$

Pure torsion
 $M=0$

from (i) & (ii)

$$\frac{32T}{\cancel{\pi} d_1^3} = \frac{16T}{\cancel{\pi} d_2^3}$$

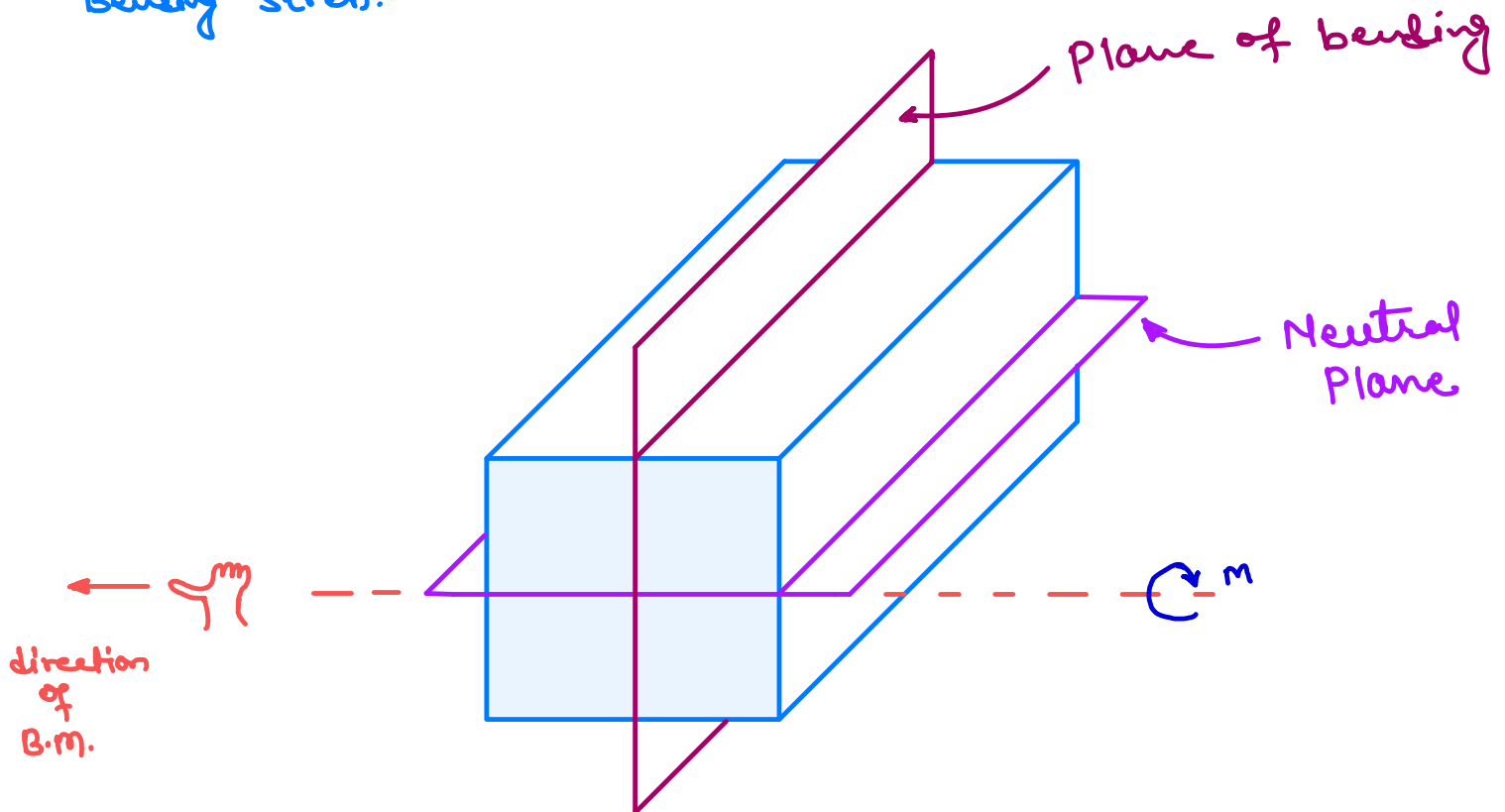
$$\left(\frac{d_1}{d_2}\right)^3 = \frac{32T}{16T}$$

$$\frac{d_1}{d_2} = (2)^{1/3} = \underline{\underline{1.2599}} \approx 1.26.$$

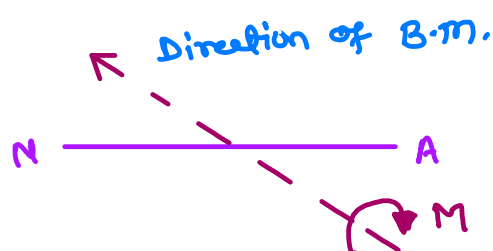


BENDING STRESS

- Bending moment at a section tends to bend or deflect the beam and internal stresses resisting this bending is termed as Bending Stress.

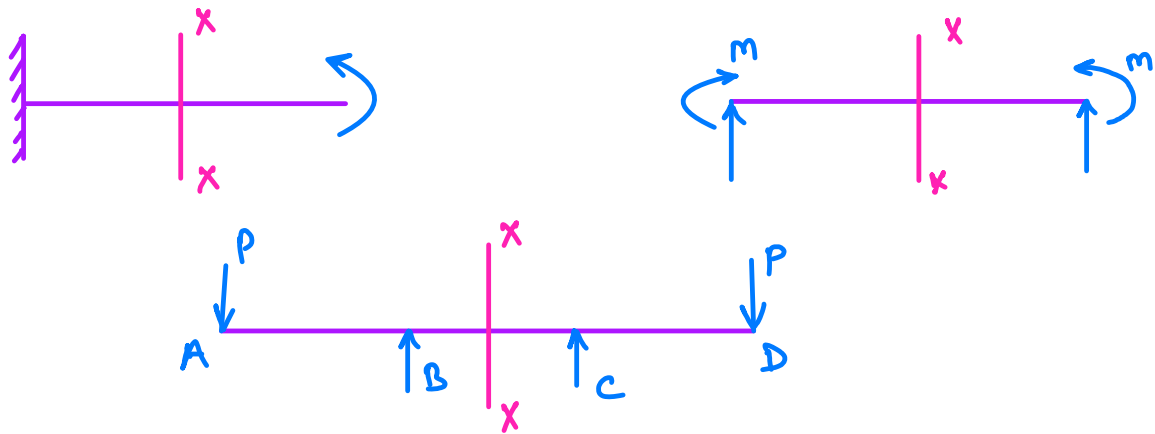


- The direction of B.M. is given by right hand thumb rule.
- when the direction of B.M. is parallel to the Neutral axis (N.A.) Bending is said to occur in plane of Bending, if however direction of B.M. & N.A. are different then bending does not take place in the plane of Bending.

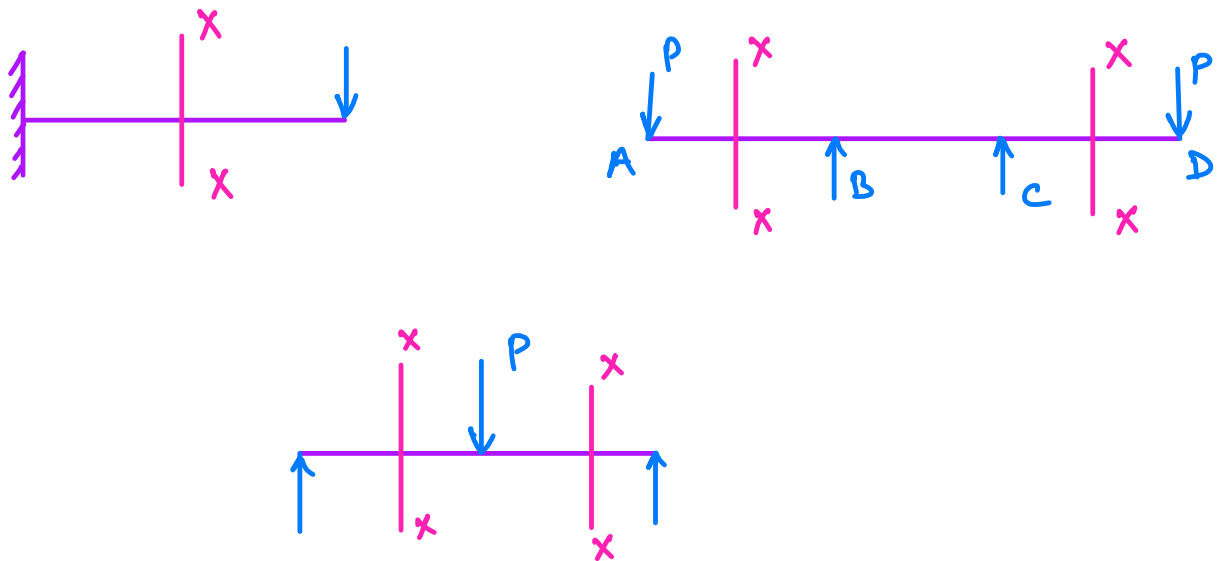


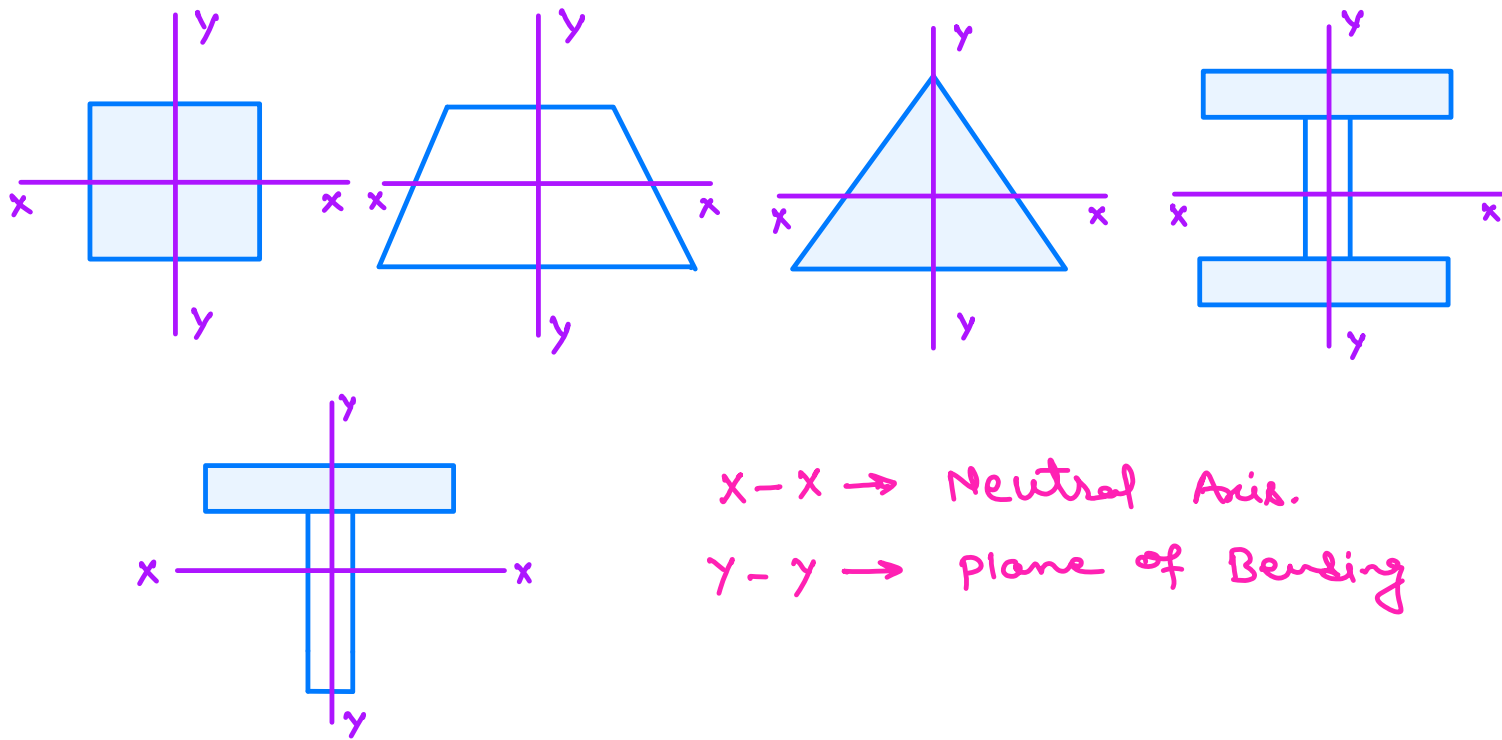
- If Bending of the beam takes place under constant B.M (under zero shear force). It is termed as pure Bending.
- But if the bending of beam takes place in the presence of shear force it is termed as Non-uniform Bending.
- for eg:-

a.) Pure Bending:-



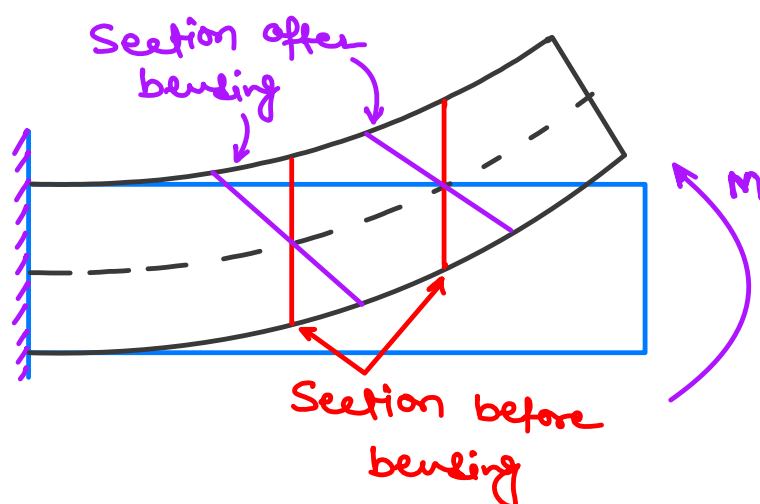
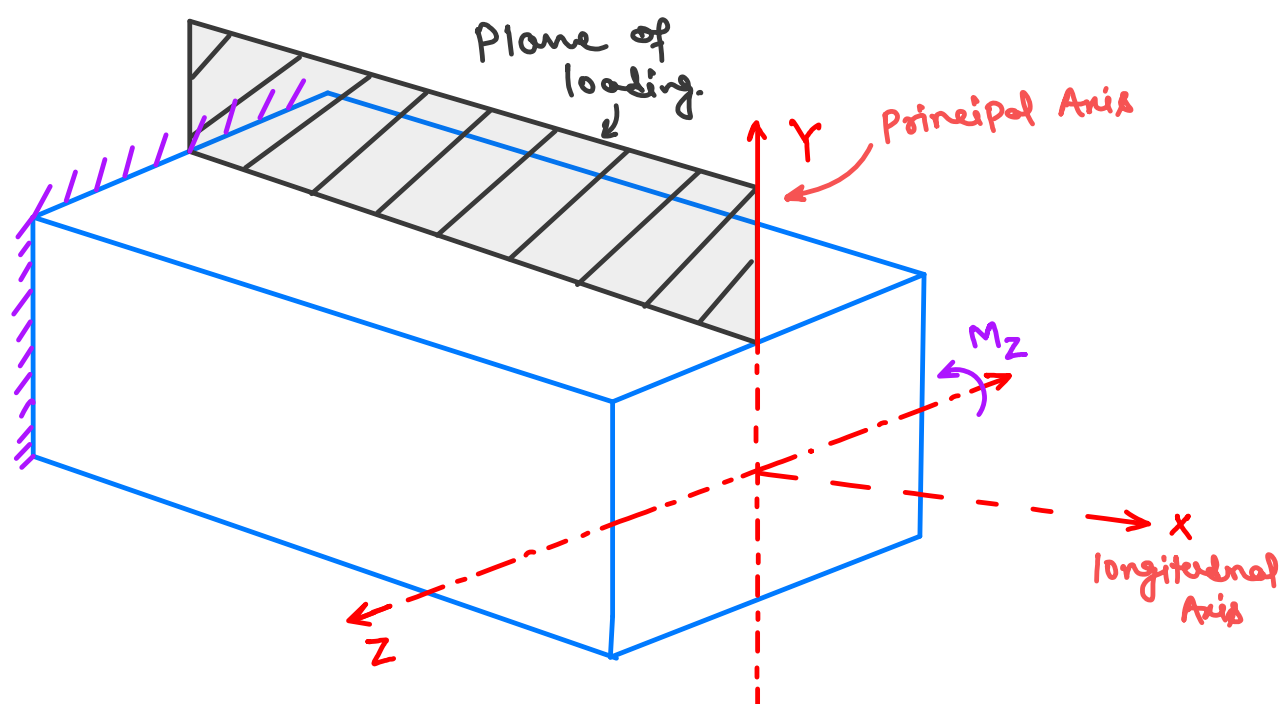
b.) Non-uniform Bending





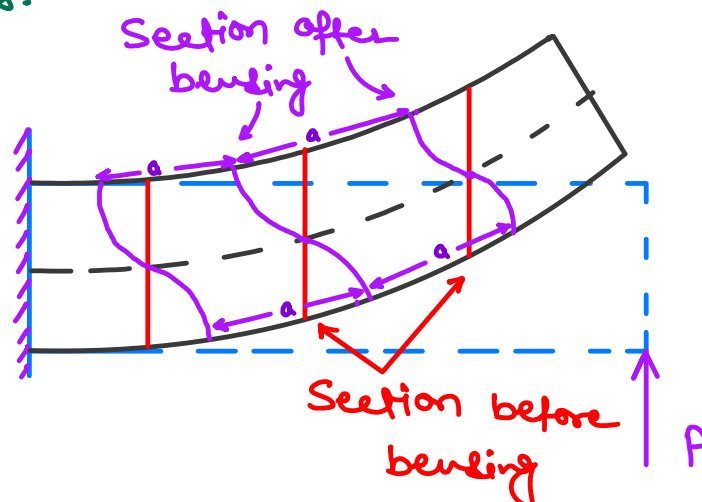
Assumptions in theory of pure Bending:-

- i.) plane section before bending remains plane after Bending.
- ii.) material is homogeneous and isotropic.
- iii.) material obeys Hooke's law.
- iv.) modulus of Elasticity in tension & Compression is Same.
- v.) Beam is initially straight and has constant cross-section throughout its length (Beam is prismatic).
- vi.) The plane of loading must contain a principal axis of cross-section of the beam & load must be perpendicular to the longitudinal axis of the beam.

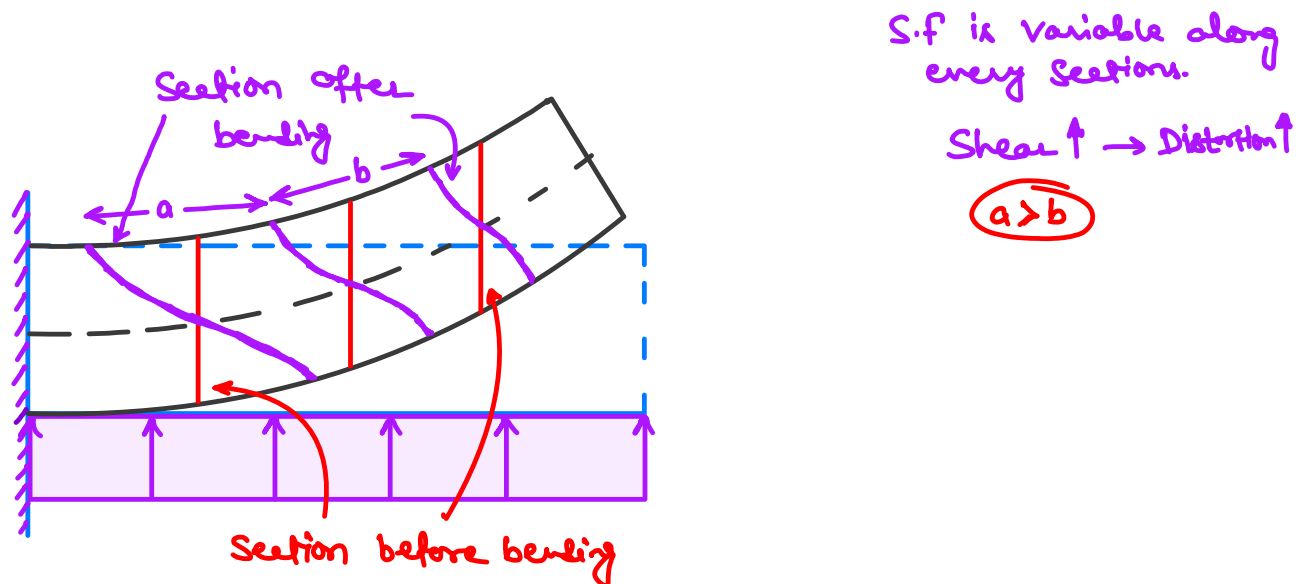


Note:-

Since in this case shear force/shear stress is zero (pure Bending) at all the sections, no distortion of Section occurs.



Extent of distortion is same i.e. " a ".

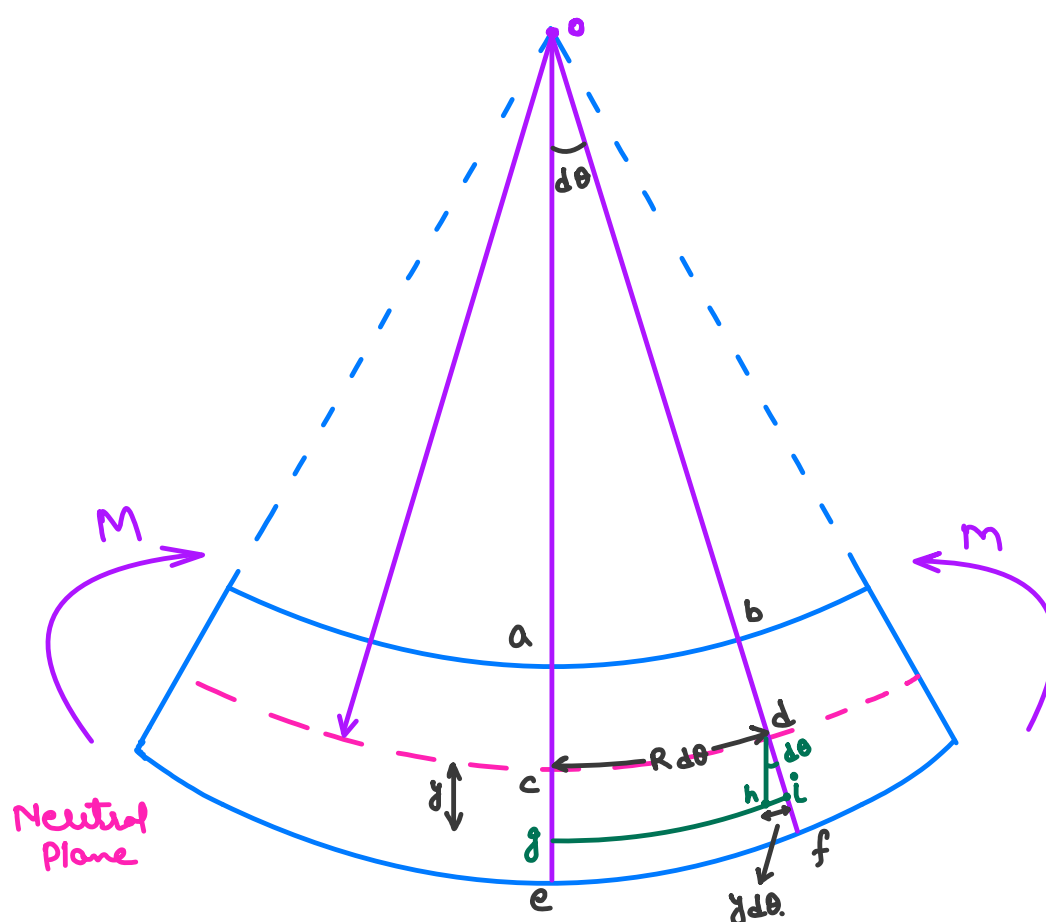
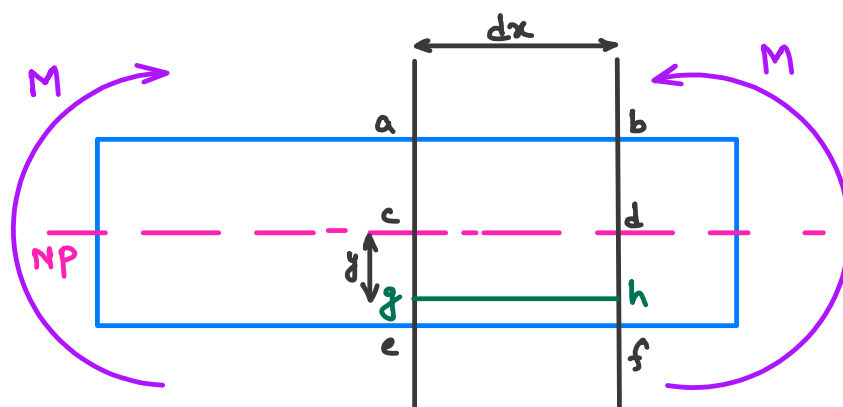


- In the presence of shear plane section before bending does not remain plane after bending because warping of the section takes place in this case.
- Since shear stress variation across the section is parabolic, amount of shear stress does not remain same at every point of section, hence extent of distortion is different at different points in a section.
- Quantity of warping/distortion at corresponding point is same at all the cross-section if shear force is constant along the length but it varies if shear force varies along the length.

Note:-

However the effect of warping on Normal stress is considered negligible for the span to depth ratio of beams normally used in field thus, even if shear force is acting on the beam, we consider it to be case of pure Bending.

Flexure formula



- Strain in the element "gh" at the distance of "y" from neutral plane.

$$\epsilon = \frac{hi}{gh} = \frac{y d\theta}{R d\theta}$$

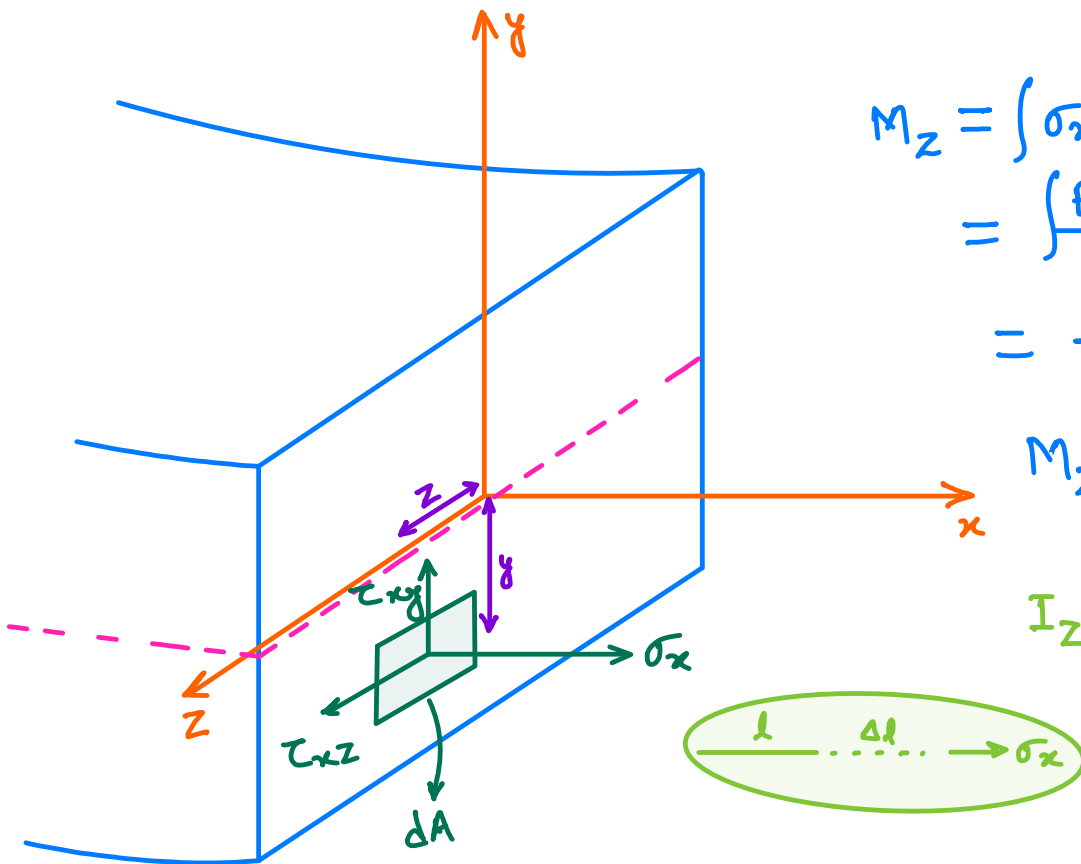
$$\epsilon = \frac{y}{R} \text{ --- (i)}$$

Note:-

variation of strain is linear wot the depth. ($\epsilon \propto y$)

— Since Hooke's law is valid, Bending Stress, $\sigma = E\epsilon$

$$\sigma = E \frac{y}{R} \text{ --- (ii)}$$



$$\begin{aligned} M_z &= \int \sigma_x dA y \\ &= \int \frac{E y}{R} \cdot y dA \\ &= \frac{E}{R} \int y^2 dA \end{aligned}$$

$$M_z = \frac{EI_z}{R} \text{ --- (iii)}$$

I_z = Moment of inertia about which Bending occurs.

from (i), (ii) & (iii)

$$\boxed{\frac{M_z}{I_z} = \frac{\sigma_x}{y} = \frac{E}{R}} \text{ --- (A)}$$

Also, $\sum F_x = 0$

$$\int \sigma_x dA = 0 \Rightarrow \int \frac{E y}{R} dA = 0 \Rightarrow \frac{E}{R} \int y dA = 0$$

$$\Rightarrow \frac{E}{R} A \bar{y} = 0$$

$$\Rightarrow \boxed{\bar{y} = 0} \text{ --- (B)}$$

$$\boxed{\bar{y} = \frac{\int y dA}{\int dA} = \frac{\int y dA}{A}}$$

$\bar{y}$ = distance of Centroidal Axis from NA.

Note:-

If Hooke's law is valid, then Centroidal axis is N.A.

Also, $M_x = -(\tau_{xy} \cdot z + \tau_{xz} \cdot y) dA = 0$ ————— (C)

for section symmetrical about y-axis & having loading in symmetrical plane, the element under consideration will have symmetrically placed counter element hence, summation of moment becomes zero.

Note:-

Hence, for symmetrical section, if loading is in plane of symmetry, twisting moment is zero.

Also, $M_y = \int \sigma_x dA z \Rightarrow \int \frac{E y}{R} \cdot z dA \Rightarrow \frac{E}{R} \int y z dA$

$$M_y = \frac{E}{R} I_{yz}$$

$$[I_{yz} = \text{product of moment of inertia}]$$

Here $I_{yz} = 0$, Since y-axis is the axis of symmetry.

$$M_y = 0 \text{ ————— (D)}$$

Note:-

for equally strong section in tension & compression, maximum bending stress at a section depends upon section-modulus.

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\sigma = \frac{M}{(I/y)}$$

for $\sigma = \sigma_{\max}$, $y = y_{\max} \Rightarrow \sigma_{\max} = \frac{M}{(I/y_{\max})} = \frac{M}{Z}$

Here Z = Section modulus about bending axis.

- Here maximum Bending moment resisted by a section without undergoing failure is called Moment of Resistance.

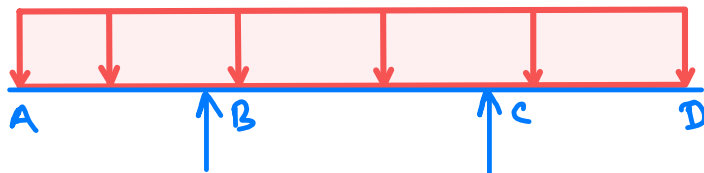
$$\frac{M}{I} = \frac{\sigma}{y}$$

$$M = \sigma Z$$

Hence, $MR = \sigma_{\text{permissible}} Z$

or, $MR = \frac{\sigma_y}{FOS} \cdot Z$

Hence, larger is the Section modulus, Stronger is the beam.

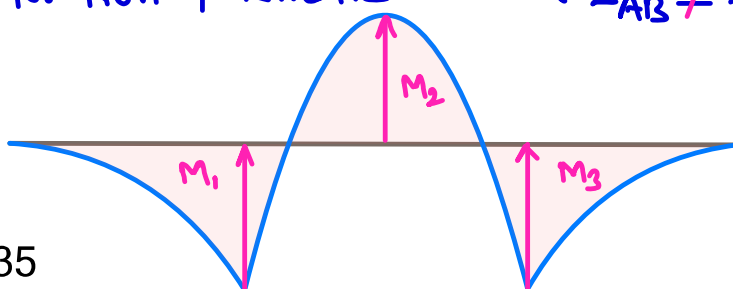


Prismatic

$\rightarrow MOR = ?$
 $M_{\max}$ taken to design

Non-prismatic beam.

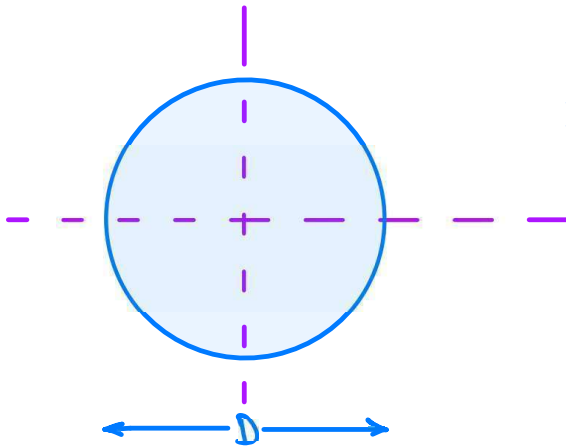
- for prismatic beam design: $Z_{AB} = Z_{BC} = Z_{CD}$
- for Non-prismatic " " : $Z_{AB} \neq Z_{BC} \neq Z_{CD}$



AB Section — M_1
 BC " — M_2
 CD " — M_3 } Taken to design

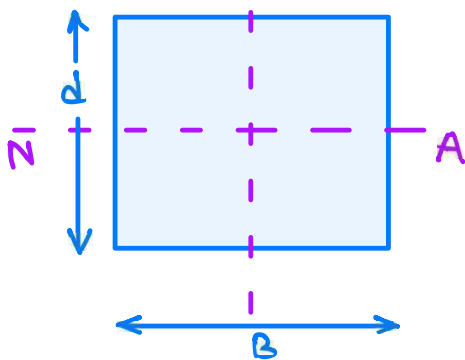
Section modulus (Z)

a.)



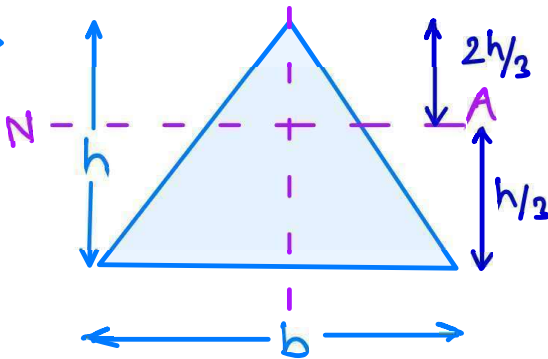
$$Z = \frac{I}{y_{\max}} = \frac{\frac{\pi D^4}{64} \cdot \frac{2}{D}}{\frac{D}{2}} = \frac{\pi D^3}{32} = \frac{A \cdot D}{8}$$

b.)



$$Z = \frac{I}{y_{\max}} = \frac{\frac{B h^3}{12} \cdot \frac{2}{h}}{\frac{h}{2}} = \frac{B h^2}{6} = \frac{B \cdot h \cdot h}{6} = \frac{A h}{6}$$

c.)

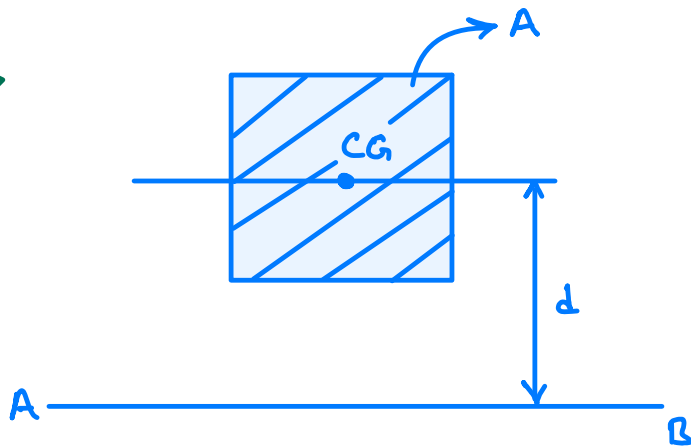


$$Z = \frac{I}{y_{\max}} = \frac{\frac{b h^3}{36} \cdot \frac{3}{2h}}{\frac{h}{3}} = \frac{b h^2}{24} = \frac{b h}{2} \cdot \frac{h}{12} = \frac{A h}{12}$$

NOTE:-

following theorems are available to compute MOI about different axis :-

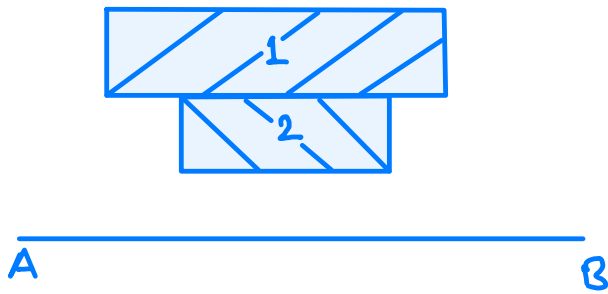
a.>



$$I_{AB} = I_{C.G} + Ad^2$$

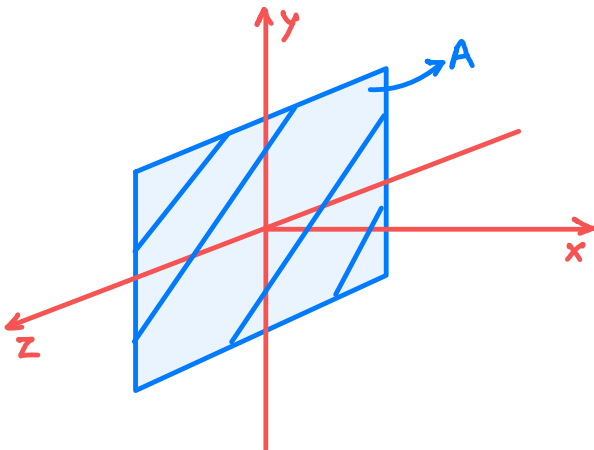
(Parallel Axis theorem)

b.>



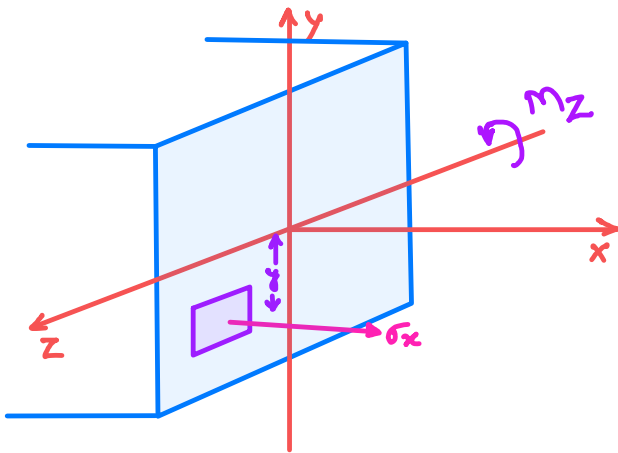
$$I_{AB} = I_{1AB} + I_{2AB}$$

c.> perpendicular Axis theorem



$$I_{xx} = I_{yy} + I_{zz}$$

Anticlastic Curvature



- when beam bends about z-axis the radius of curvature is "R" and longitudinal strain $\epsilon_x = \frac{y}{R}$
- Now due to poisson's effect strain would also be observed in lateral direction.

$$\epsilon_z = -\mu \epsilon_x$$

$$\epsilon_z = -\mu \frac{y}{R} \quad \dots (i)$$

- Here, -ve sign indicates lateral strain would be opposite of longitudinal strain

hence below N.A., $\epsilon_z = \text{Compressive}$

and, above N.A., $\epsilon_z = \text{Tensile}$

- let the radius of curvature in transverse bending be R' .

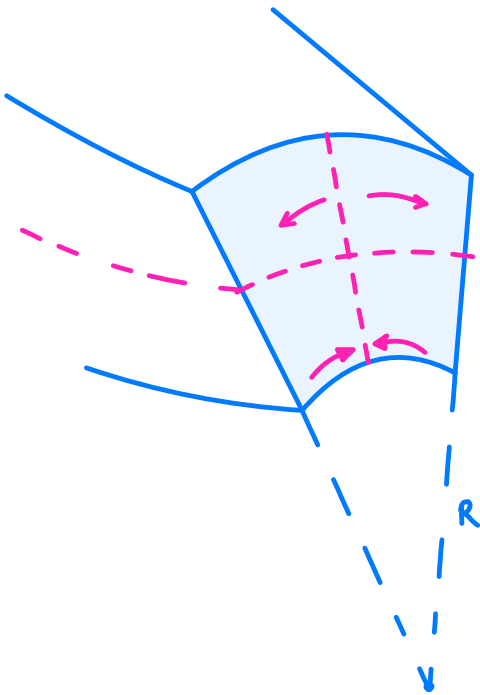
$$\epsilon_z = \frac{y}{R'} \quad \dots (ii)$$

from (i) & (ii)

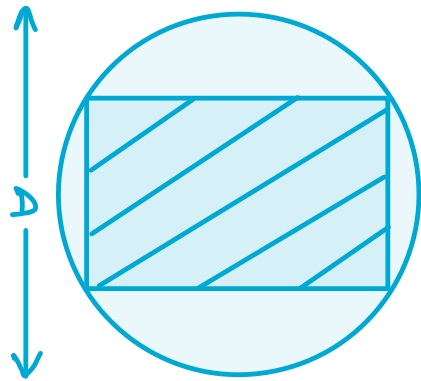
$$-\mu \frac{y}{R} = \frac{y}{R'}$$

$$R' = -\frac{R}{\mu}$$

- Hence shape of section after bending is as follows:-



Q. > find the Strongest section that can be cut from a circle of diameter "d".



Solution

$$Z = \frac{xy^2}{6}$$

$$Z = \frac{x(D^2 - x^2)}{6}$$

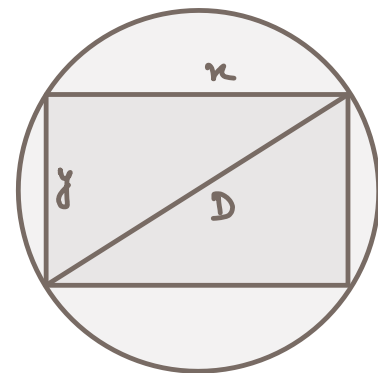
for max^m Z,

$$\frac{dZ}{dx} = 0 \Rightarrow \frac{d[x(D^2 - x^2)]}{dx} = 0 \Rightarrow D^2 - 3x^2 = 0$$

$$x = \frac{D}{\sqrt{3}}$$

$$y = \sqrt{\frac{2}{3}} D$$

$$Z_{\max} = \frac{D^3}{9\sqrt{3}}$$



$$x^2 + y^2 = D^2$$

$$y^2 = D^2 - x^2$$

Q. > Compare the flexural strength of the following beam sections of equal weight, same material with equal length of the beam.

i) I-Section of 300 mm X 160 mm wide flanges with 20 mm thick and web 16 mm thick

ii) Rectangular section having depth 2 times with width

iii) circular solid section.

Solution:- Flexural strength, $M_Z = \sigma_{\text{permissible}} Z$

$$M_Z \propto Z$$

Z - axis
Z - Section modulus

$$Wt. = \text{Volume} \times \text{Unit wt.}$$

$$Wt_I = Wt_R = Wt_C$$

$$V_I = V_R = V_C$$

$$V_I = V_R = V_C$$

$$A L_I = A L_R = A L_C$$

$$A_I = A_R = A_C$$

$$1) \quad I = \frac{160(300)^3}{12} - \frac{144(260)^3}{12}$$

$$I = 149 \times 10^6 \text{ mm}^4$$

$$Z_I = \frac{149 \times 10^6}{150} = 9.933 \times 10^5 \text{ mm}^3$$

$$11) \quad A_I = 160 \times 300 - 144 \times 260 =$$

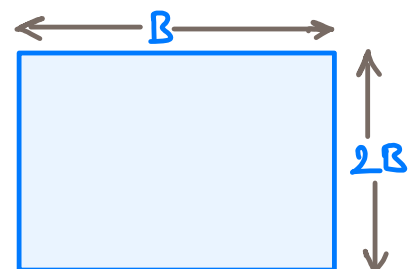
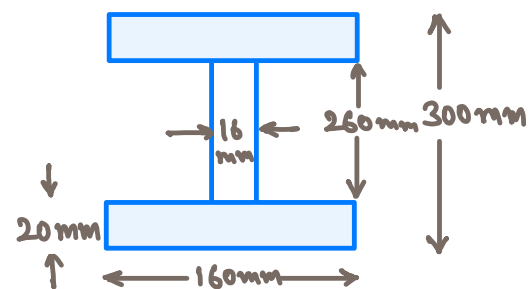
$$A_I = 10560 \text{ mm}^2$$

$$A_I = A_R$$

$$10560 = B \cdot 2B$$

$$\Rightarrow B = 72.66 \text{ mm}$$

$$D = 145.32 \text{ mm.}$$



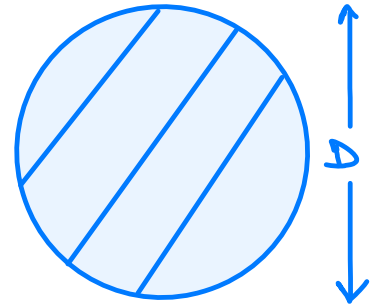
$$Z_R = \frac{BD^2}{6} = \frac{72.66 \times (145.32)^2}{6} = 2.5 \times 10^5 \text{ mm}^3$$

iii> $A_c = A_I = 10560 \text{ mm}^2$

$$\frac{\pi D^2}{4} = 10560$$

$$D^2 = \sqrt{\frac{10560 \times 4}{\pi}}$$

$$D = 116 \text{ mm}.$$



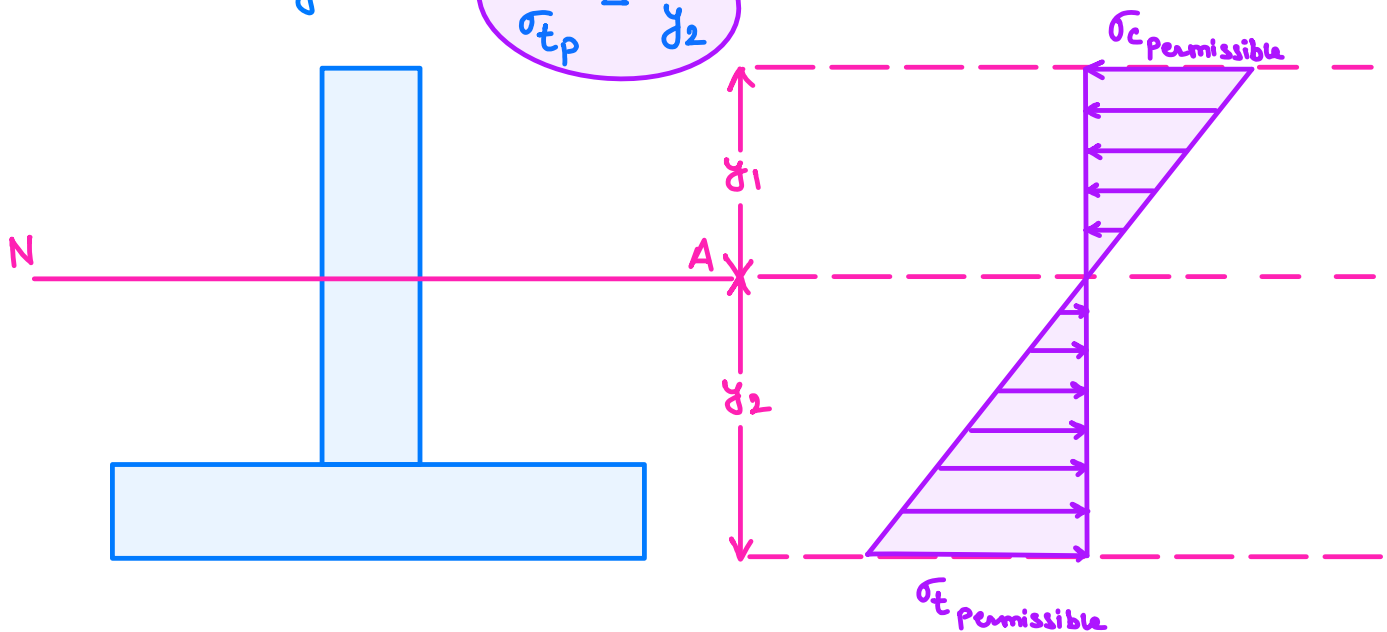
$$Z_c = \frac{\pi D^3}{32} = \frac{\pi \times (116)^3}{32} = 1.5 \times 10^5 \text{ mm}^3$$

Note:-

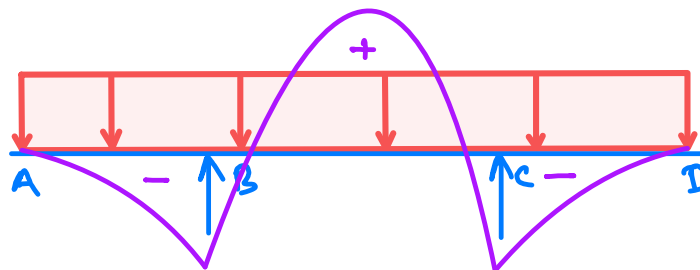
- Since the flexural strength of the section depends on section modulus (Z), that is further dependent on MoI (I).
- Hence, for given y , Z is more if I is more. and I is more if more area is located away from N.A.
- Thereby, Flexural strength of I-section is more than Rectangular section which is further more than solid circular section.

Section not Equally Strong in Tension and Compression:-

- In this case section dimension is selected such that the ratio of distances from N.A. to Extreme fibres in tension and compression is exactly same as the ratio of allowable stresses in tension and compression.
- Under this provision, the section is most economically utilised as in this case maximum permissible tensile and compressive stress at a section would reach simultaneously and $\frac{\sigma_{cp}}{\sigma_{tp}} = \frac{y_1}{y_2}$

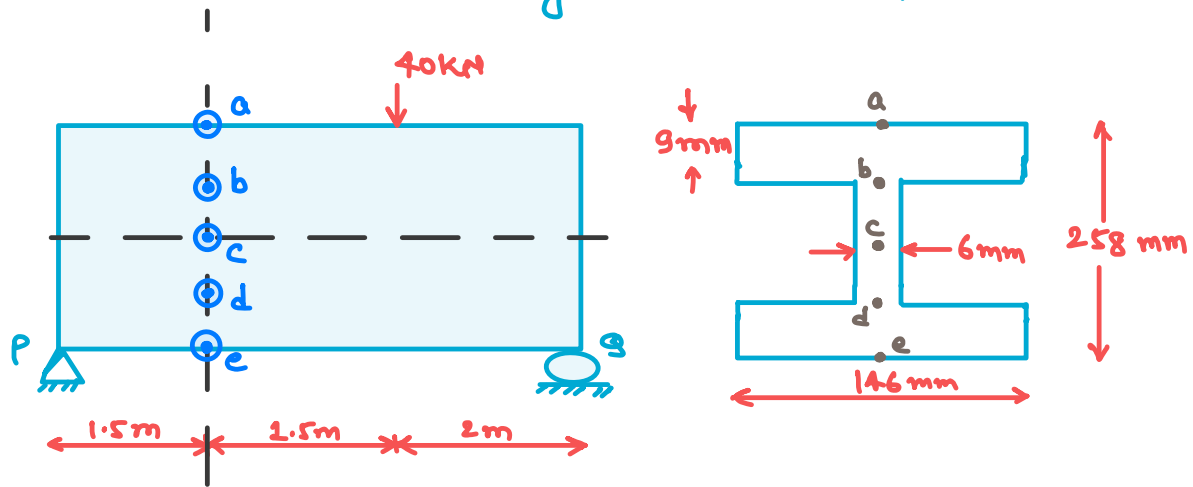


eg:-

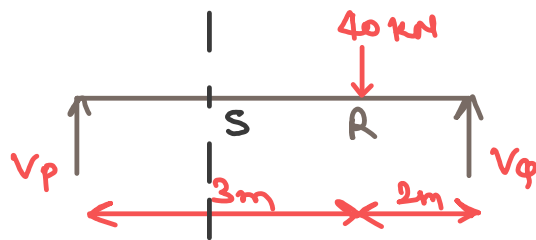


- However if there is reversal of curvature it is not necessary that maximum Bending tensile & compressive stress will occur at same section in such case it will not be possible to utilise the material economically.

Q.7 A simply supported beam having I-Cross-section of length 5m in which transverse load of 40kN is acting as shown in figure. Find out the bending stresses at point a, b, c, d and e.



Solution :-



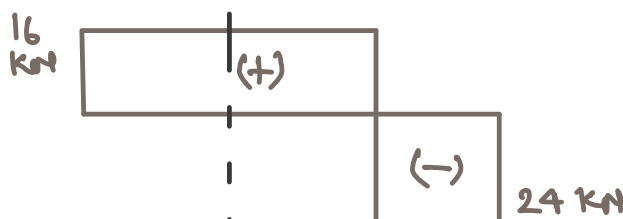
$$\sum f_y = 0 \Rightarrow V_p + V_q = 40$$

$$\sum M_p = 0 \Rightarrow -V_q \cdot 5 + 40 \times 3 = 0$$

$$V_q = 24 \text{ kN}$$

$$V_p = 16 \text{ kN}$$

SFD $\Rightarrow$



$$\Delta Bm = \text{area of SFD}$$

$$Bm_R - Bm_p = 16 \times 3$$

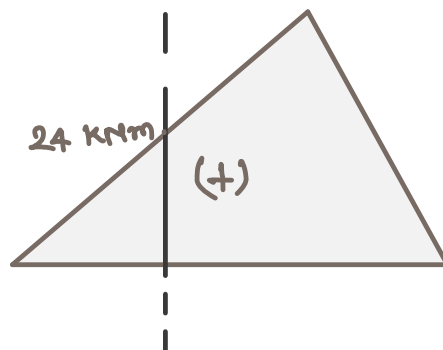
$$Bm_R = 48 \text{ kN}$$

$$\Delta Bm = \text{area of SFD}$$

$$Bm_s - Bm_p = 16 \times 1.5$$

$$Bm_s = 24 \text{ kN-m}$$

BMD $\Rightarrow$



$$I = \frac{146(258)^3}{12} - \frac{140(140)^3}{12}$$

$$I = 4.766 \times 10^7 \text{ mm}^4$$

$$\sigma = \frac{M}{I} y \Rightarrow$$

$$\text{At point "a", } \sigma_a = \frac{-My_a}{I} = \frac{-24 \times 10^3 \times 10^3 \times \frac{258}{2}}{4.766 \times 10^7}$$

$$\sigma_a = -64.95 \text{ N/mm}^2 \quad (-ve \rightarrow \text{comp.})$$

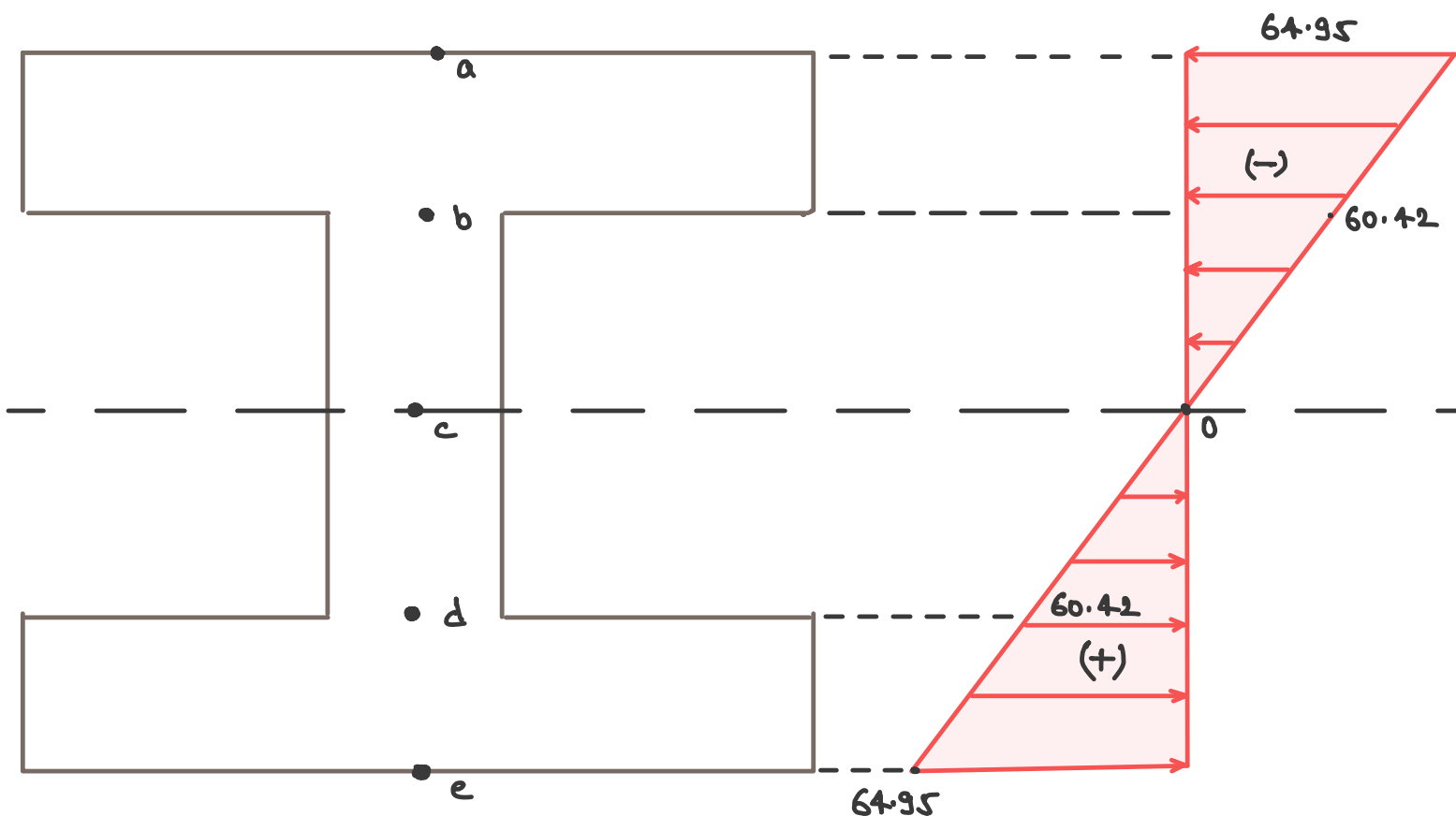
$$\text{Similarly at point "e", } \sigma_e = \sigma_a = 64.95 \text{ N/mm}^2$$

$$\text{At point "b", } \sigma_b = \frac{-My_b}{I} = \frac{-24 \times 10^6 \times \left(\frac{258}{2} - 9\right)}{4.766 \times 10^7}$$

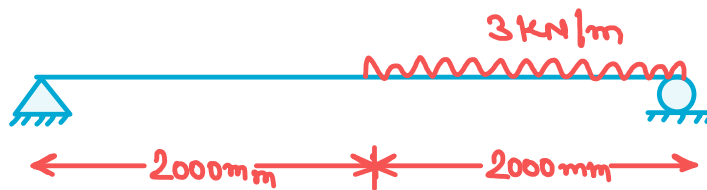
$$\sigma_b = -60.30 \text{ N/mm}^2$$

$$\text{Similarly at point "d", } \sigma_d = 60.30 \text{ N/mm}^2$$

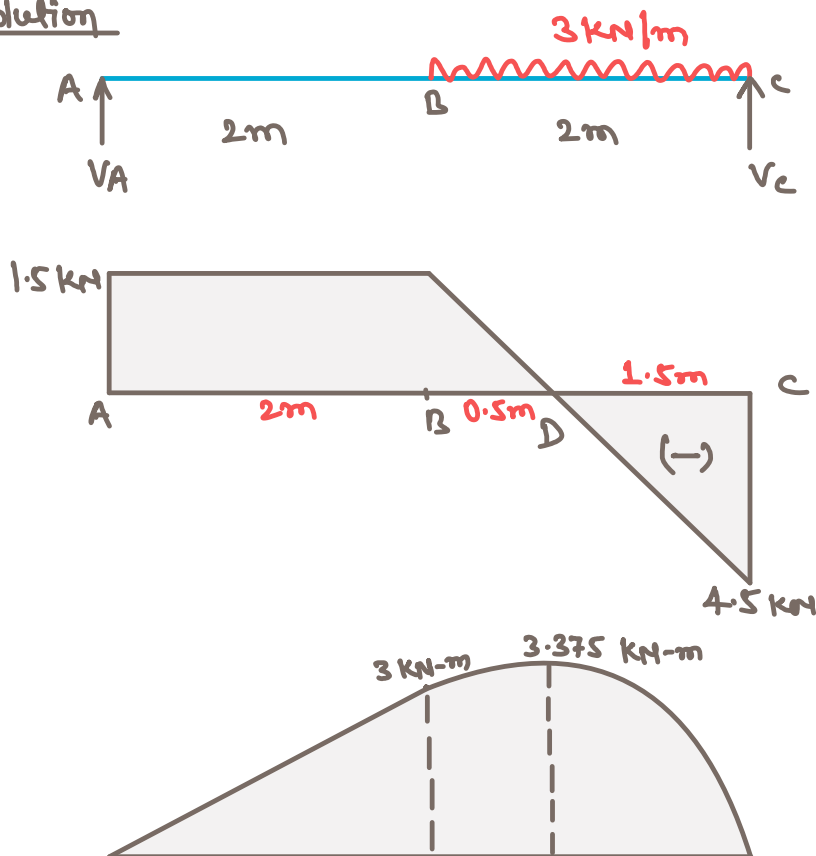
$$\sigma_c = \frac{My_c}{I} = 0$$



- Q. > A horizontal beam has a loading pattern as shown in fig. A beam is of Rectangular cross-section with width of 30 mm & height of 100 mm. Determine
- Maximum B.M and its location from the left end.
 - Maximum magnitude of Bending Stress.



Solution



$$\therefore B.M_{max} = 2.5m$$

$$\begin{aligned} \text{b. > } \sigma_{max} &= \frac{M_{max} \cdot y_{max}}{I} \\ \sigma_{max} &= \frac{3.375 \times 10^6 \times 50}{\frac{30(100)^3}{12}} \\ &= 67.5 \text{ N/mm}^2 \end{aligned}$$

$$\begin{aligned} \text{a. > } \sum F_y &= 0, \quad V_A + V_C = 6 \text{ kN} \\ \sum M_A &= 0, \end{aligned}$$

$$-V_C \times 4 + 3 \times 2 \times 3 = 0$$

$$V_C = 4.5 \text{ kN}$$

$$V_A = 1.5 \text{ kN}$$

$\Delta SF = \text{Area of loading diag}$

$$SF_C - SF_B = -3 \times 2$$

$$SF_C = -6 + 1.5 = -4.5 \text{ kN}$$

$\Delta SF = \text{Area of loading diag}$

$$SF_D - SF_B = -3 \cdot x$$

$$0 - 1.5 = -3x$$

$$x = 0.5 \text{ m.}$$

$\Delta BM = \text{Area of SFD}$

$$BM_D - BM_A = \frac{1}{2} (2 + 2.5) \times 1.5$$

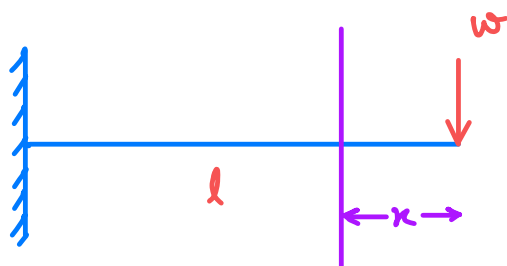
$$BM_D = 3.375 \text{ kNm}$$

→ BM_{max}

Beam of constant Strength / fully Stressed Beam.

- To minimise the quantity of the material and thereby to have the lightest possible beam, we can vary the dimensions [width, height or both] of the cross-section such that maximum stress of every section of the beam is equal to maximum allowable bending stress.
- The beam so obtained is termed as beam of constant strength or fully stressed beam.
- Eg:- Plate girder, Tapered poles.

Case-(i) Rectangular Beam loading



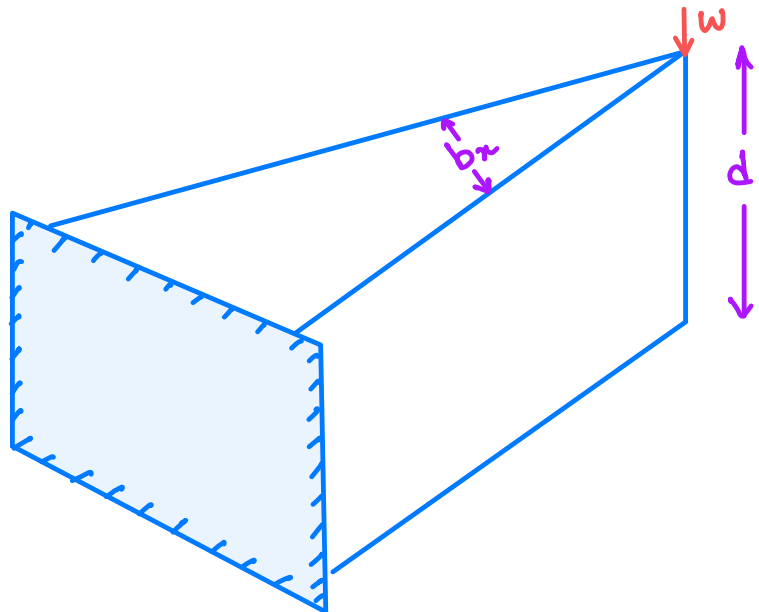
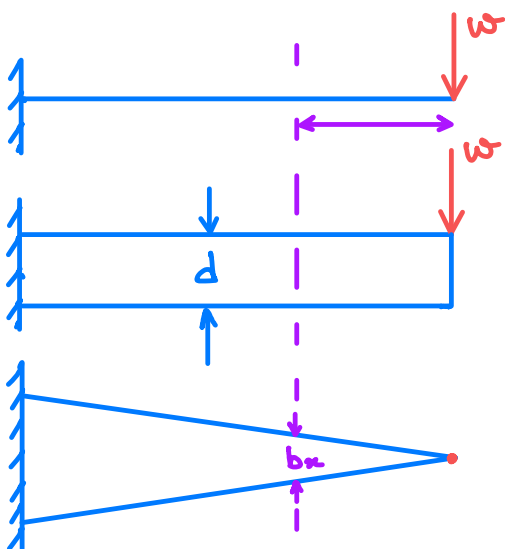
a> when depth is kept constant

$$\sigma = \sigma_{\text{permissible}} = \frac{M y_{\text{max}}}{I}$$

$$\sigma = \frac{W x (d/2)}{\frac{b_x d^3}{12}} = \frac{6 W x}{b_x d^2}$$

$$b_x = \frac{6 W}{\sigma d^2} \cdot x \quad (b_x \propto x)$$

- Hence, if depth is kept constant width would increase linearly.



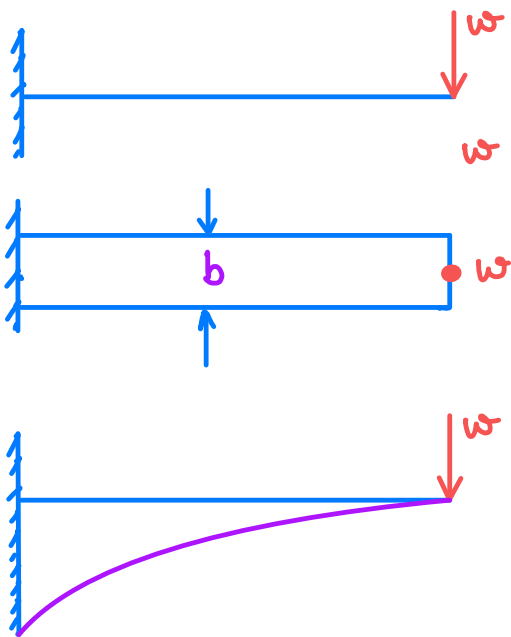
Case - (ii) : when width is constant

$$\sigma = \frac{M y_{\max}}{I}$$

$$\sigma_{\max} = \frac{W x \frac{dx}{2}}{\left(\frac{b d_x^3}{12}\right)} = \frac{6 W x}{b d_x^3}$$

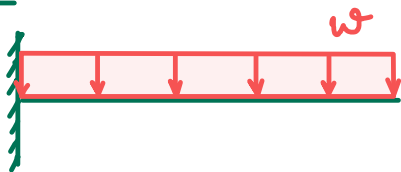
$$dx = \sqrt{\frac{6 W x}{\sigma b}}$$

$$(dx \propto \sqrt{x})$$



→ If width is constant, thickness varies parabolically with distance from free end.

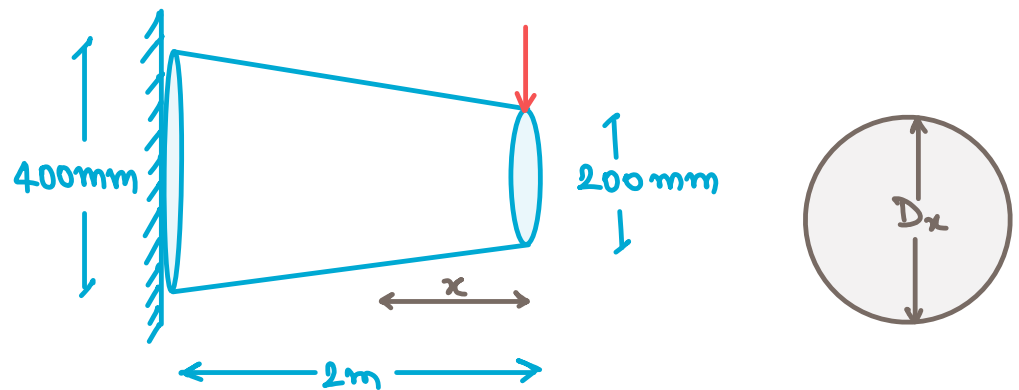
Note :-



- Here if b is constant, $d_x \propto x$.
and if d is constant, $b_x \propto x^2$.

- If simply supported beam is analysed for similar loading condition the results are also same as that of cantilever beam.

Q. > A cantilever of circular solid cross-section is fixed at one end and it carries a concentrated load at the free end. The dia of the free end is 200 mm and increases uniformly to 400 at fixed end of length 2m as shown in figure. At what distance from free end, the bending stress in the cantilever is maximum. Calculate Bending stress if $P = 30 \text{ kN}$.



Solution :-

Let at "x" mm distance from free end σ is max.

$$\sigma_x = \frac{M y_{\text{max}}}{I_x} = \frac{P x D_x / 2}{\left(\frac{\pi D_x^4}{64} \right)} = \frac{32 P x}{\pi D_x^3}$$

$$D_x = Kx + c$$

$$\text{At } x = 0, \quad D_x = 200 \text{ mm}$$

$$D_x = 200 = 0 + c \quad c = 200 \text{ mm}$$

$$\text{At } x = 2000 \text{ mm}, \quad D_x = 400 \text{ mm}$$

$$400 = K(2000) + 200$$

$$K = \frac{200}{2000} = 0.1$$

$$D_x = 0.1x + 200 \quad (x \text{ mm})$$

$$\sigma_x = \frac{32 P x}{\pi (0.1x + 200)^3}$$

$$\text{for } \sigma_{\max} = \frac{d\sigma_x}{dx} = 0.$$

$$\frac{d}{dx} \left(\frac{32 P x}{\pi (0.1x + 200)^3} \right) = 0$$

$$\frac{32 P}{\pi} \left[\frac{(0.1x + 200)^3 \times 1 - x \times 3 (0.1x + 200)^2 \times 0.1}{(0.1x + 200)^6} \right] = 0$$

$$\frac{(0.1x + 200)^2}{(0.1x + 200)^6} \left[0.1x + 200 - 0.3x \right] = 0.$$

$$0.1x + 200 - 0.3x = 0.$$

$$0.2x = 200$$

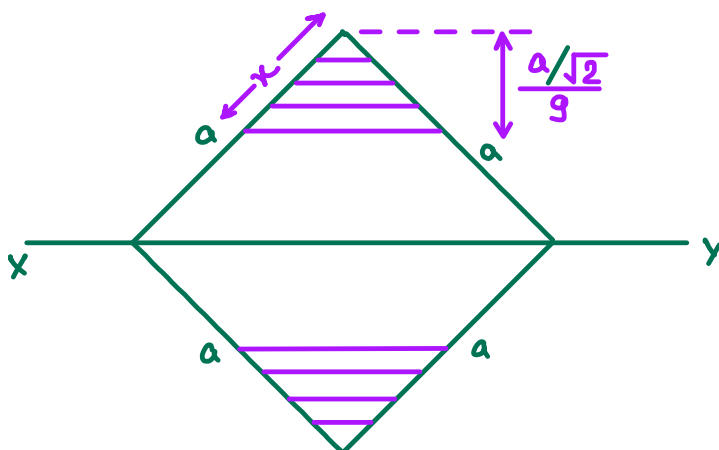
$$x = \frac{200}{0.2}$$

$$x = 1000 \text{ mm} / 1 \text{ m}$$

$$\sigma_{\max} = \frac{32 \times 30 \times 10^3 \times 1000}{\pi \times (0.1 \times 1000 \times 200)^3} = \underline{\underline{11.317 \text{ MPa}}}$$

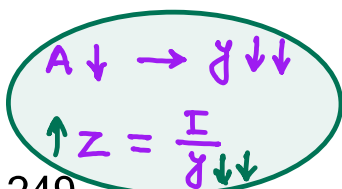
Note :-

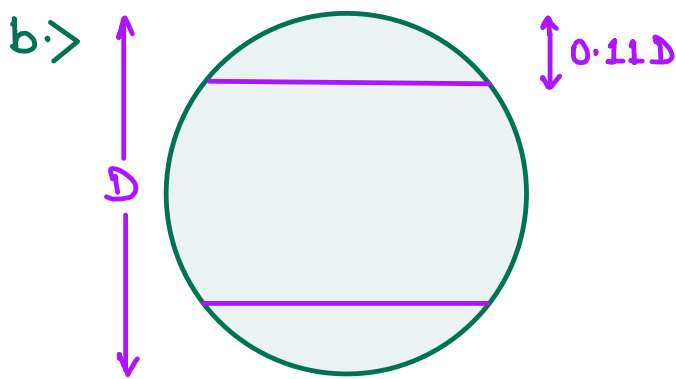
a.)



$$x = \frac{a}{9}$$

By cutting $x = \frac{a}{9}$ or depth of $0.078a$, the beam can be made stronger by approx: 5.35% .



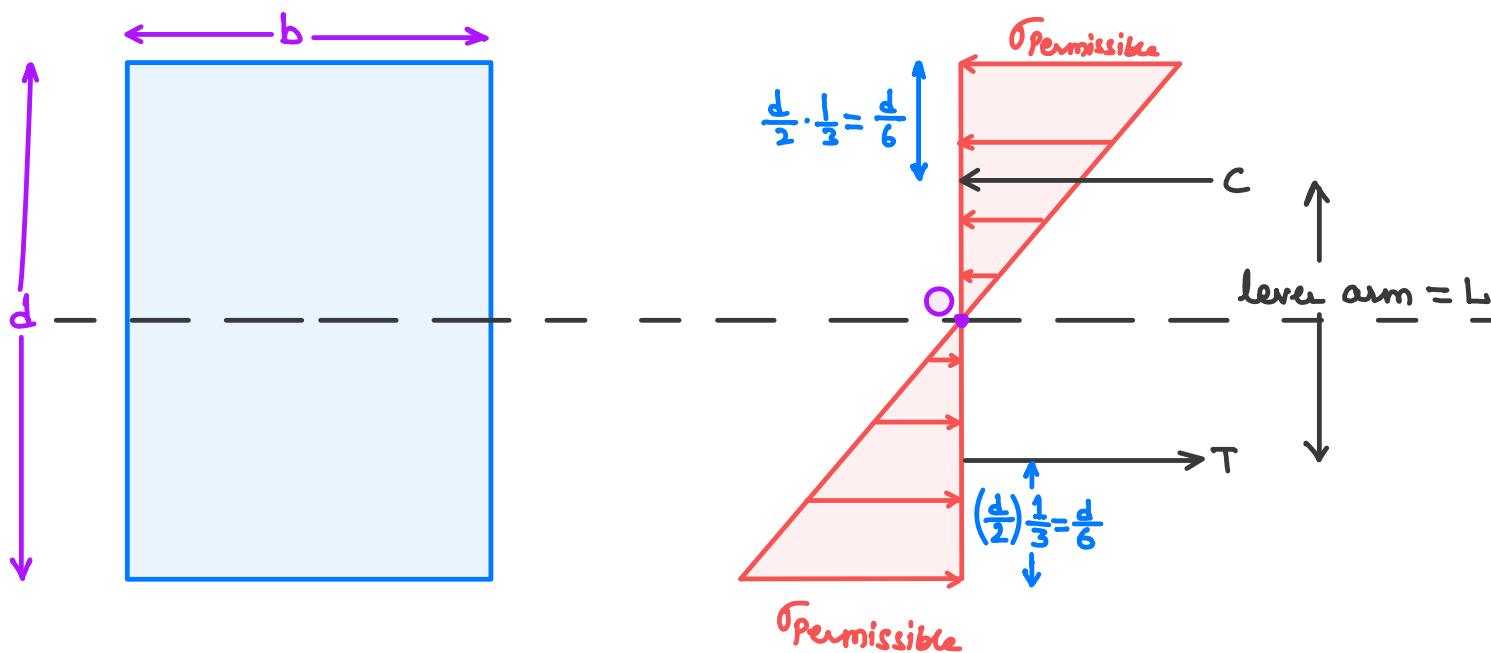


By cutting $0.11D$ from top and bottom strength can be increased by 0.7% in bending.

Analysis of Moment of Resistance

— It is the maximum bending moment resisted by a section without undergoing failure.

$$M_{OR}/M_R = \sigma_{\text{Permissible}} Z$$



$$M_R = \text{Compressive force (C)} \times \text{lever arm (L)}$$

or

$$M_R = \text{Tensile force (T)} \times \text{lever arm (L)}$$

$$M_R = C \times L = T \times L$$

$$- C = T = \frac{1}{2} \sigma_p \cdot \frac{d}{2} \cdot b = \frac{\sigma_p b d}{4}$$

or

$$- C = T = \sigma_{avg} \cdot \text{Area} \quad \text{Section.}$$

$$= \left(\frac{0 + \sigma_p}{2} \right) \times \left(b \cdot \frac{d}{2} \right) = \frac{\sigma_p b d}{4}$$

or

$$- C = T = \int \sigma_x A_x = \int_0^{d/2} \frac{2 \sigma_p x (b dx)}{d}$$

$$= 2 \sigma_p \frac{b}{d} \left| \frac{x^2}{2} \right|_0^{d/2}$$

$$C = T = \sigma_p \frac{b}{d} \frac{d^2}{4} = \frac{\sigma_p b d}{4}$$

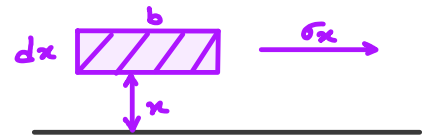
$$M_R = C/T \cdot L$$

$$= \sigma_p \frac{b d}{4} \cdot \left(d - \frac{2d}{6} \right)$$

$$\boxed{M_R = \frac{\sigma_p b d^2}{6}} \Rightarrow \sigma_p \cdot Z_{Reet.}$$

$$T = C = \text{Area of Stress distribution} \times \text{Width}$$

$$C = T = (\text{Stress})_{avg} \times \text{Area of Section}$$

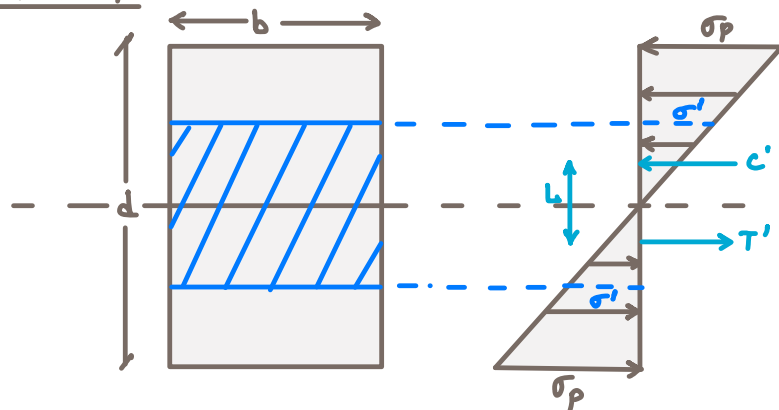


$$\frac{\sigma_p}{d/2} = \frac{\sigma_x}{x}$$

$$\sigma_x = \frac{2 \sigma_p \cdot x}{d}$$

Q. > If a rectangular beam of width "b" & depth "d" is subjected to B.M (M) at any section, then find out the B.M resisted by middle $\frac{1}{3}$ rd section.

Solution.



$$M = \sigma_p Z$$

$$M = \sigma_p \cdot \frac{bd^2}{6}$$

$$M' = C'L \text{ or } T'L$$

$$C' = T' = \left\{ \frac{1}{2} \sigma' \left(\frac{d}{3} \right) \cdot \frac{1}{2} \right\} \cdot b = \frac{\sigma' bd}{12}$$

$$C' = T' = \frac{\sigma_p}{3} \cdot \frac{bd}{12} = \frac{\sigma_p bd}{36}$$

$$L = \frac{d}{3} - 2 \cdot \frac{1}{3} \left(\frac{d}{3} \cdot \frac{1}{2} \right) = \frac{2d}{9}$$

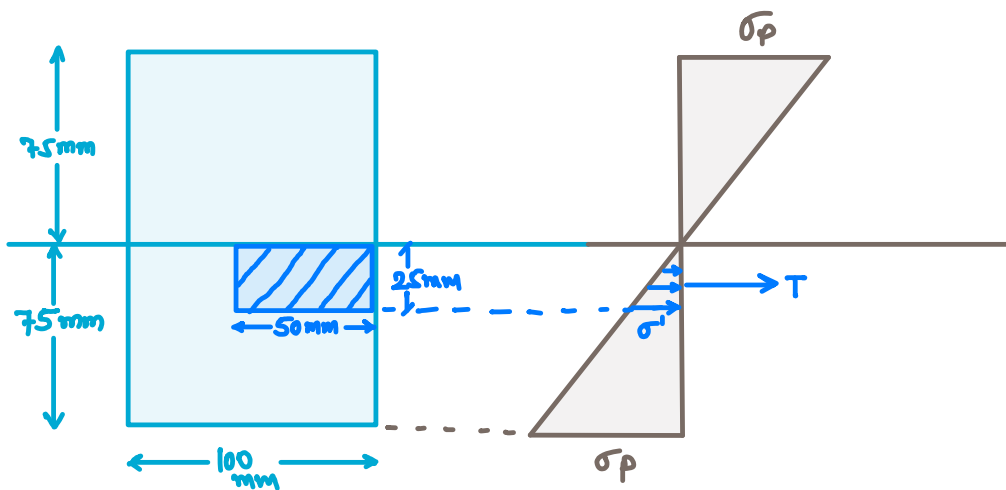
$$M' = \sigma_p \frac{bd}{36} \cdot \frac{2d}{9} = \sigma_p \frac{bd^2}{6} \cdot \frac{1}{27}$$

$$M' = \frac{M}{27}$$

$$\frac{\sigma_p}{d/2} = \frac{\sigma'}{\left(\frac{d}{3} \right) \cdot \frac{1}{2}}$$

$$\sigma' = \frac{\sigma_p}{3}$$

Q. > A beam with the cross-section given below is subjected to positive B.M causing compression at the top of 16 kN-m acting around the horizontal axis. The tensile force acting on the hatched area of the cross-section is ??



Solution

$$T = \frac{1}{2} \sigma' \times (25) \times (50)$$

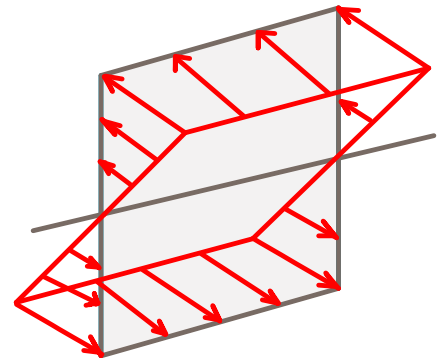
$$\frac{\sigma_p}{75} = \frac{\sigma'}{25} \Rightarrow \sigma' = \frac{\sigma_p}{3}$$

$$M = \sigma_p Z$$

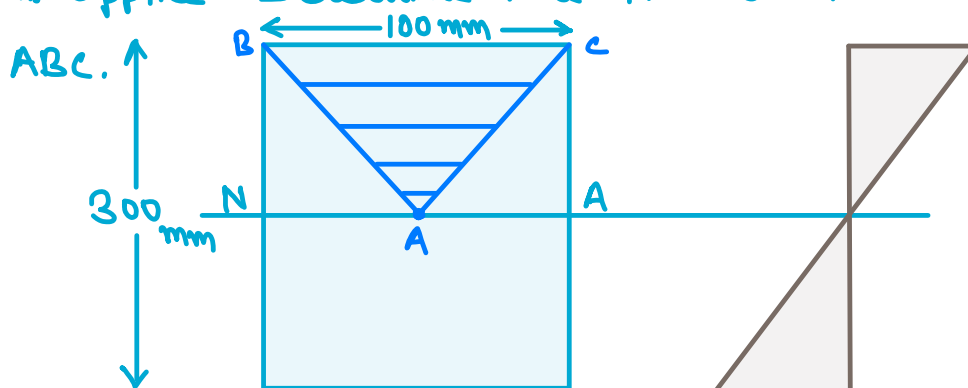
$$\sigma_p = \frac{M}{Z} = \frac{16 \times 10^3 \times 10^3}{100 \times \frac{(150)^2}{6}} = 42.67 \text{ N/mm}^2$$

$$T = \frac{1}{2} \times \frac{42.67}{3} \times 25 \times 50 \times 10^{-3}$$

$$T = 8.89 \approx 8.9 \text{ kN.}$$



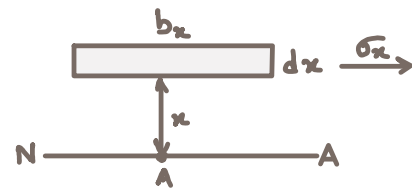
Q. > A beam with Rectangular section in which Sagging B.M 16 kNm is applied. Determine the Normal force developed by region ABC.



Solution:-

$$\begin{aligned}
 C &= \int \sigma_x \cdot b_x dx \\
 &= \int_0^{150} 10.67 \frac{x}{150} \times \frac{2}{3} x \cdot dx \\
 &= \frac{10.67 \times 2}{150} \int_0^{150} x^2 dx \\
 C &= 0.047 \left| \frac{x^3}{3} \right|_0^{150} \times 10^{-3}
 \end{aligned}$$

$$C = 53.3 \text{ KN.}$$



$$\frac{100}{150} = \frac{b_x}{x}$$

$$b_x = \frac{2}{3} x.$$

$$\frac{\sigma_p}{150} = \frac{\sigma_x}{x}$$

$$\sigma_x = \frac{\sigma_p \cdot x}{150}$$

$$\sigma_p = \frac{M}{Z}$$

$$= \frac{16 \times 10^6}{100 \cdot \frac{(300)^2}{6}}$$

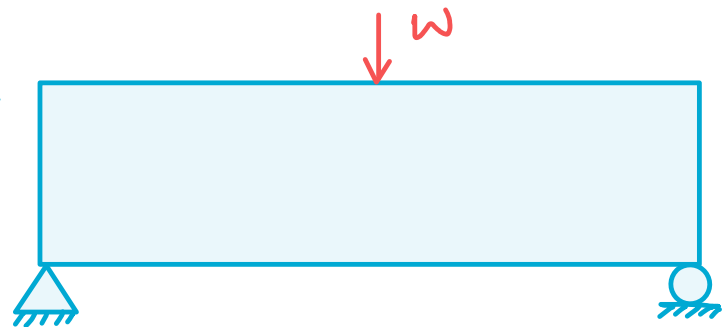
$$\sigma_p = 10.67 \text{ N/mm}^2$$

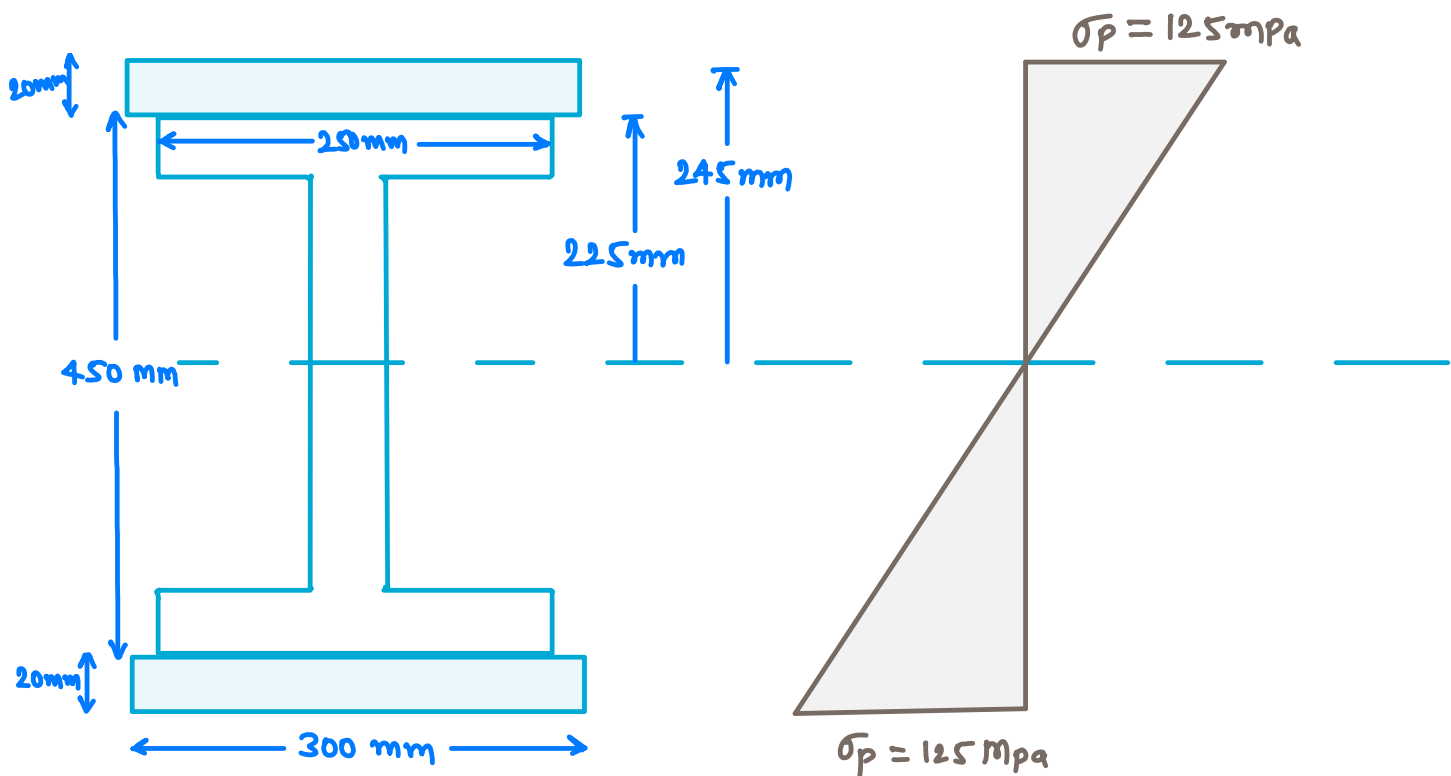
$$\sigma_x = 10.67 \times \frac{x}{150}$$

Q. A Simply Supported beam made on rolled Steel Joist with I-Cross-section $450 \times 250 \text{ mm}$. has a span of 5 m . It carries a load of $W \text{ KN}$. The flanges are strengthened by $300 \times 20 \text{ mm}$ plate. The moment of inertia of the Joist about the N.A. is 35060 cm^4 . Calculate

(a) The greatest central load the will carry if the bending stress in $300 \times 20 \text{ mm}$ plate is not to exceed 125 MPa .

(b) compute the minimum length of 300 mm plate required to resist the maximum bending stress in the flanges of the joist to 125 MPa .





$$\sigma_p = \frac{M_{\max}}{I} y_{\max}$$

$$\sigma_p = 125 \text{ MPa}$$

$$M_{\max} = \frac{wl}{4}$$

$$I = I_{\text{Joist}} + 2 I_{\text{Plate}}$$

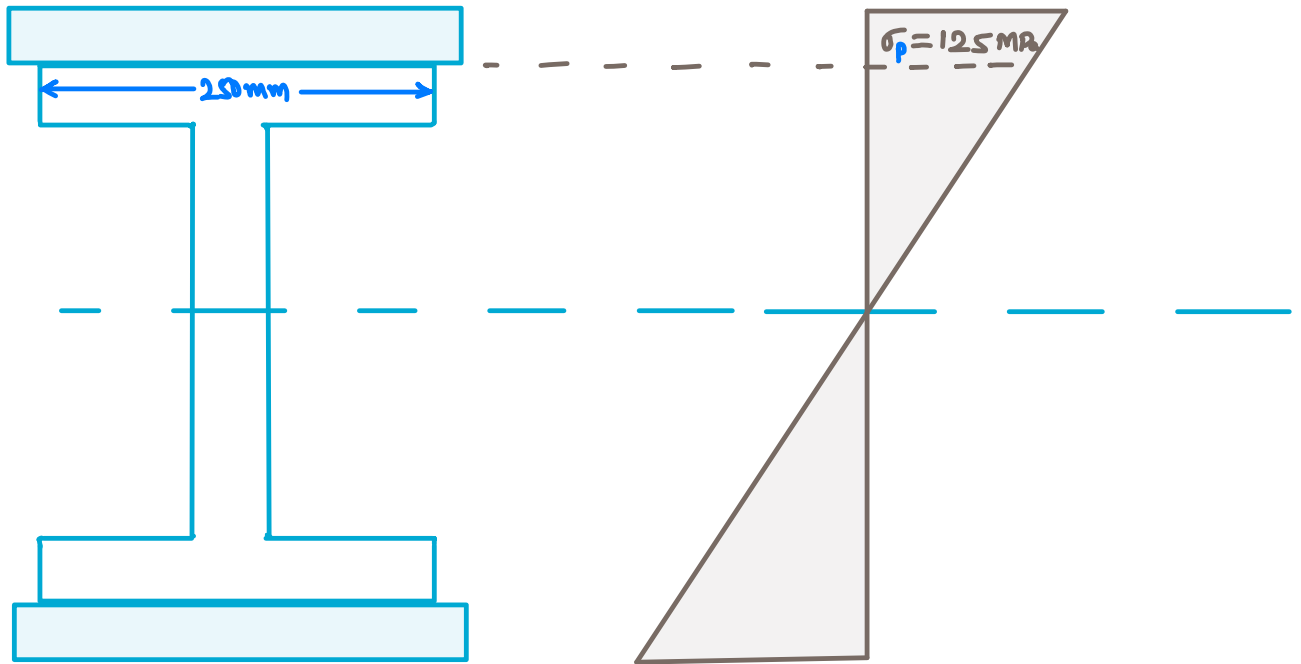
$$= 35060 \times 10^4 + 2 \left[300 \times \frac{20^3}{12} + 300 \times 20 \times \left(225 + \frac{20^2}{2} \right)^2 \right]$$

$$I = 1.013 \times 10^9 \text{ mm}^4$$

$$\sigma_p = \frac{M_{\max}}{I} \cdot y_{\max} = \frac{\frac{w \times 5}{4} \times 10^6 \times 245}{1.013 \times 10^9} = 125$$

$$w = 413.46 \text{ N or } 0.413 \text{ kN.}$$

(b)



$$R_A = R_B = \frac{w}{2} \text{ (from force equation)}$$

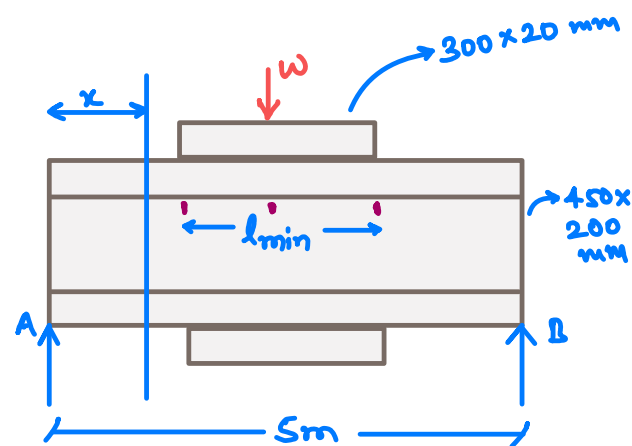
$$BM_x = \frac{w}{2} x$$

$$\sigma_p = \frac{BM_x}{I_{\text{Joist}}} y_{\text{max}}$$

$$125 = \frac{413.46/2 \cdot x \times 10^3}{35060 \times 10^4} \times 225$$

$$x = 942 \text{ mm} = 0.942 \text{ m}$$

$$\begin{aligned} l_{\text{min}} &= 5 - 2x \\ &= 5 - 2 \times 0.942 \\ &= \underline{\underline{3.116 \text{ m}}} \end{aligned}$$



longitudinal view.

only 3 section $\rightarrow$ 125 mpa will occur
other than this $< 125 \text{ mpa}$.

- Connected Composite Section is that for which component material will have the same radius of bending i.e. they will bend together

$$M_{R1} = \sigma_1 Z_1 = \frac{\sigma_1 b_1 d_1^2}{6}$$

$$M_{R2} = \sigma_2 \text{Max } Z_2 = \sigma_{2\text{max}} \frac{b d^2}{6}$$

Total moment to be Resisted, $M = M_{R1} + M_{R2}$

Note:-

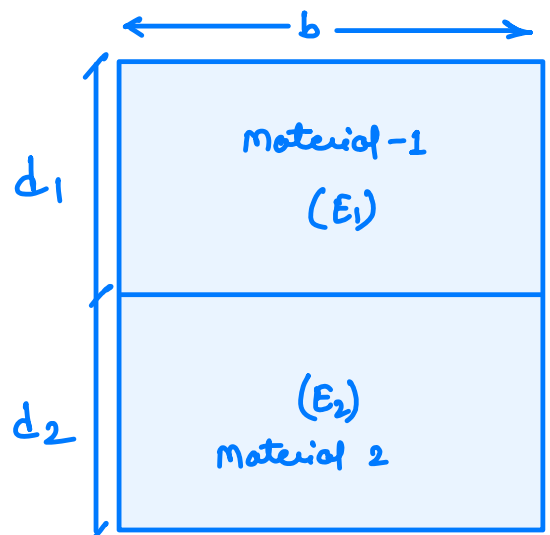
The above Symmetrical Sections are also termed as Flitched beams.

ii) Unsymmetrical Section

- when 2 or more materials are placed over each other and are rigidly connected it is termed as Unsymmetrical section
- Analysis of this case is done as follows:-

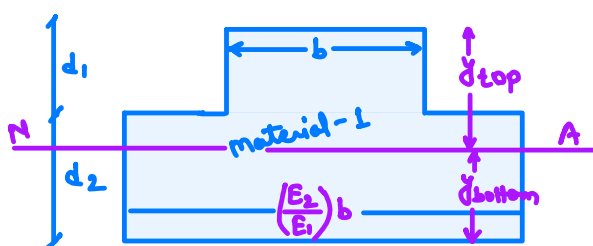
(i) N.A. is found out by drawing the transformed section.

- Transformed section is drawn by converting one material into the another equivalent material equivalent material.

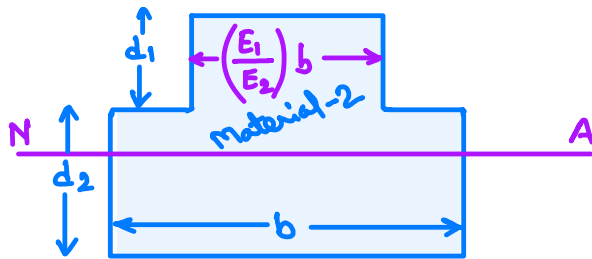


$$E_2 > E_1$$

(c) material 2 converted to 1.



b.) material 1, converted to 2.



- for finding the equivalent section only width is changed and not the depth.
- for transformed section N.A. is found by using the concept of moment of Area.

III.) From Neutral Axis Stress distribution is calculated for transformed section.

IV.) From stress distribution in transformed section stress in both the material is calculated by multiplying the stress of transformed section with width modification factor.

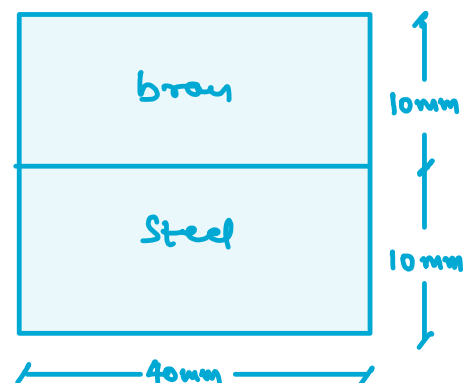
Q.) For the given rigidly connected unsymmetrical section find maximum load "P" that can be applied over it.

Assume $M_0 E_{\text{Steel}} = 2.05 \times 10^5 \text{ N/mm}^2$

$E_{\text{brass}} = 0.82 \times 10^5 \text{ N/mm}^2$

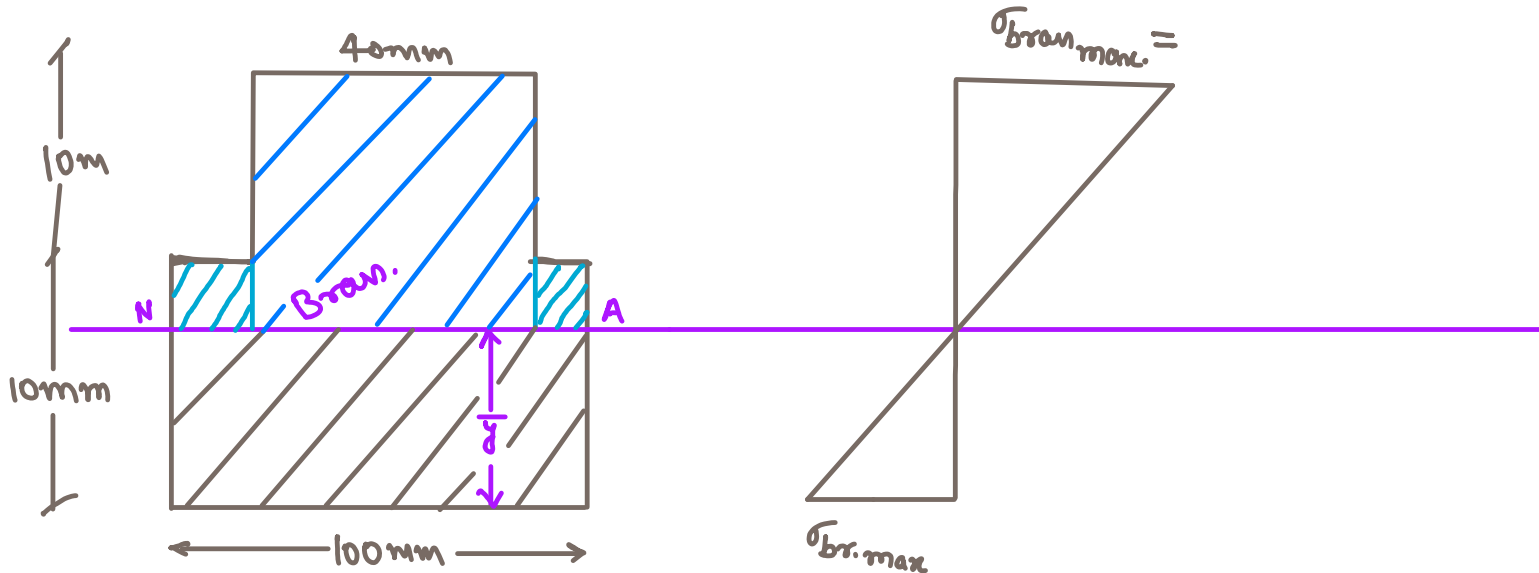
$\sigma_{\text{brass}} = 75 \text{ N/mm}^2$

$\sigma_{\text{St max}} = 112.5 \text{ N/mm}^2$



Solution : Transforming Steel into Brass.

$$b_{eq} = \frac{E_{st}}{E_{bran}} \cdot b = \frac{2.05 \times 10^5}{0.87 \times 10^5} \times b = 2.5b = 100 \text{ mm}$$



$$(40 \times 10 + 100 \times 10) \bar{y} = 40 \times 10 (10 + 5) + 100 \times 10 \times \frac{10}{2}$$

$$\bar{y} = 7.857 \text{ mm}$$

$$\boxed{bd^3/3}$$

$$I = \frac{1}{3} [100 \times 7.857^3 + 40 \times (10 + 10 - 7.857)^3 + 60 \times (10 - 7.857)^3]$$

$$= 40238 \text{ mm}^4$$

$$BM_{max} = \frac{Pl}{4} = \frac{P}{4} \text{ N-m}$$

$$\sigma_{br,max} = 75 = \frac{M}{I} y_{max}$$

$$75 = \frac{P/4 \times 10^2}{40238} \times (20 - 7.857)$$

$$P = 994.103 \text{ N}$$

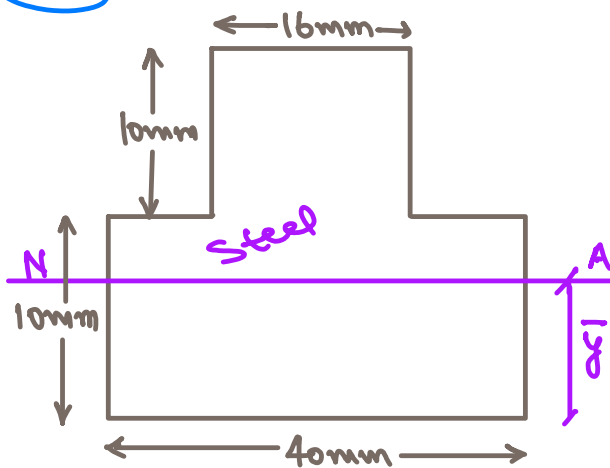
$$\sigma_{st,max} = 112.5 = \left(\frac{M}{I} y_{max} \right) \frac{E_{st}}{E_{bran}} = \left(\frac{P/4 \times 10^2}{40238} \times 7.857 \right) 2.5$$

$$P = 921.82 \text{ N}$$

$$P_{man} = \min(994.103, 921.82)$$

$$\underline{P_{man} = 921.82 \text{ N}}$$

(OR)



for converting brass into steel.

$$b_{eq} = \frac{E_{brass}}{E_{steel}} b = \frac{0.82}{2.05} b$$

$$= 0.4 b = 0.4 \times 40 = 16 \text{ mm}$$

$$(16 \times 10 + 10 \times 40) \bar{y} = 10 \times 16 \times 15 + 10 \times 10 \times 5$$

$$\bar{y} = 7.857 \text{ mm}$$

$$I = \frac{1}{2} \left[40 \times (7.857)^3 + 16 \times (20 - 7.857)^3 + 24 \times (10 - 7.857) \right] \leftarrow \begin{matrix} \text{base} \\ \text{about} \\ \text{N.A.} \end{matrix}$$

$$I = 16095.23 \text{ mm}^4.$$

$$\sigma_{br max.} = 75 = \left(\frac{M}{I} y_{max} \right) \frac{E_{br}}{E_{st.}}$$

$$75 = \left(\frac{Pl \times 10^3}{4 \times 16095.23} \times (20 - 7.857) \right) \times 0.4$$

$$P = 994.117 \text{ N}$$

$$\sigma_{st max} = 112.5 = \frac{M}{I} y_{max}$$

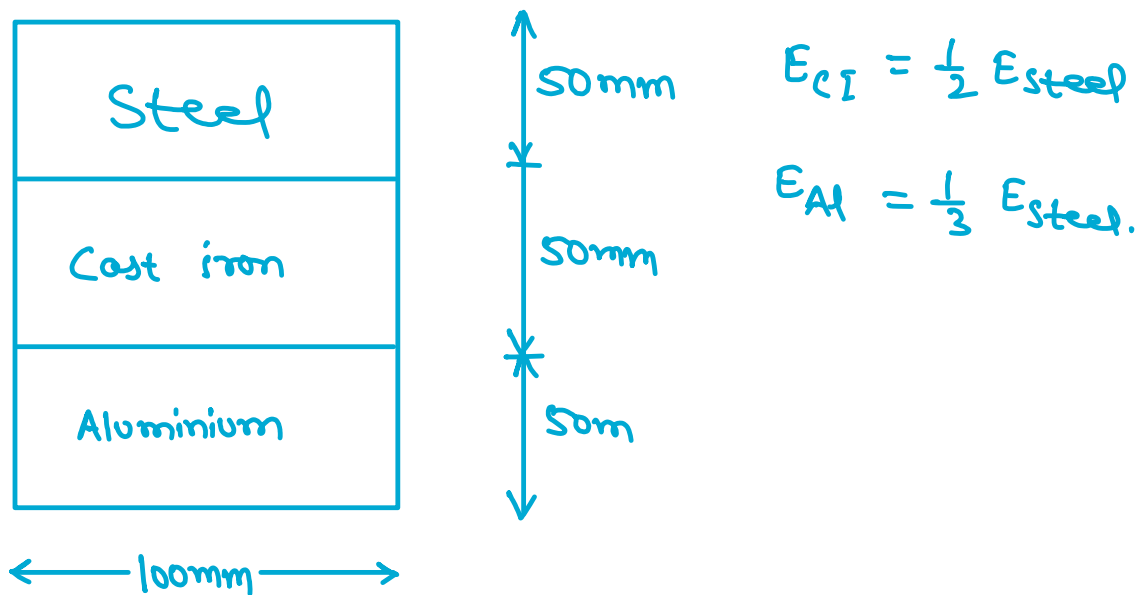
$$112.5 = \left(\frac{Pl}{4} \times \frac{10^3}{16095.23} \times 7.857 \right) \cdot 1 \rightarrow \frac{E_{steel}}{E_{steel}} = 1.$$

$$P = 921.82 \text{ N}$$

$$P_{max} = \min(994.117, 921.82)$$

$$P_{max} = 921.82$$

Q. > A Composite beam is made up of 3 metals of equal depth and width as shown. Determine the
 (a) position of NA from top surface for composite section and draw the Bending Stress and Strain distribution diagram for Composite Section.



Solution

$$\frac{E_s}{E_{Al}} = 3$$

$$E_{steel} = 3 E_{Al} \quad , \quad E_{st} = 2 E_{CI}$$

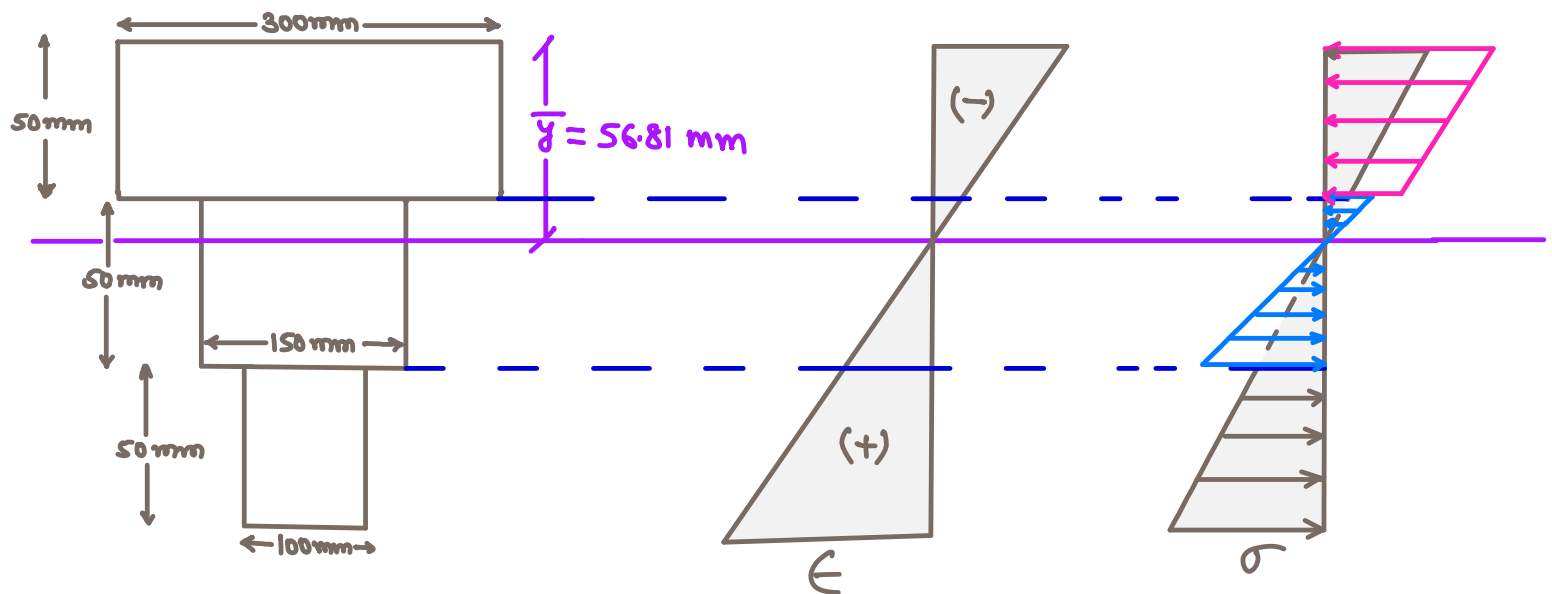
$$3 E_{Al} = 2 E_{CI}$$

$$\frac{E_{CI}}{E_{Al}} = 1.5$$

Converting Steel and Cast iron in equivalent Aluminium section

$$B_{Al \text{ for steel}} = 3 \times 100 = 300 \text{ mm.}$$

$$B_{Al \text{ for CI}} = 1.5 \times 100 = 150 \text{ mm.}$$



$$\bar{y} (300 + 150 + 100) 50 = (300 \times 25 + 150 \times 75 + 100 \times 125) 50$$

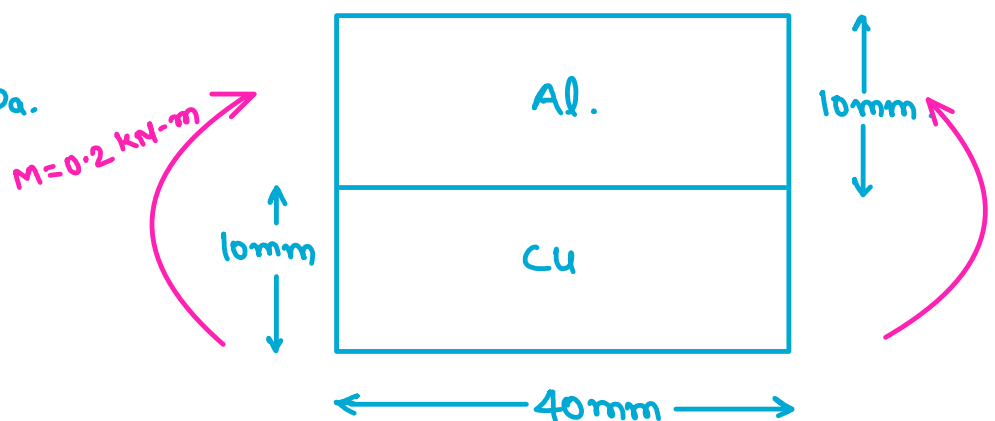
Assuming moment to be sagging.

$$\bar{y} = 56.81 \text{ mm.}$$

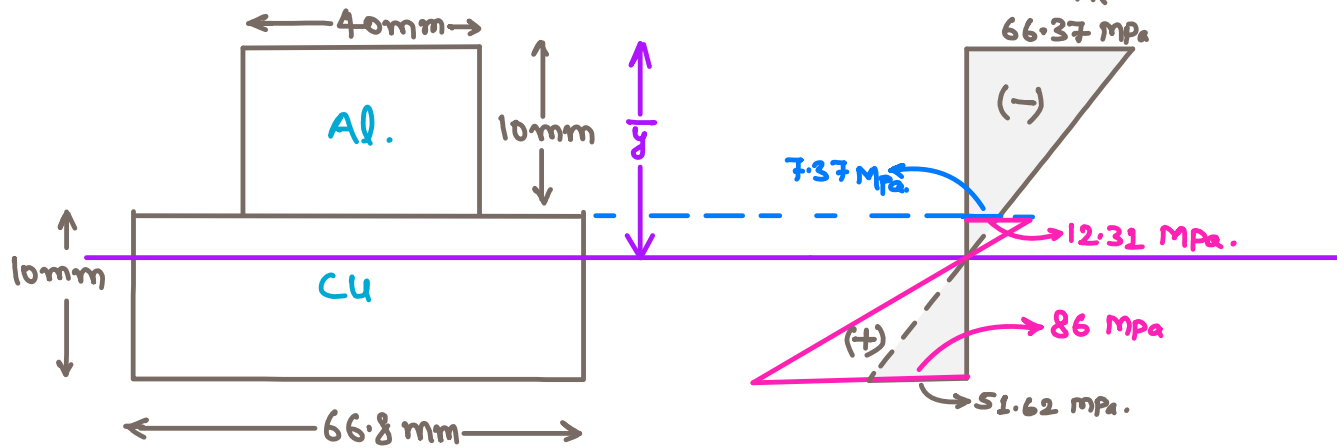
Q. A Strip of Copper with 40 mm width and 10 mm thickness is bonded with another Strip of Aluminium of same width and 20 mm depth. The strip is subjected to pure Bending of 0.2 kN-m. (Sagging) as shown in figure. Calculate the radius of Curvature of the Strip and maximum tensile and Compressive stresses developed in metals.

$$E_{Cu} = 10^5 \text{ mpa.}$$

$$E_{Al} = 0.6 \times 10^5 \text{ Mpa.}$$



Solution. width modification factor $= \frac{E_{cu}}{E_{Al.}} = \frac{105}{0.6 \times 10^5} = 1.67$



$$B_{Al.} \text{ for } Cu = 1.67 \times 40 = 66.8 \text{ mm.}$$

$$(40 + 66.8) 10 \cdot \bar{y} = (40 \times 5 + 66.8 \times 15) 10$$

$$\bar{y} = 11.25 \text{ mm from Top.}$$

$$\bar{y} = 8.75 \text{ mm from bottom.}$$

$$I = \frac{1}{3} [66.8 (20 - 11.25)^3 + 40 \times 11.25^3 + (66.8 - 40) (11.25 - 10)^3]$$

$$I = 3.39 \times 10^4 \text{ mm}^4$$

$$\text{Now, } \frac{M}{I} = \frac{E}{R} \Rightarrow R = \frac{EI}{M} = \frac{0.6 \times 10^5 \times 3.39 \times 10^4}{0.2 \times 10^3 \times 10^3}$$

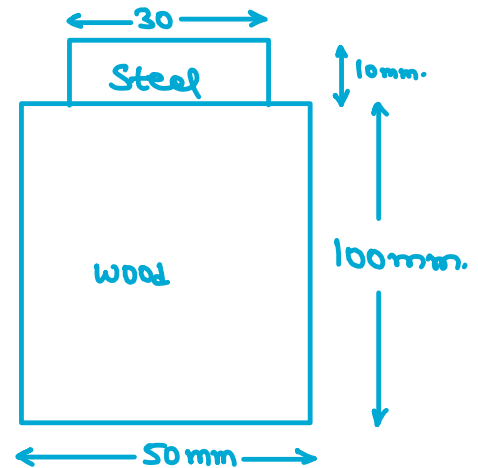
$$R = 10175 \text{ mm} = 10.17 \text{ m.}$$

$$\sigma_{\max Al.} = \frac{-M}{I} y_{\max} = \frac{0.2 \times 10^6 \times 11.25}{3.39 \times 10^4} \times 1 = -66.37 \text{ MPa. (Compression)}$$

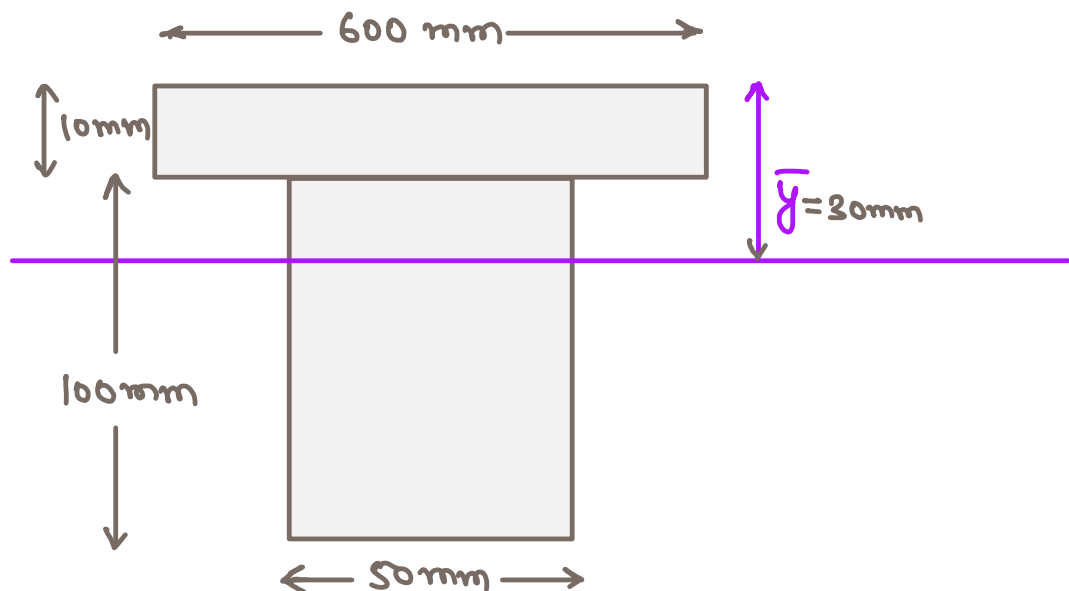
$$\begin{aligned} \sigma_{\max Cu} &= \frac{-M}{I} y_{\max} \left(\frac{E_{cu}}{E_{Al.}} \right) = \frac{-0.2 \times 10^6 \times (11.25 - 10) \times 1.67}{3.39 \times 10^4} \\ &= -7.37 \times 1.67 = -12.31 \text{ MPa. (comp.)} \end{aligned}$$

$$\begin{aligned} \sigma_{\max Cu} &= \frac{+M}{I} y_{\max} \left(\frac{E_{cu}}{E_{Al.}} \right) = \frac{0.2 \times 10^6 \times (20 - 11.25) \times 1.67}{3.39 \times 10^4} \\ &= 51.62 \times 1.67 = 86 \text{ MPa (Tensile)} \end{aligned}$$

Q.7 A timber beam of Rectangular c/s having 50 mm width & 100 mm depth. The wooden section is strengthened by Steel strip attached at the top of the wood & Symmetrically placed as shown in fig. The Allowable Stress in wood is 6 MPa. Find out the moment of Resistance of Composite Section. if modular ratio is 20.



Soln
width modification factor = $M = 20$
 $B_{\text{wood for St.}} = 30 \times 20 = 600 \text{ mm}$



$$(600 \times 10 + 50 \times 100) \bar{y} = 600 \times 10 \times 5 + 50 \times 100 \times 60$$

$$\bar{y} = 30 \text{ mm}$$

$$M = \sigma_{\max} \cdot Z$$

$$y_{\max} = 80 \text{ mm.}$$

$$M = \sigma_{\max} \cdot \frac{I}{y_{\max}}.$$

$$I = \left[\frac{50 \times 100^3}{12} + 50 \times 100 (80 - 50)^2 \right] + \left[\frac{600 \times 10^3}{12} + 600 \times 10 (30 - 5)^2 \right]$$

$$I = 12.46 \times 10^6 \text{ mm}^4$$

$$M = \frac{6 \times 12.46 \times 10^6}{80} \times 10^{-6} = 0.93 \text{ kNm.}$$

Q. > A timber beam 150 mm wide & 200 mm depth is reinforced by steel flitched each of 150 mm width & 12.5 mm thickness find out the Moment of resistance of Composite section if.

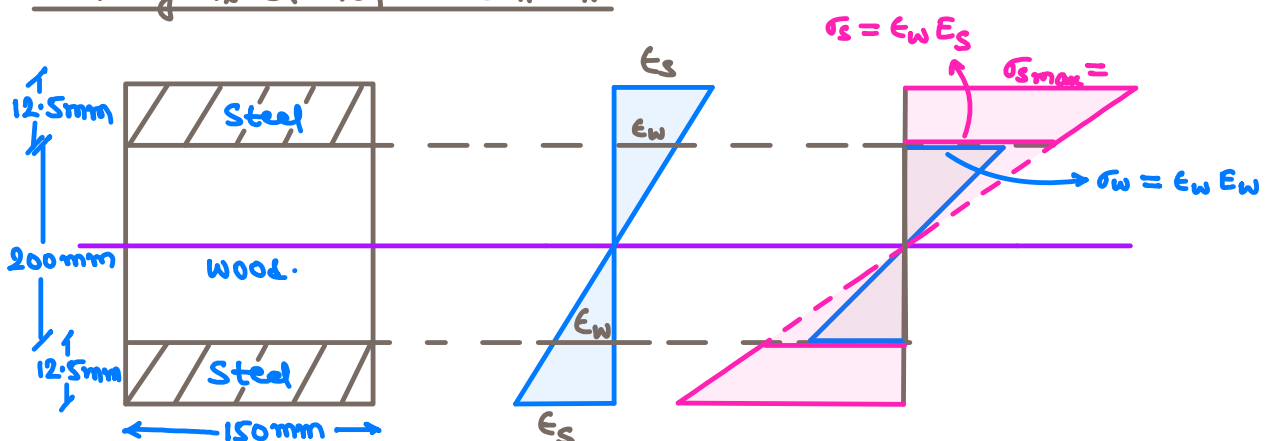
a) Flitching is done at top & bottom.

b) Flitching is at sides

Assume $m = 20$ & Allowable timber = 6 MPa.

Solution :-

a) Flitching is at top & Bottom.



$$M_R = M_{Rw} + M_{Rs}$$

$$M_{Rw} = \sigma_w \sigma_z = 6 \times \frac{150 \times 200^2}{6} \times 10^{-6} = 6 \text{ kN-m}$$

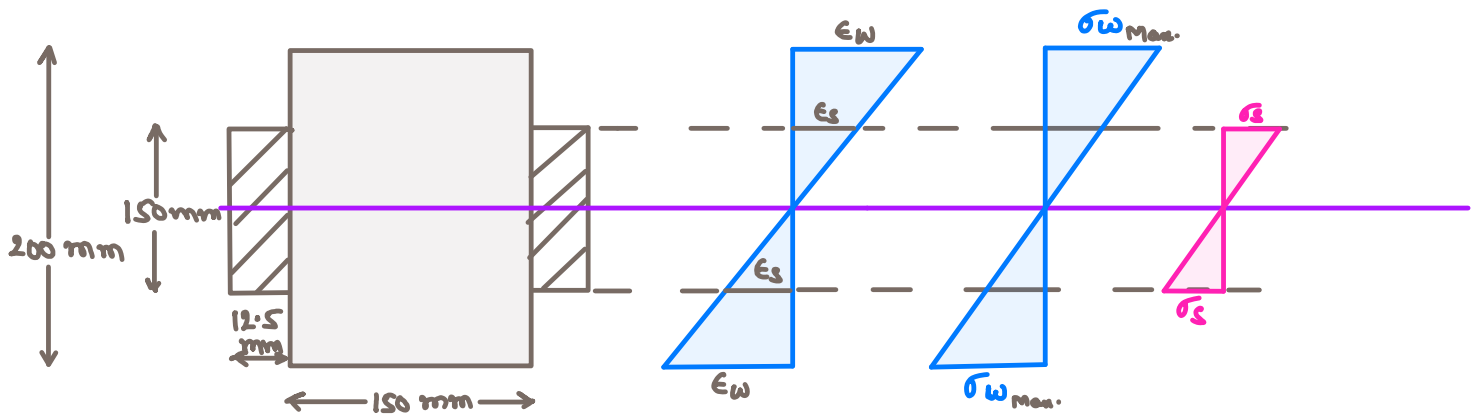
$$\sigma_s = \sigma_w m = 6 \times 20 = 120 \text{ mpa} \quad \left\{ \frac{E_s}{E_w} = m \right\}$$

$$\frac{\sigma_{s_{max}}}{112.5} = \frac{\sigma_s}{100} \Rightarrow \sigma_{s_{max}} = \frac{120 \times 112.5}{100} = 135 \text{ mpa}$$

$$\begin{aligned} M_{R_s} &= \sigma_{s_{max}} \cdot \frac{b(d+2t)^2}{6} - \sigma_s \cdot \frac{bd^2}{6} \\ &= \left[135 \times \frac{150(200+2 \times 12.5)^2}{6} - 120 \times \frac{150 \times 200^2}{6} \right] \times 10^{-6} \\ &= 50.85 \text{ KN-m.} \end{aligned}$$

$$M_R = 6 + 50.85 = \underline{\underline{56.85 \text{ KNm}}}$$

(b.) flitching is at sides:-



$$M_R = M_{R_w} + M_{R_s}$$

$$M_{R_w} = \sigma_{w_{max}} \cdot Z = 6 \times \frac{150 \times 200^2}{6} \times 10^{-6} = 6 \text{ KN-m}$$

$$\frac{\sigma_{w_{max}}}{100} = \frac{\sigma_w}{75} \Rightarrow \sigma_w = 6 \times \frac{75}{100} = 4.5 \text{ MPa.}$$

$$\sigma_s = \sigma_w \cdot m = 4.5 \times 20 = 90 \text{ mpa}$$

$$M_{R_s} = \sigma_s Z = 90 \times \frac{(2 \times 12.5) \times 150^2}{6} \times 10^{-6} = 8.43 \text{ KN-m}$$

$$M_R = 6 + 8.43 = \underline{\underline{14.43 \text{ KN-m.}}}$$

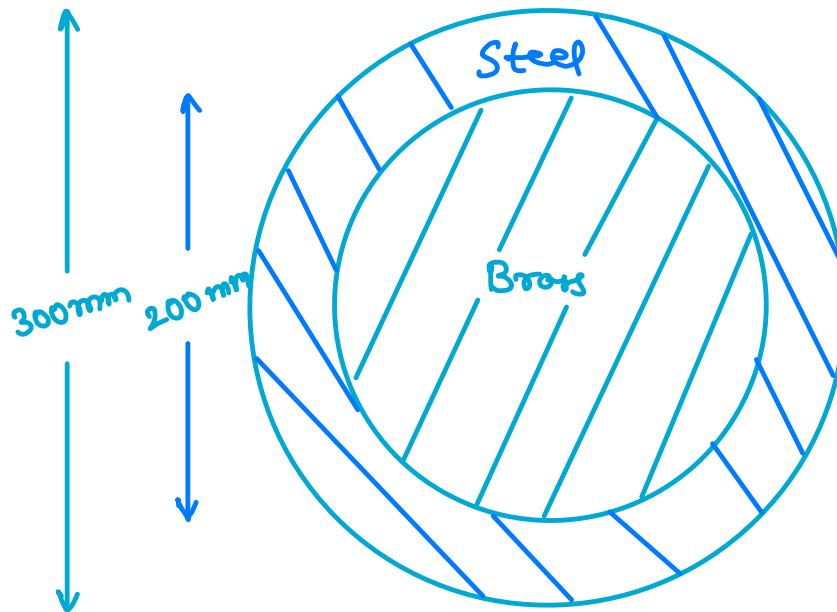
Note:-

Top and Bottom flitched beam is (3-5) times stronger than side flitched beams with same cross-sectional Area.

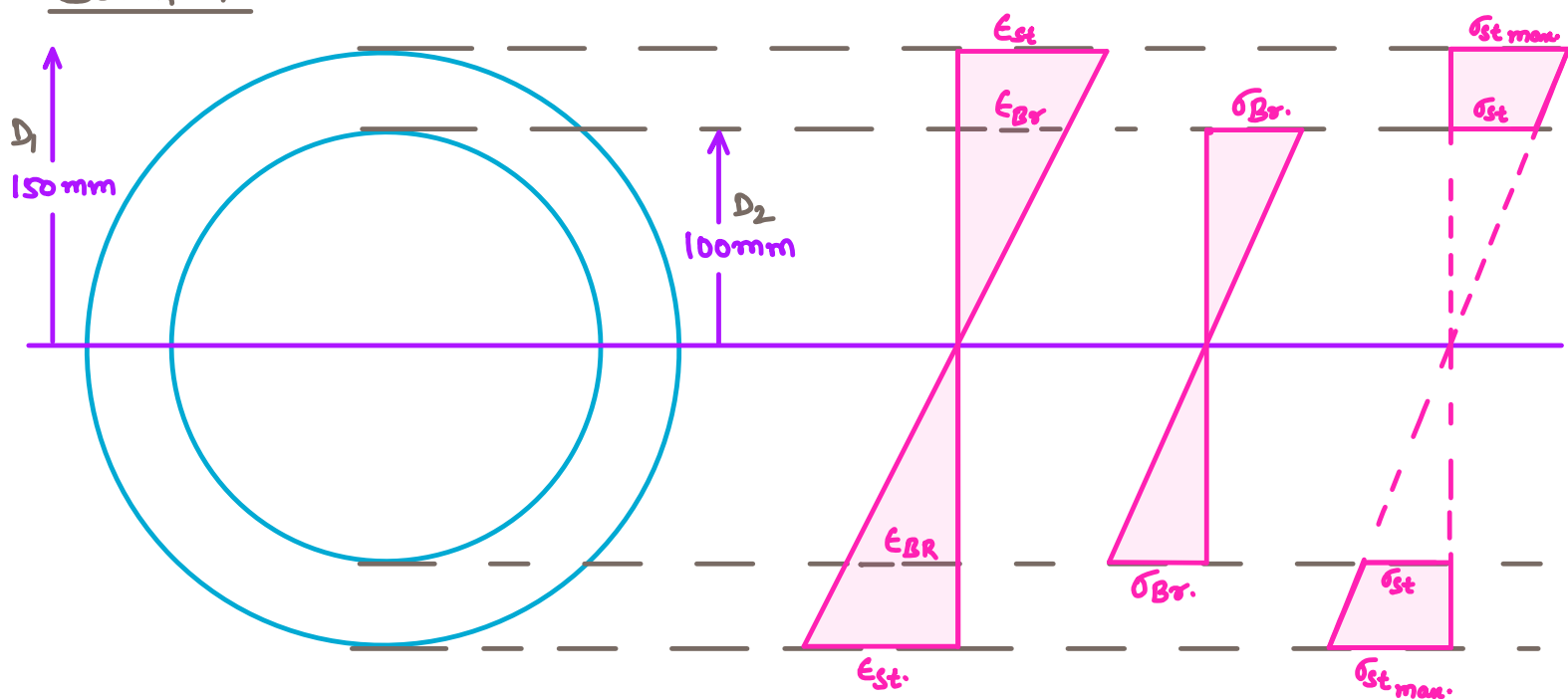
Q. > find the MoR of the given section

$$\sigma_{per, brn} = 70 \text{ mpa} , E_{brn} = 0.8 \times 10^5 \text{ mpa}.$$

$$\sigma_{per, steel} = 100 \text{ mpa.}, E_{steel} = 2 \times 10^5 \text{ mpa}$$



Solution



$$M_R = M_{br} + m_{st}.$$

Case-(i) - if steel reaches its permissible stress first

$$\sigma_{st_{max}} = 100 \text{ mpa}.$$

$$\frac{\sigma_{st\max}}{150} = \frac{\sigma_{st}}{100} \Rightarrow \sigma_{st} = \sigma_{st\max} \times \frac{100}{150} \Rightarrow 100 \times \frac{100}{150}$$

$$\sigma_{st} = 66.67 \text{ mpa.}$$

$$\sigma_{st} = 66.67 \text{ mpa.}$$

$$\sigma_{st} = m \sigma_{Br.}$$

$$\sigma_{Br} = \frac{\sigma_{st}}{m} = \frac{66.67}{2 \times 10^5} \times 0.8 \times 10^5 = 26.67 \text{ mpa} < \sigma_{per. Br.}$$

$$M_{R_{Br.}} = \sigma_{Br} \cdot Z = 26.67 \times \frac{\pi (200)^3}{32} \times 10^{-6}$$

$$= 20.94 \text{ KN-m.}$$

$$\left(\frac{\pi d^4}{64} \times \frac{2}{d} \right)$$

$$M_{R_{st}} = \sigma_{st\max} \cdot Z = \sigma_{st\max} \cdot \frac{\pi D_1^3}{32} - \sigma_{st} \frac{\pi D_2^3}{32}$$

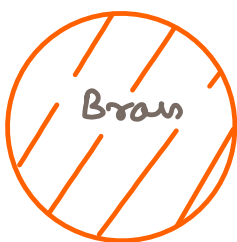
$$= \left[100 \cdot \pi \times \frac{(300)^3}{32} - 66.67 \frac{\pi \times (200)^3}{32} \right] \times 10^{-6}$$

$$= 212.70 \text{ KN-m}$$

$$M_R = 20.94 + 212.70$$

$$= 233.64 \text{ KN-m.}$$

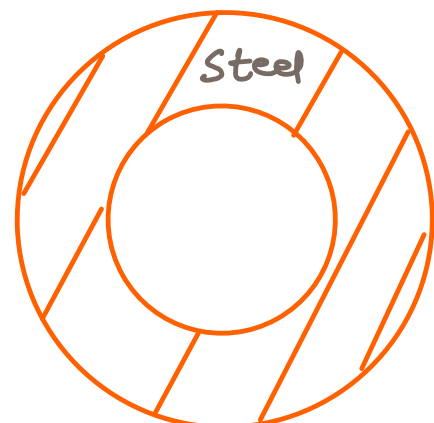
Alternate method



$$\frac{M_{Br}}{I_{Br}} = \frac{E_{Br}}{R} \quad \text{--- (i)}$$

$$\frac{M_{Br}}{I_{Br} E_{Br}} = \frac{M_{st}}{I_{st} E_{st}} \quad \text{--- (1)}$$

$$M_R = M_{Br} + M_{st} \quad \text{--- (1)}$$



$$\frac{m_{st}}{I_{st}} = \frac{E_{st}}{R} \quad \text{--- (ii)}$$

$$M_R = M_{Br} + \frac{M_{Br}}{I_{Br} \cdot E_{Br}} I_{St} E_{St}$$

$$M_{Br} = \frac{M_R (I_{Br} \cdot E_{Br})}{I_{Br} E_{Br} + I_{St} E_{St}}, \quad M_{St} = \frac{M_R (I_{St} E_{St})}{I_{Br} E_{Br} + I_{St} E_{St}}$$

$$\sigma_{Per, Brn} = \frac{M_{Br} \cdot y_{Br, max.}}{I_{Brn}}$$

$$\Rightarrow M_R = M_1$$

$$\sigma_{Per, steel} = \frac{M_{St} \cdot y_{St, max.}}{I_{steel}}$$

$$\Rightarrow M_R = M_2$$

Minimum value

$$M_R = \min \{ M_1, M_2 \}$$

Solution.

$$M_{Br} = \frac{M_R}{1 + 2.5 \times \left[\frac{(300)^4 - (200)^4}{(200)^4} \right]} = 0.089 M_R$$

$$M_{St.} = \frac{M_R}{1 + \frac{1}{2.5} \times \left[\frac{(200)^4}{(300)^4 - (200)^4} \right]} = 0.91 M_R$$

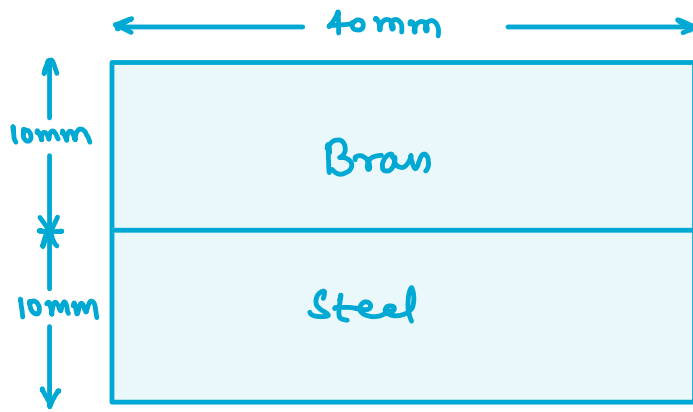
$$\text{Brn} \rightarrow 70 = \frac{0.089 M_R \times 100}{\pi \frac{(200)^4}{64}} = 617.7 \text{ KN-m.}$$

$$\text{Steel} \rightarrow 100 = \frac{0.91 M_R \times 150}{\pi (300)^4 - (200)^4} \times 64 = \underline{233.75 \text{ KN-m}}$$

Minimum

$$M_R = 233.75 \text{ KN-m.}$$

Q.7 find MOR of the given section.



$$E_{st} = 2 \times 10^5 \text{ MPa.}$$

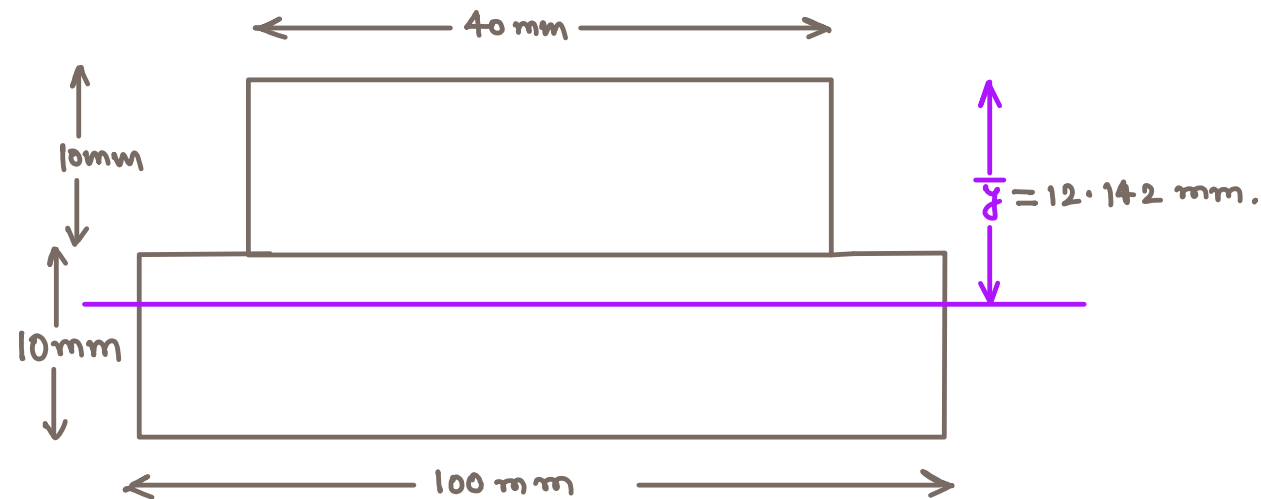
$$E_{Br} = 0.8 \times 10^5 \text{ MPa}$$

$$\sigma_{st\ per} = 112.5 \text{ MPa}$$

$$\sigma_{Br\ per} = 75 \text{ MPa}$$

Soln

$$B_{Br\ for\ steel} = 40 \times \frac{E_{st}}{E_{Br}} = 40 \times \frac{2}{0.8} = 100 \text{ mm.}$$



$$(40 + 100)10 \cdot \bar{y} = (40 \times 5 + 100 \times 15) \times 10$$

$$\bar{y} = 12.142 \text{ mm.}$$

$$\sigma_{per, Br} = \frac{M_1 \bar{y}_{max Br}}{I_{transformed}} \Rightarrow 75 = \frac{M_1 \times 12.142}{40176.4} \Rightarrow M_1 = 0.248 \text{ kN-m}$$

$$I_{transformed} = \frac{1}{3} [100 \times (20 - 12.142)^3 + 40 \times 12.142^3 + (100 - 40)(12.142 - 10)^3]$$

$$I_{trans} = 40176.4 \text{ mm}^4$$

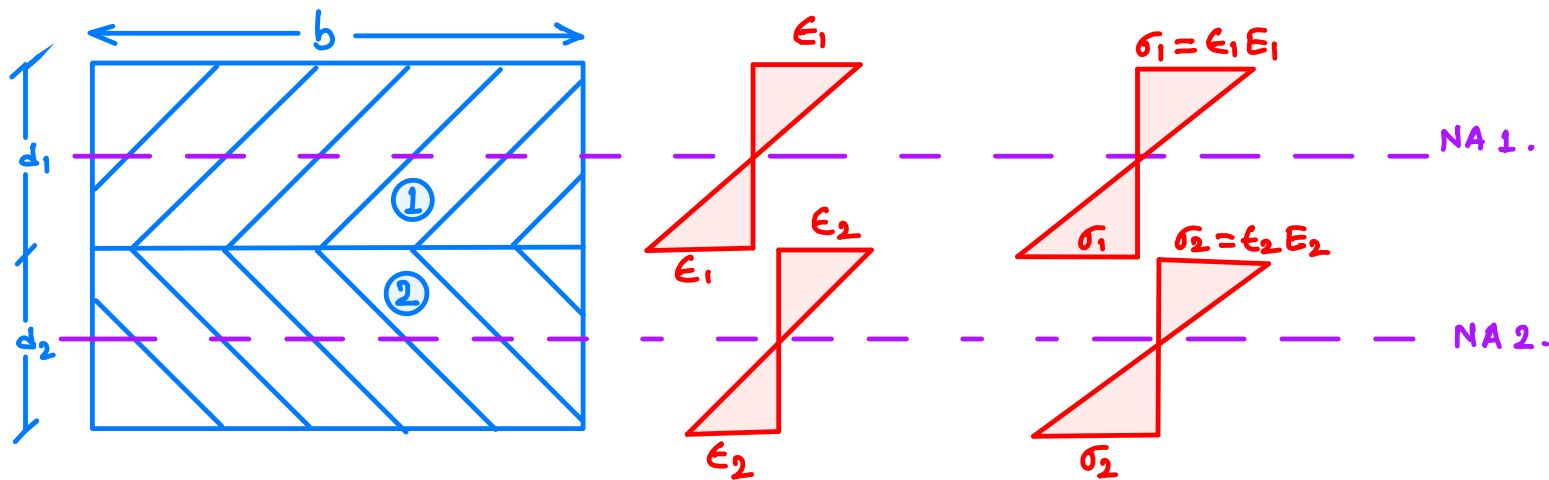
$$\sigma_{per, st.} = \frac{M_2 \times \bar{y}_{max steel}}{I_{transformed}} \times \left(\frac{2 \times 10^5}{0.8 \times 10^5} \right) \Rightarrow \frac{M_2 \times (20 - 12.142)}{40176.4} \times 2.5 = 112.5$$

$$M_2 = 0.230 \text{ kN-m.}$$

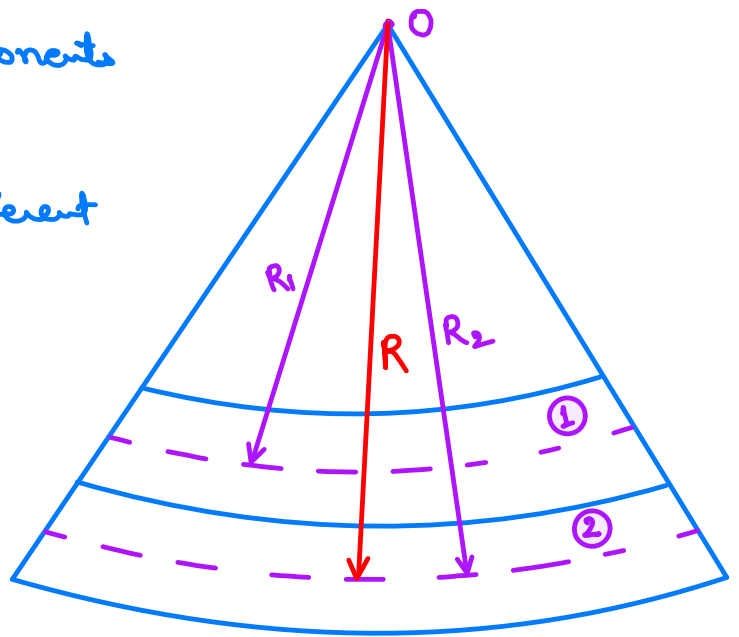
$$MOR = \min(0.248, 0.230) \text{ kN-m}$$

$$= 0.230 \text{ kN-m.}$$

b> when two sections are simply placed over each other.
(Not connected rigidly)



- In this case both the components bend independently.
- Both the components have different radius of bending (R_1, R_2) in actual but it is approximated to be same (R) for simplicity in calculation.



- $M_R = M_1 + M_2$ ---- (i)
- $R_1 = R_2 = R$ ---- (ii)

$$\frac{M_1}{I_1 E_1} = \frac{M_2}{I_2 E_2} \Rightarrow M_2 = \frac{M_1 I_2 E_2}{I_1 E_1}$$

$$M_R = M_1 + \frac{M_1 I_2 E_2}{I_1 E_1}$$

$$M_1 = \frac{M_R (I_1 E_1)}{I_1 E_1 + I_2 E_2}$$

$$M_2 = \frac{M_R (I_2 E_2)}{I_1 E_1 + I_2 E_2}$$

Note:- If $E_1 = E_2 = E$

$$\sigma_1 = \frac{M_1 y_{1max}}{I_1}$$

$$\sigma_2 = \frac{M_1 y_{2max}}{I_2}$$

$$\sigma_1 = M_R \frac{I_2}{(I_1 + I_2)} \cdot \frac{d_1}{2}, \quad \sigma_2 = M_R \frac{I_1}{(I_1 + I_2)} \cdot \frac{d_2}{2}$$

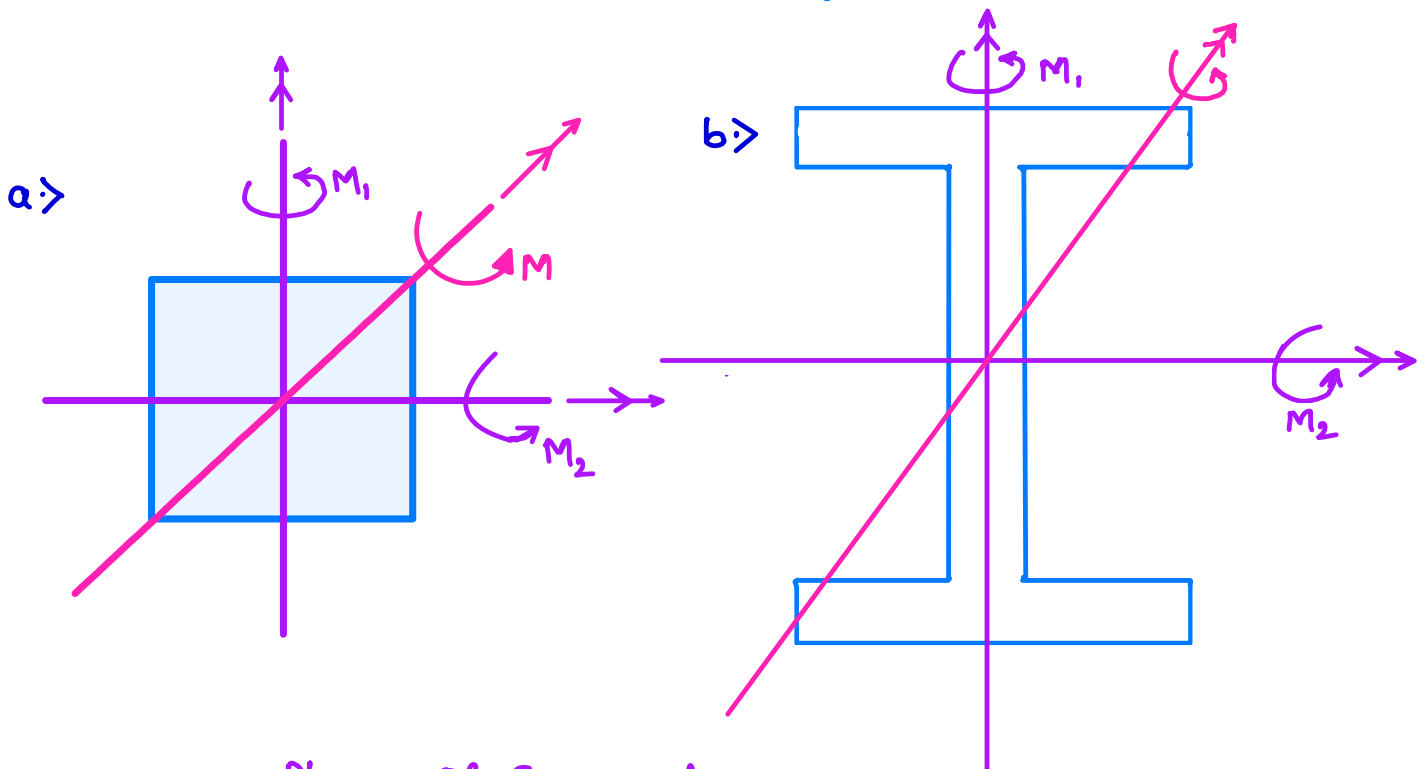
$$\frac{\sigma_1}{\sigma_2} = \frac{M_R \frac{d_1}{2}}{(I_1 + I_2)} \cdot \frac{(I_1 + I_2)}{M_R \cdot \frac{d_2}{2}}$$

$$\boxed{\frac{\sigma_1}{\sigma_2} = \frac{d_1}{d_2}}$$

→ when E for both the material is same.

Unsymmetrical Bending

- when the bending couple does not act in plane of symmetry of member either because there is no plane of symmetry or because couple acts in different plane the bending is termed as Unsymmetrical Bending.



— Plane of Symmetry

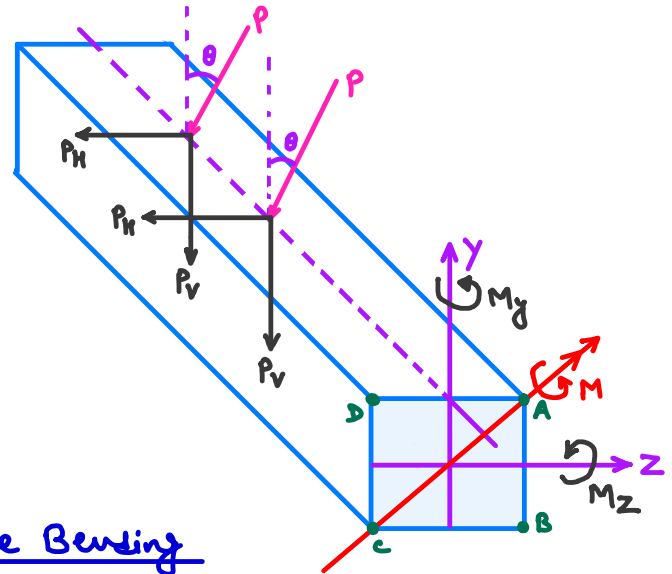
→ Bending couple

⇒ Unsymmetrical Bending can be analysed in following cases:-

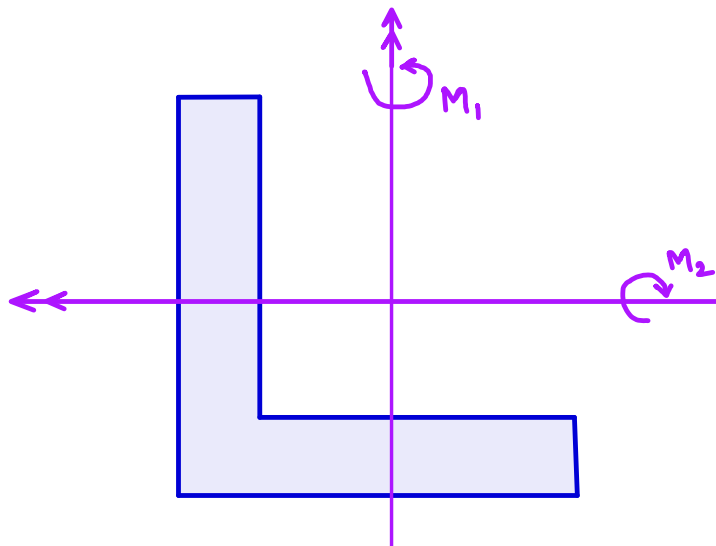
a) Doubly Symmetric Beam with skew loading

Note :-

If the section is having at least one axis of symmetry then that axis and axis perpendicular to it are "Principal Axes".

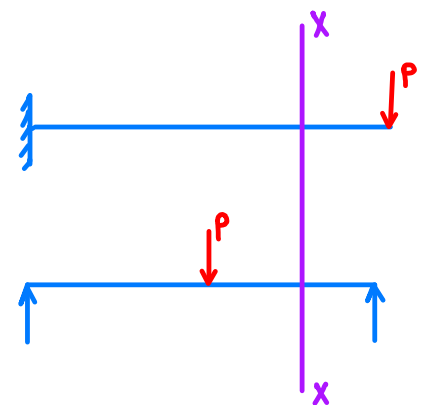
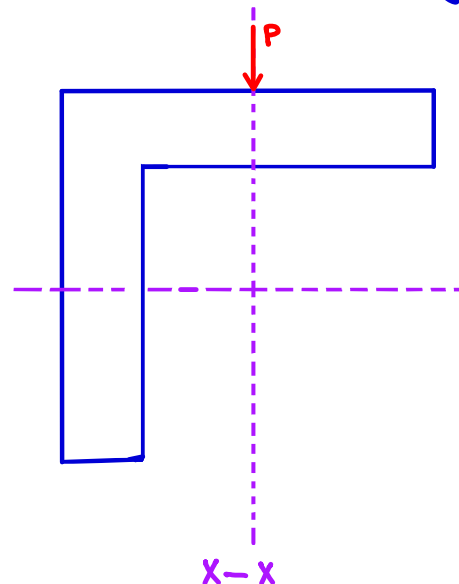


b) Unsymmetric beam with Pure Bending



	$\frac{M_y}{I_y}$	$\frac{M_z}{I_z}$
A →	-	-
B →	-	+
C →	+	+
D →	+	-

c) Unsymmetric beam with load causing moment & shear.



loading will lead to shear & Bending and will cause Unsymmetrical Bending.

Analysis of Unsymmetrical Bending (with no twisting)

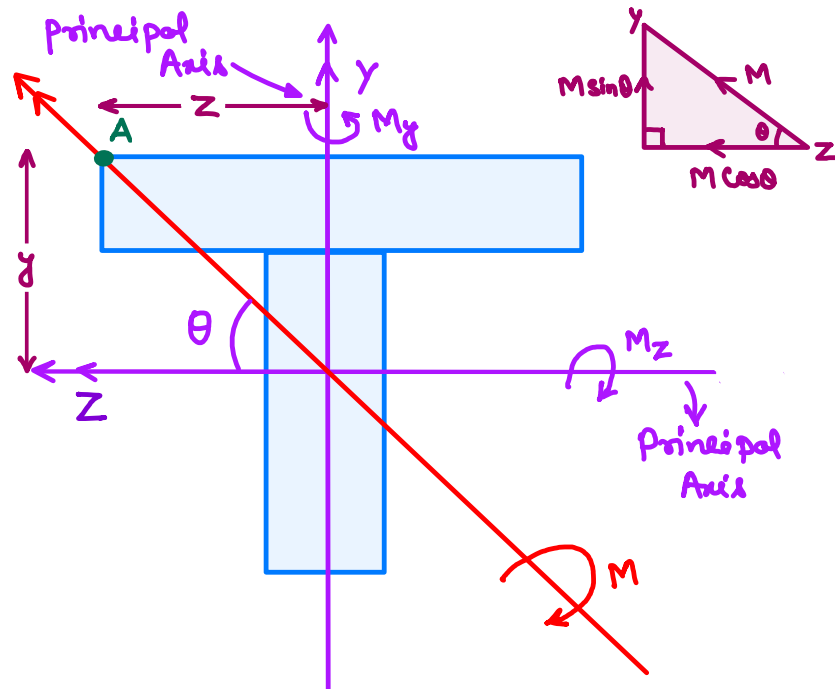
— In order to find out bending stress at any point in the cross-section under unsymmetric bending the following procedure is adopted.

Step-1. Locate the principal axis of cross-section.

Step-2. find the component of moment about principal axis.

$$M_y = M \sin \theta$$

$$M_z = M \cos \theta$$



Step-3. Stress at a point is given by

$$\sigma_{Ax} = \frac{-M_z \cdot y}{I_z} + \frac{M_y \cdot z}{I_y}$$

(-): Compressive

(+): Tension

— In previous equation substitute the value of y & z with sign

Step-4. Neutral axis is located by substituting $\sigma_x = 0$

$$\sigma_x = 0 \Rightarrow \frac{-M_z \cdot y}{I_z} + \frac{M_y \cdot z}{I_y} = 0$$

$$y = \left(\frac{M_y}{M_z} \cdot \frac{I_z}{I_y} \right) z$$

$$y = \left(\frac{M \sin \theta}{M \cos \theta} \cdot \frac{I_z}{I_y} \right) z$$

$$y = \left(\frac{I_z}{I_y} \cdot \tan \theta \right) z$$

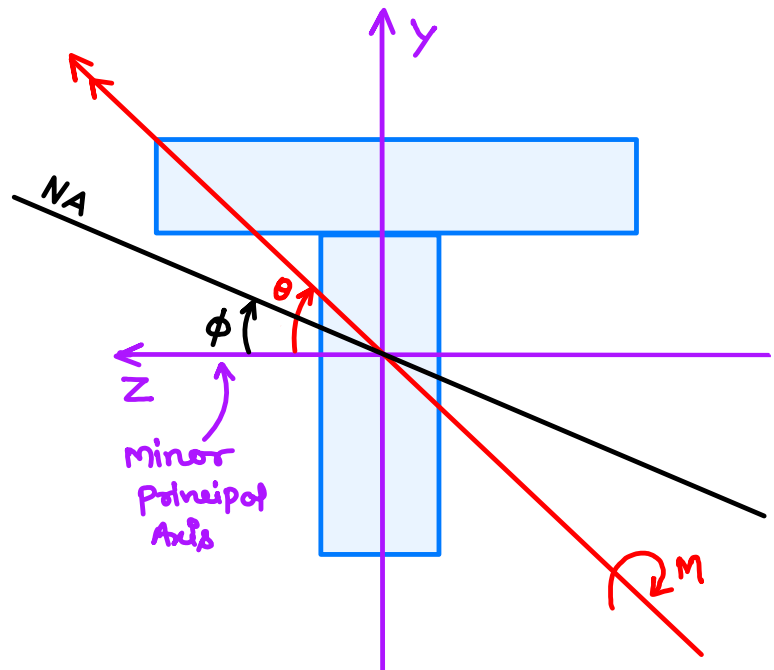
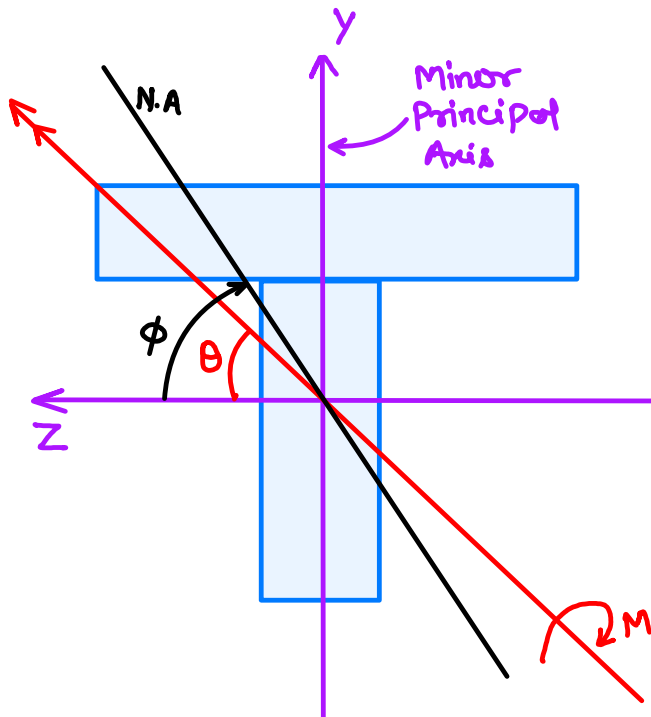
$$\{ y = mx + c \}$$

Here, $m = \tan \phi = \text{slope of N.A.}$

$$\tan \phi = \frac{I_z}{I_y} \tan \theta$$

$$\text{If } I_z > I_y \Rightarrow \phi > \theta$$

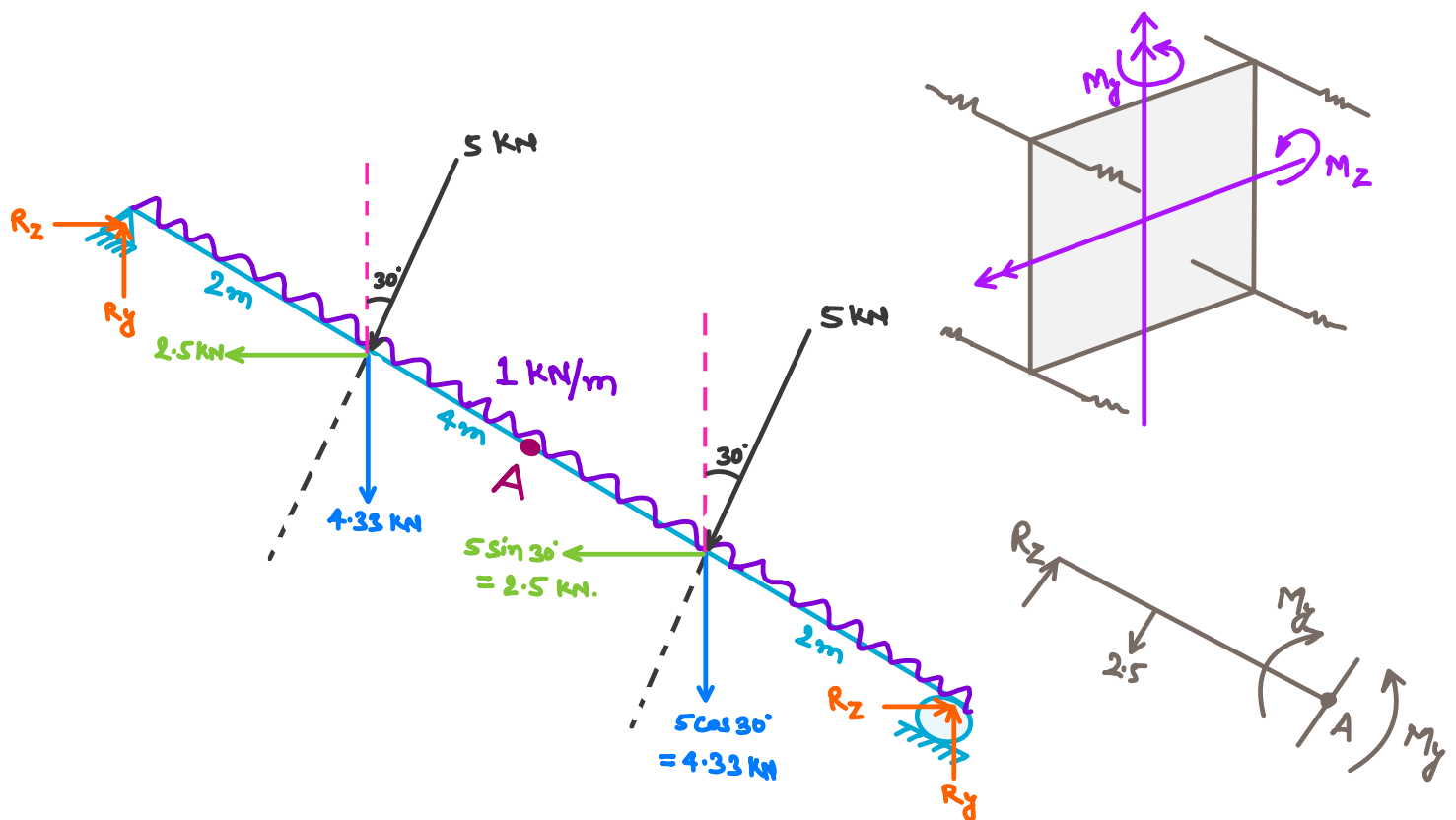
$$\text{If } I_y > I_z \Rightarrow \phi < \theta$$



Note :-

Thus N.A. is always located between the direction of moment Couple and minor Principal Axis.

Q. > A rectangular Steel beam 180 X 260 mm in cross-section is used as simply supported beam over a span of 8m and is subjected to a UDL of 1 kN/m and a concentrated load of 5 kN at 2m from each support inclined at an angle of 30° to the vertical axis. Determine bending stresses at four corners of the beam and location of N.A. of the cross-section at mid-span.



$$\sum f_y = 0 \Rightarrow 1 \times 8 + 4.33 \times 2 - 2R_y = 0$$

$$\Rightarrow R_y = 8.33 \text{ kN.}$$

$$\sum f_z = 0 \Rightarrow 2 \times 2.5 - 2R_z = 0$$

$$\Rightarrow R_z = 2.5 \text{ kN.}$$

$$M_z = R_y \times 4 - 1 \times 4 \times \frac{4}{2} - 4.33 \times 2 \Rightarrow 8.33 \times 4 - 8 - 8.66$$

$$M_z = 16.67 \text{ kN}\cdot\text{m.}$$

$$M_y = 2.5 \times 4 - 2.5 \times 2 = 5 \text{ kN}\cdot\text{m}$$

$$\tan \phi = \frac{M_y}{M_z} = \frac{5}{16.67} = 0.3$$

$$\theta = \tan^{-1}(0.3) = 16.7^\circ$$

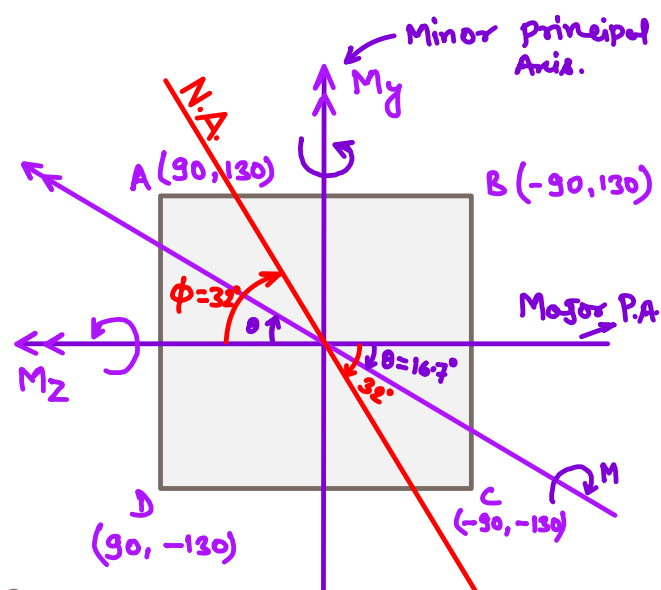
$$\sigma_x = \frac{-M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$I_z = \frac{180 \times (260)^3}{12} =$$

$$= 263.64 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{260 \times (180)^3}{12} =$$

$$= 126.36 \times 10^6 \text{ mm}^4$$



At point 'A',

$$\sigma_x = \left(\frac{-16.67 \times 130}{263.64 \times 10^6} + \frac{5 \times 90}{126.36 \times 10^6} \right) \times 10^6 = -8.22 + 3.56$$

$$\sigma_x = -4.65 \text{ N/mm}^2 \text{ (compression)}$$

At point 'B'

$$\sigma_x = -8.22 - 3.56 = -11.87 \text{ N/mm}^2 \text{ (comp)}$$

At point 'C'

$$\sigma_x = +8.22 - 3.56 = 4.65 \text{ N/mm}^2 \text{ (Tension)}$$

At point 'd'

$$\sigma_x = +8.22 + 3.56 = +11.78 \text{ N/mm}^2 \text{ (Tension)}$$

$$\tan \phi = \frac{I_z}{I_y} \tan \theta$$

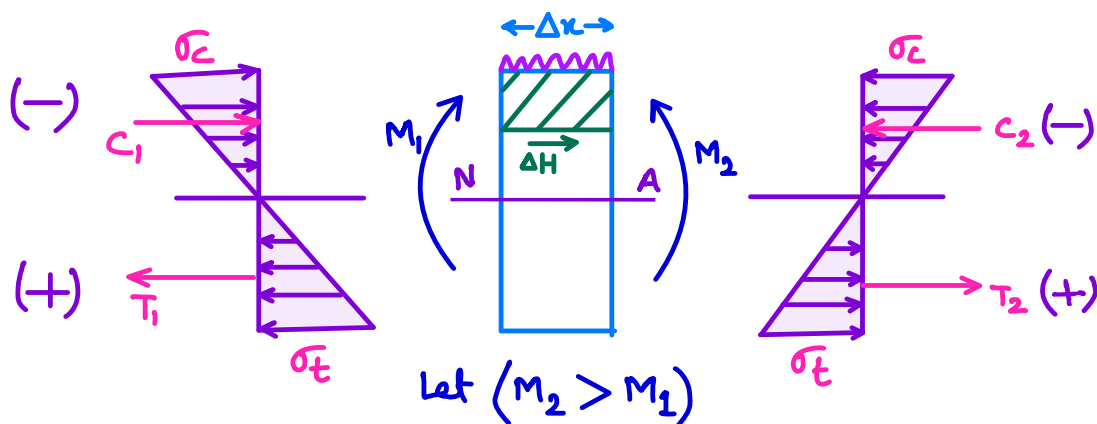
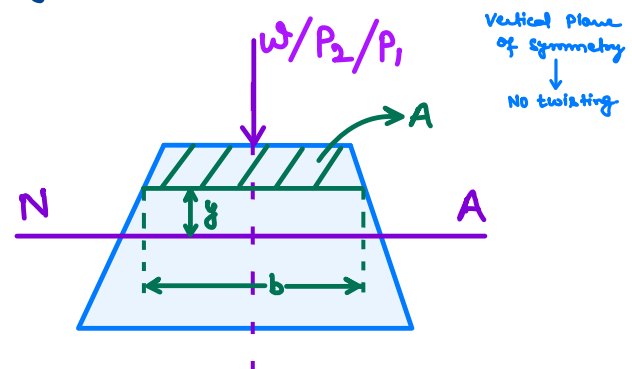
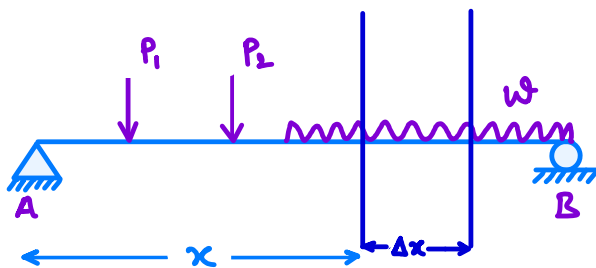
$$\phi = \tan^{-1} \left[\left(\frac{263.64 \times 10^6}{126.36 \times 10^6} \right) 0.3 \right]$$

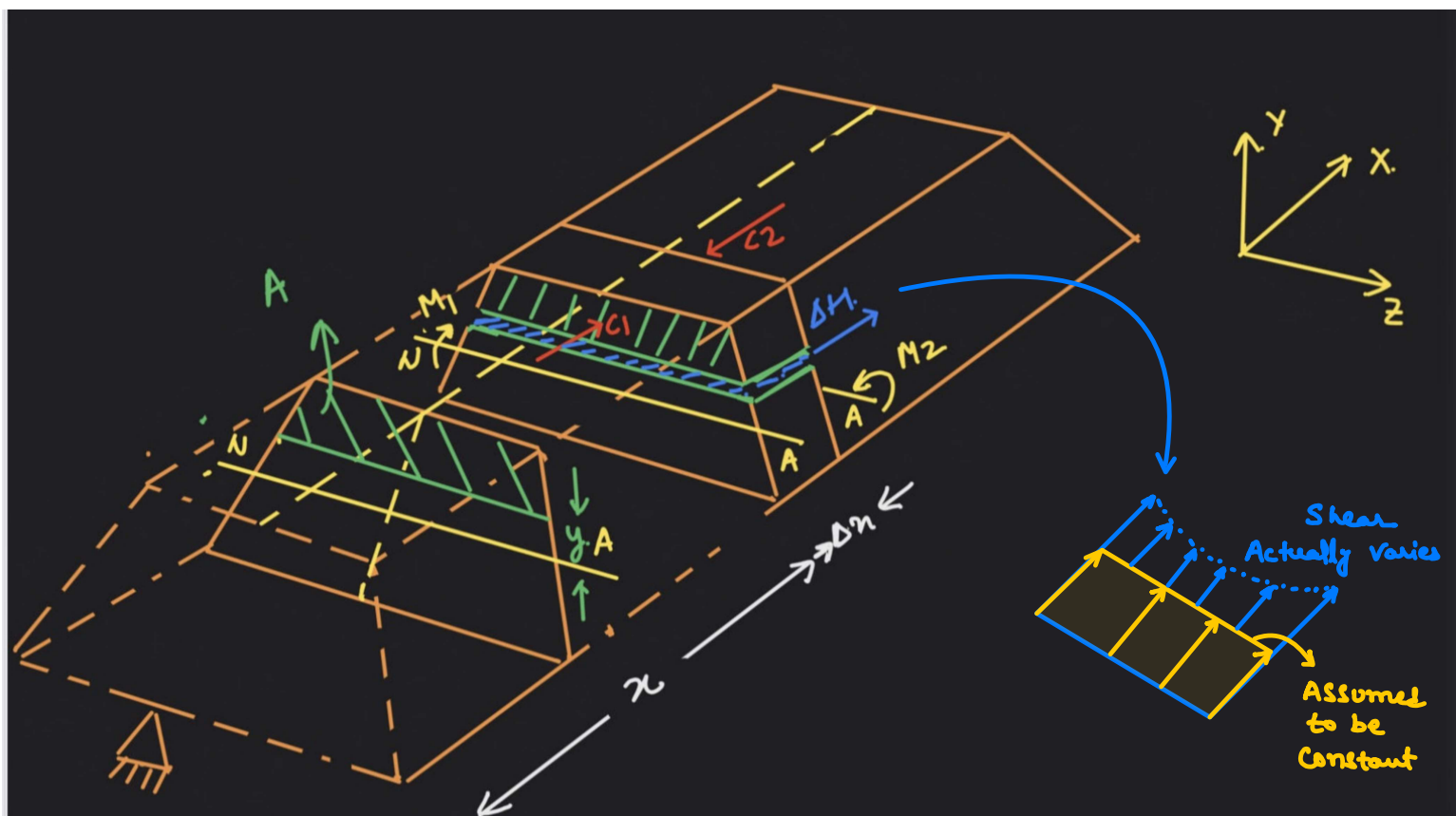
$$\phi = 32^\circ$$

SHEAR STRESS IN BEAM

- when transverse loading is applied to a beam it results in normal and Shearing Stresses in beam
- Normal Stresses are produced by Bending moment & Shearing Stresses are produced by Shear force.
- The vertical Shearing Stresses are accompanied by horizontal Shearing Stresses of equal magnitude termed as Complementary Shearing Stress.

Shear Stress in prismatic beam having loading in the vertical Plane of Symmetry.





for equilibrium, $\sum f_x = 0$

$$C_1 + \Delta H - C_2 = 0$$

$$\Delta H = C_2 - C_1$$

$$\tau = \frac{\text{Shear force}}{\text{Area}} = \frac{\Delta H}{b \Delta x} = \frac{C_2 - C_1}{b \Delta x}$$

Note:-

Here, Shear force per unit length is termed as Shear flow.

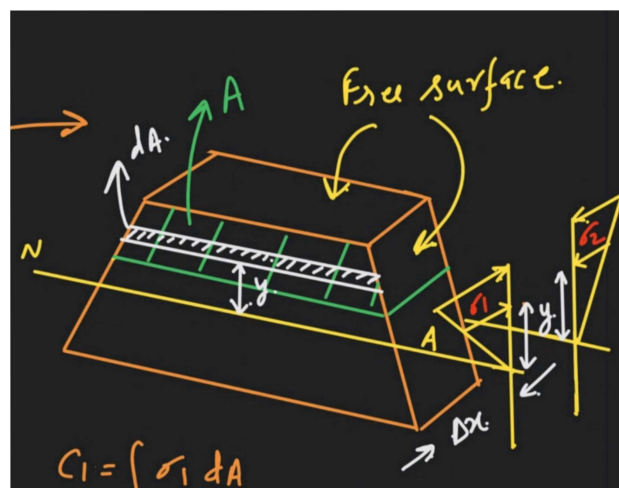
$$q = \frac{\Delta H}{\Delta x} = \frac{C_2 - C_1}{\Delta x} = \tau \cdot b$$

$$C_1 = \int_A \sigma_1 dA$$

$$C_2 = \int_A \sigma_2 dA$$

$$C_2 - C_1 = \int_A (\sigma_2 - \sigma_1) dA$$

$$\frac{M_1}{I} = \frac{\sigma_1}{y} \quad \text{and} \quad \frac{M_2}{I} = \frac{\sigma_2}{y}$$



$$C_2 - C_1 = \int_A \left(\frac{M_2 y}{I} - \frac{M_1 y}{I} \right) dA \Rightarrow \frac{1}{I} \int_A (M_2 - M_1) y dA$$

$$= \frac{1}{I} \int_A \Delta M y dA \Rightarrow \frac{1}{I} \int_A \frac{\partial M}{\partial x} \Delta x \cdot y dA$$

$$= \frac{1}{I} \int_A V \Delta x y dA \Rightarrow \frac{V \cdot \Delta x}{I} \int y dA$$

$$C_2 - C_1 = \frac{V \cdot \Delta x}{I} A \bar{y}$$

Hence

$\Delta H = C_2 - C_1 = \frac{V \cdot \Delta x}{I} A \bar{y}$
$q = \frac{\Delta H}{\Delta x} = \frac{V A \bar{y}}{I}$
$\tau = \frac{\Delta H}{\Delta x \cdot b} = \frac{V A \bar{y}}{I b}$

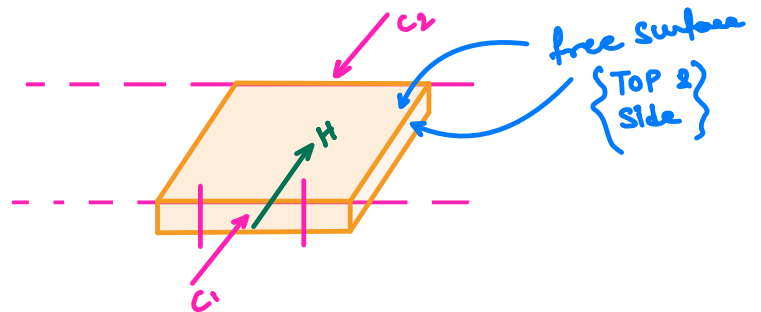
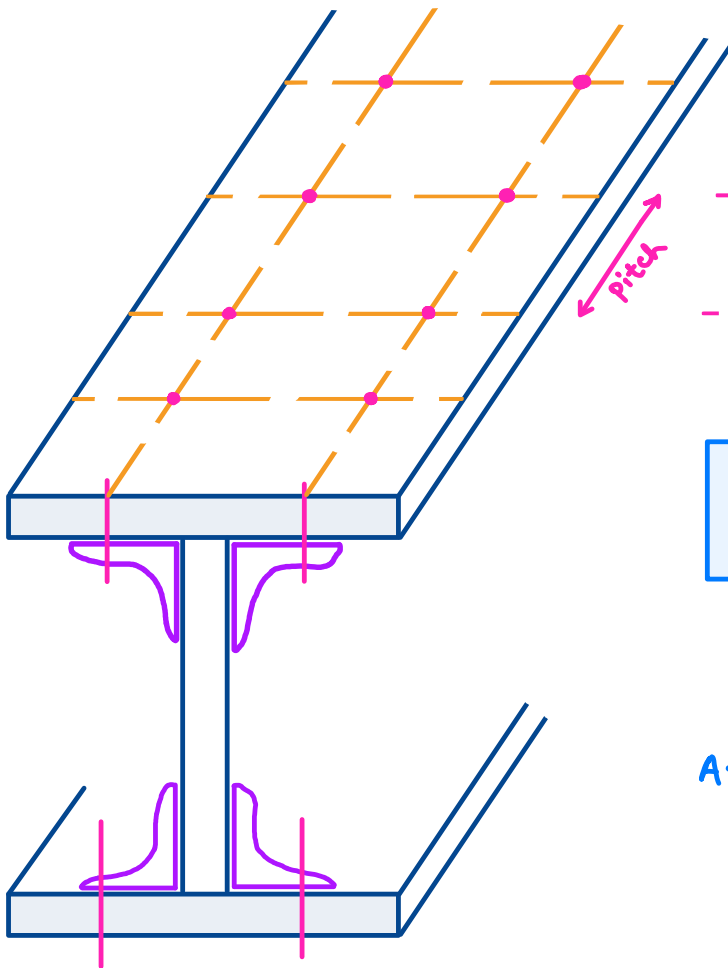
$A \bar{y}$ = moment of Area of section above the level at which shear stress is to be found.

I = moment of inertia of complete section.

b = width of section at level where shear stress is required.

q = shear flow.

Application of Shear Stress concept

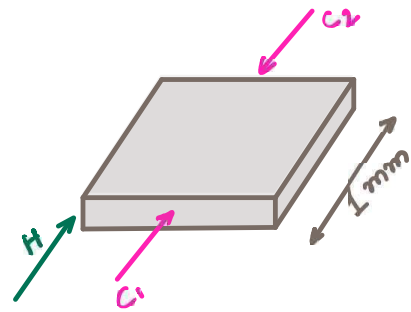
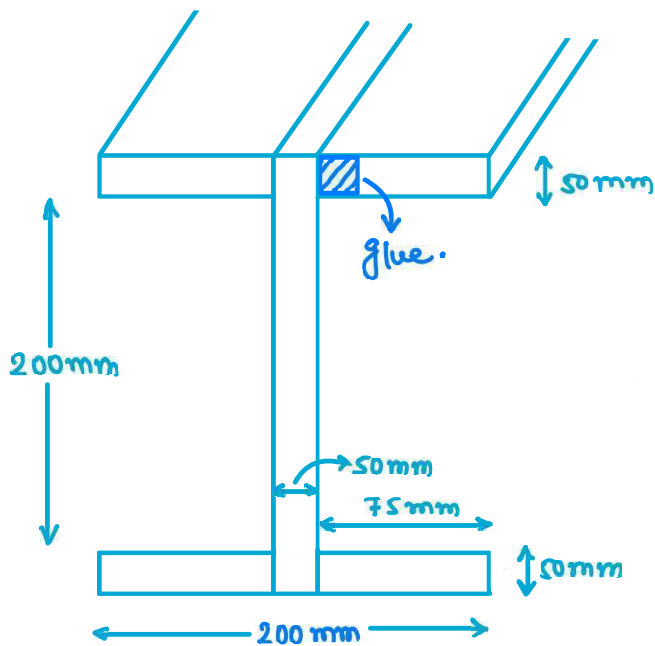


$$H = \frac{VA\bar{y}P}{I} = 2 \left[\tau_{per} \frac{\pi d^2}{4} \right]$$

No. of bolts.

At free surface, NO Shear Stress will be developed.

Q. Find the shear force resisted by one glue Junction per mm length inside if the shear force at Section is 2000N



Soln

$$H = c_2 - c_1 = \frac{VA\bar{y}\Delta x}{I}$$

$$A = 750 \times 50 = 3750 \text{ mm}^2$$

$$\bar{y} = 100 + 25 = 125 \text{ mm}$$

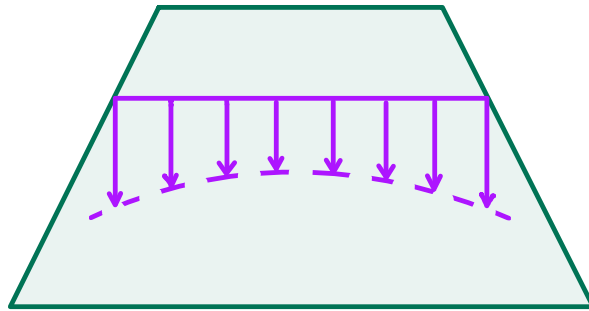
$$I = \frac{200 \times 300^3}{12} - \frac{150 \times 200^3}{12} = 350 \times 10^6 \text{ mm}^4$$

$$H = \frac{2000 \times 3750 \times 125 \times 1 \text{ mm}}{350 \times 10^6}$$

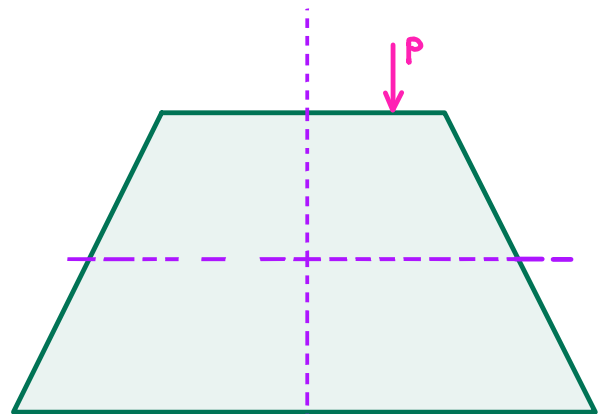
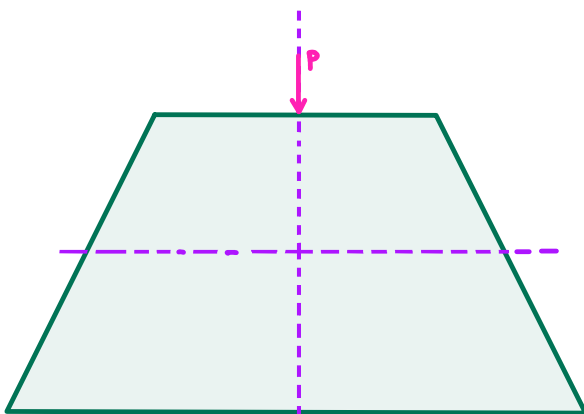
$$H = 4.01 \text{ N/mm.}$$

Note:-

- Shear stress, $\tau = \frac{VA\bar{y}}{Ib}$ has been derived assuming the shear stress to be constant across the width of the member.
- However, Shear stress varies across the width also. Hence, the above expression is valid for narrow section.

Note -:

- Shear stress, $\tau = \frac{VA\bar{y}}{Ib}$ is valid only when load passes through vertical axis of symmetry (i.e. there is no twisting in the section).



Note:-

- As moment of area of complete section about Neutral axis is equal to zero.

$$A = A_1 + A_2$$

$$A\bar{y} = A_1\bar{y}_1 + A_2(-\bar{y}_2)$$

$$\bar{y} = \frac{A_1\bar{y}_1 - A_2\bar{y}_2}{A_1 + A_2}$$

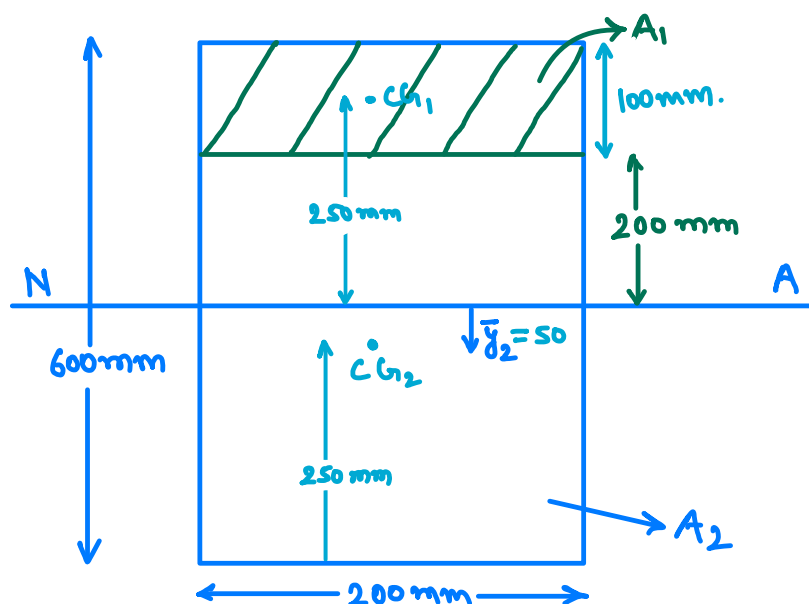
$$\text{As } \bar{y} = 0$$

$$A_1\bar{y}_1 - A_2\bar{y}_2 = 0$$

$$A_1\bar{y}_1 = A_2\bar{y}_2$$

$$\left[\begin{array}{l} \text{M of Shaded} \\ \text{Area} \end{array} \right] = \left[\begin{array}{l} \text{M of Plane} \\ \text{Area} \end{array} \right] \text{ About N.A.}$$

- Hence, for computation of shear stress moment of Area about N.A. of the area above the level where shear stress is required ($A_1\bar{y}_1$) may be considered or Moment of Area about N.A. of area below the level where shear stress is required ($A_2\bar{y}_2$) can also be considered.

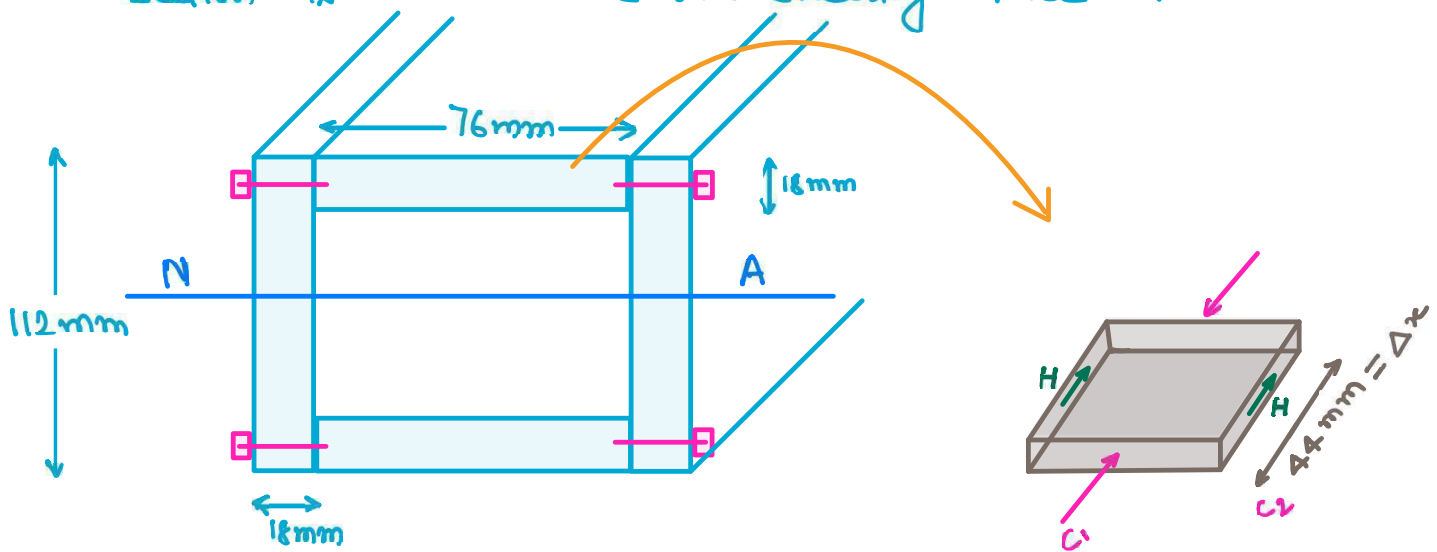


$$A_1\bar{y}_1 = 200 \times (300 - 200) (200 + 50)$$

$$A_1\bar{y}_1 = 5 \times 10^6 \text{ mm}^3$$

$$A_2\bar{y}_2 = 200 \times (300 + 200) (300 - 250) = 5 \times 10^6$$

Q.7 If spacing between rails is 44mm and shear force at the section is 2.5 kN. Find the shearing force at each rail.



Solution.

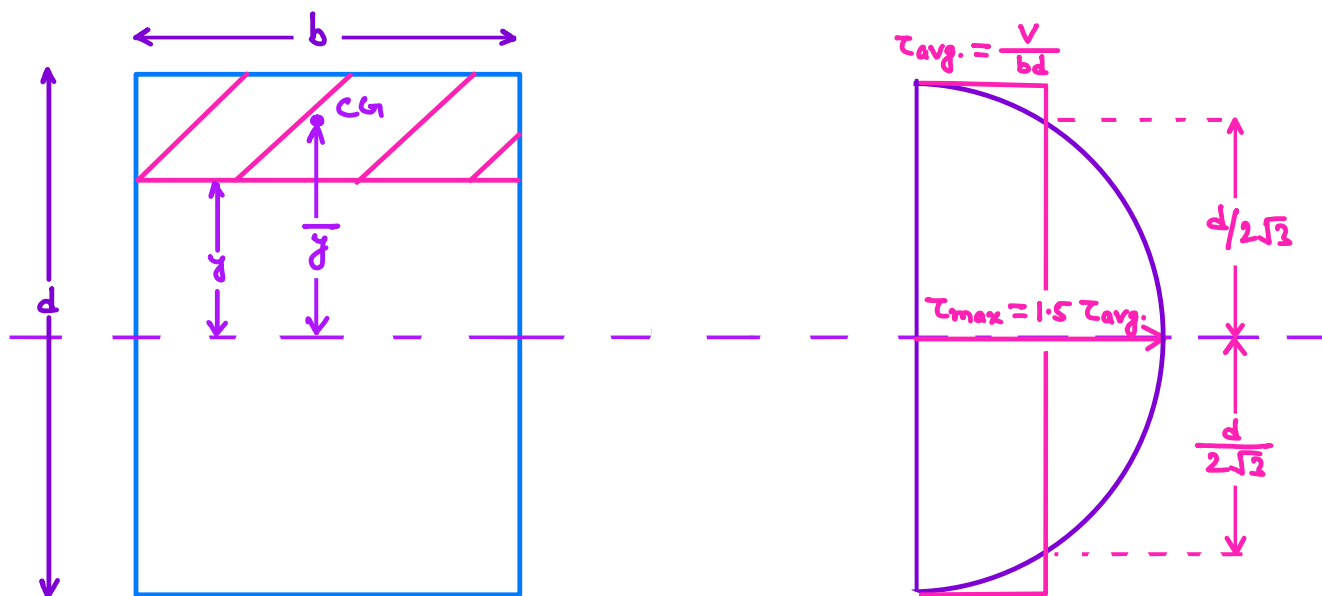
$$2H = C_2 - C_1 = \frac{VA \bar{y} \Delta H}{I}$$

$$\text{force carried by each bolt} = \frac{V A_f \bar{\Delta} H}{2 I}$$

$$\rightarrow \frac{2.5 \times (76 \times 18) 47 \times 44}{\left[\left(112 \times \frac{(112)^2}{12} \right) - \left(\frac{76 \times (76)^3}{12} \right) \right] \times 2}$$

$$= 342.25 \text{ N.}$$

Shear Stress in Rectangular Section



$$\tau = \frac{VA\bar{y}}{Ib}, \quad I = \frac{bd^3}{12}$$

$$A\bar{y} = b\left(\frac{d}{2} - y\right)\left(y + \frac{\frac{d}{2} - y}{2}\right) = \frac{b}{2}\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right) = \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$\tau = \frac{V \cdot \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)}{\frac{bd^3}{12} \cdot b}$$

$$\tau = \frac{6V}{bd^3} \left(\frac{d^2}{4} - y^2 \right)$$

for $\tau_{\max}$, $\frac{d\tau}{dy} = 0$

$$\frac{d}{dy} \left[\frac{6V}{bd^3} \left(\frac{d^2}{4} - y^2 \right) \right] = 0$$

$$-\frac{6V}{bd^3} \cdot 2y = 0, \quad y = 0$$

Hence, maximum shear stress occurs at $y = 0$, i.e. N.A.

$$\tau_{\max} = \frac{6V}{bd^3} \left[\frac{d^2}{4} - y^2 \right]$$

$$\tau_{\max} = \frac{3}{2} \frac{V}{bd}$$

or

$$\tau_{\max} = \frac{3}{2} \tau_{\text{avg.}}$$

$$\tau_{\text{avg.}} = \frac{V}{bd}$$

for $\tau = \tau_{\text{avg.}}$

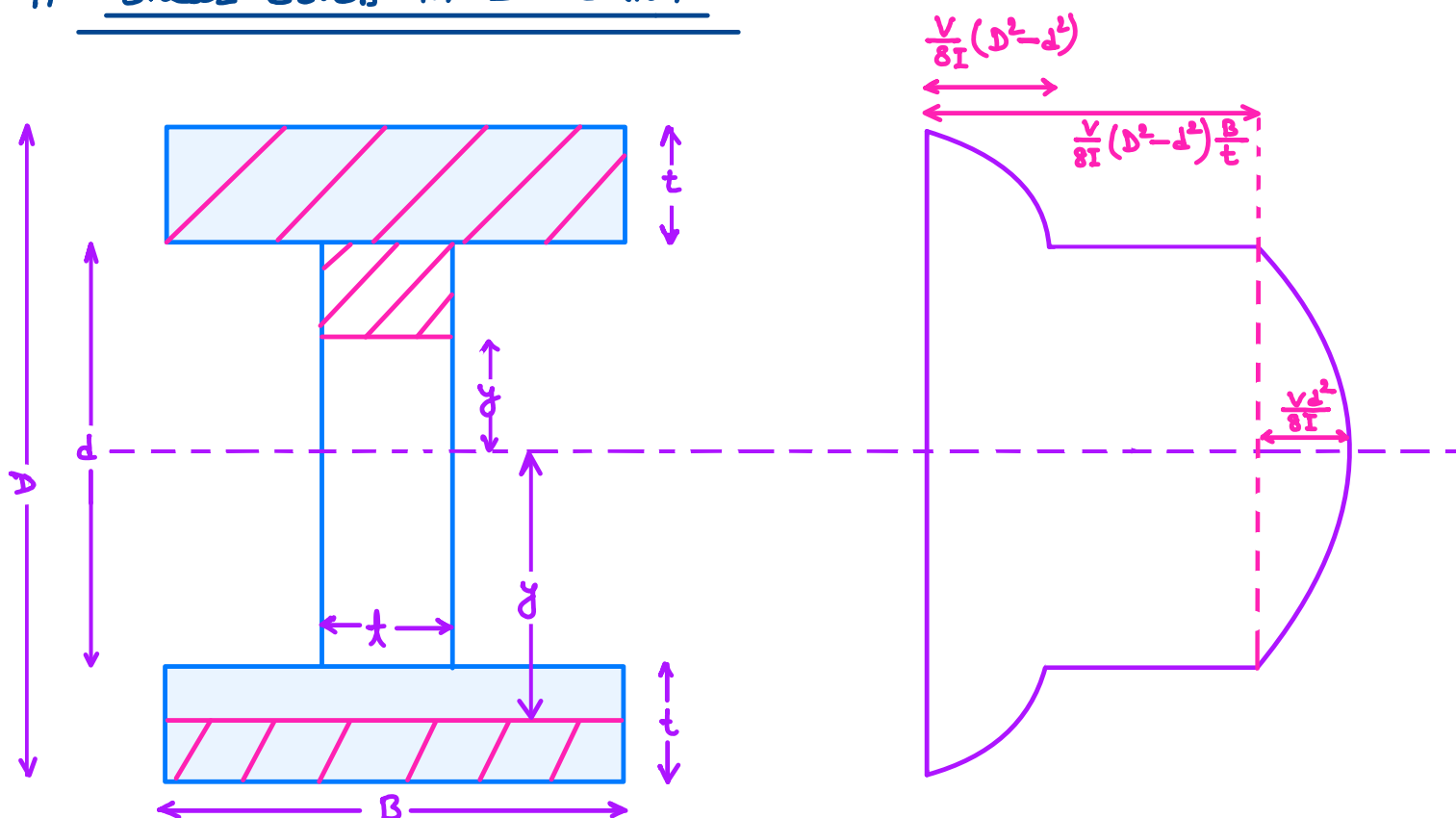
$$\frac{6V}{bd^3} \left(\frac{d^2}{4} - y^2 \right) = \frac{V}{bd}$$

$$\frac{d^2}{4} - y^2 = \frac{d^2}{6}$$

$$y^2 = \frac{d^2}{4} - \frac{d^2}{6} = \frac{d^2}{12}$$

$$y = \pm \frac{d}{2\sqrt{3}}$$

Shear Stress in I-Section



a) Shear stress in web

$$\tau = \frac{VA\bar{y}}{Ib}$$

$$A\bar{y} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4} + \frac{d}{2}\right) + t\left(\frac{d}{2} - y\right)\left(\frac{\frac{d}{2} - y}{2} + y\right)$$

$$A\bar{y} = \frac{B}{8}(D^2 - d^2) + \frac{t}{2}\left(\frac{d^2}{4} - y^2\right)$$

$$I = \frac{BD^3}{12} - \frac{(B-t)(D-2t)^3}{12}$$

$$\tau = \frac{\sqrt{\left[\frac{B}{8}(D^2 - d^2) + \frac{t}{2}\left(\frac{d^2}{4} - y^2\right)\right]}}{I \cdot t}$$

$$\tau = \frac{V}{8I}(D^2 - d^2)\frac{B}{t} + \frac{V}{2I}\left(\frac{d^2}{4} - y^2\right)$$

- maximum shear stress is at $y = 0$, i.e. at NA.

$$\tau_{web\max} = \frac{V}{8I}(D^2 - d^2)\frac{B}{t} + \frac{V}{2I}\left(\frac{d^2}{4}\right)$$

- at $y = \pm \frac{d}{2}$, i.e. at interface of flange.

$$\tau_{web} = \frac{V}{8I}(D^2 - d^2)\frac{B}{t} + \frac{V}{2I}\left(\frac{d^2}{4} - \frac{d^2}{4}\right)$$

$$\tau_{web} = \frac{V}{8I}(D^2 - d^2)\frac{B}{t}$$

b) Shear stress in Flange

$$\tau = \frac{VA\bar{y}}{I \cdot b}$$

$$A\bar{y} = B\left(\frac{D}{2} - y\right)\left(\frac{y + \frac{D}{2} - y}{2}\right) = B\left(\frac{D}{2} - y\right)\left(\frac{D}{2} + y\right)$$

$$A\bar{y} = \left(\frac{D^2}{4} - y^2\right)$$

$$\tau = \frac{V \frac{B}{2}\left(\frac{D^2}{4} - y^2\right)}{I B}$$

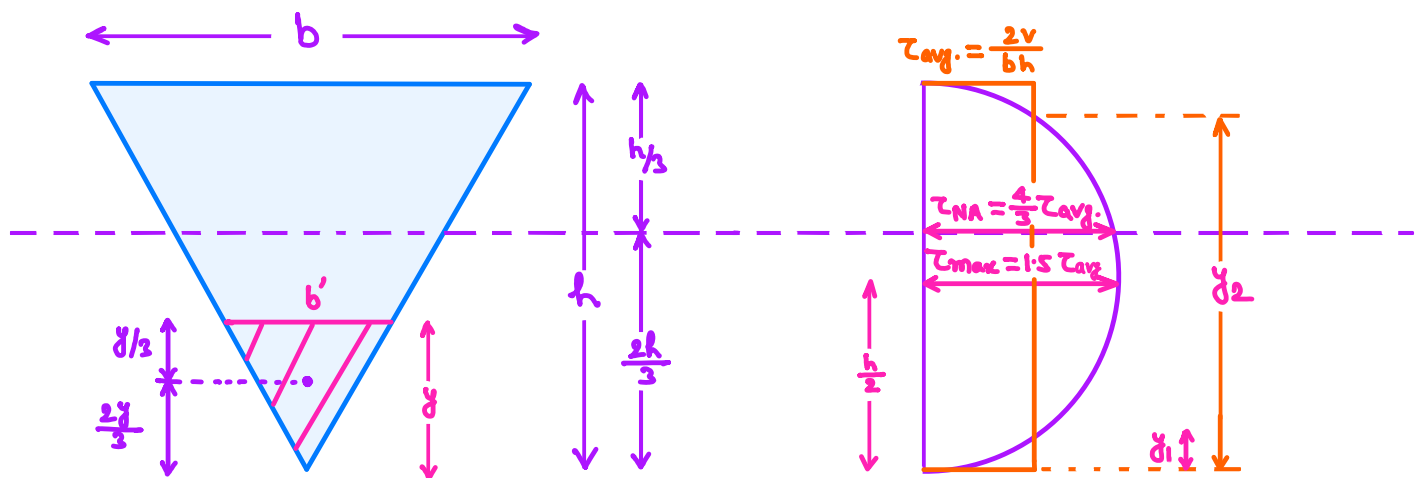
$$\tau = \frac{V}{2I}\left(\frac{D^2}{4} - y^2\right)$$

$$\tau_{\max \text{ flange}} = \frac{V}{2I} \left(\frac{D^2}{4} - \frac{d^2}{4} \right) = \frac{V}{8I} (D^2 - d^2)$$

NOTE:—

- Normally for I-section, approximately (80–85)% of shear is resisted by web.

Shear stress in Triangular Section



$$\tau = \frac{VA\bar{y}}{Ib}$$

$$A\bar{y} = \frac{1}{2} b'y \left(\frac{2}{3} h - \frac{2}{3} y \right) = \frac{b'y}{3} (h - y)$$

$$\tau = \frac{Vb'y(h-y)}{3Ib'}$$

$$\tau = \frac{Vy(h-y)}{3I}$$

for maximum τ , $\frac{d\tau}{dy} = 0$

$$\frac{d}{dy} \left(\frac{V(hy - y^2)}{3I} \right) = 0 \Rightarrow \frac{V}{3I} (h - 2y) = 0 \Rightarrow h - 2y = 0$$

$$y = \frac{h}{2}$$

Hence, Shear Stress is maximum at mid depth

$$\tau_{max} = \frac{V}{3I} \left(h \cdot \frac{h}{2} - \frac{h^2}{4} \right) = \frac{V}{3I} \frac{h^2}{4}$$

$$\tau_{max} = \frac{V h^2}{12 I}$$

$$\tau_{max} = \frac{V h^2}{12 \cdot \frac{b h^3}{36}} = \frac{V}{12 \cdot \frac{b h}{2 \cdot 18}} = \frac{3}{8} \frac{V}{\frac{b h}{2}} = \frac{3}{2} \tau_{avg.}$$

$$\tau_{max} = 1.5 \tau_{avg.}$$

$$\text{where, } \tau_{avg.} = \frac{V}{A} = \frac{V}{\left(\frac{b h}{2}\right)} = \frac{2V}{b h}$$

Now,

$$\tau_{NA} = \frac{V}{3I} y(h-y) \Big|_{y=\frac{2}{3}h}$$

$$\tau_{NA} = \frac{V}{3I} \left(h \cdot \frac{2}{3}h - \left(\frac{2}{3}h\right)^2 \right) \Rightarrow \frac{V}{3I} \left[\frac{2}{3}h^2 - \frac{4}{9}h^2 \right]$$

$$\tau_{NA} = \frac{2}{27} \frac{V h^2}{I}$$

OR

$$\tau_{NA} = \frac{2}{27} \frac{V h^2}{\frac{b h^3}{36}} = \frac{2}{27} \frac{V}{\frac{b h}{2 \cdot 18}} = \frac{4}{3} \tau_{avg.}$$

$$\tau_{NA} = \frac{4}{3} \tau_{avg.}$$

Now,

for $\tau = \tau_{avg.}$

$$\frac{V}{3I} y(h-y) = \frac{V}{\left(\frac{b h}{2}\right)}$$

$$\frac{V}{3 \cdot \frac{b h^3}{36}} y(h-y) = \frac{V}{\left(\frac{b h}{2}\right)} \Rightarrow$$

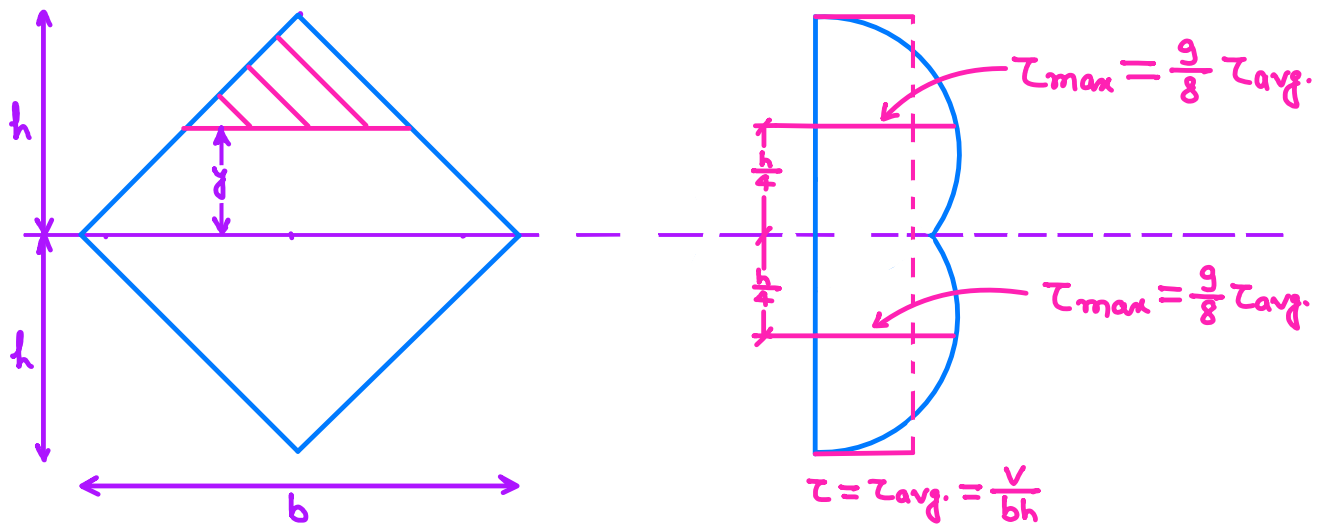
$$6y(h-y) = h^2$$

$$6y^2 - 6yh + h^2 = 0$$

$$y_2 = 0.788 h$$

$$y_1 = 0.212 h.$$

Shear Stress in Diamond Section



$$\tau = \frac{VA\bar{y}}{I \cdot b}$$

$$A\bar{y} = \frac{1}{2} \frac{b}{h} (h-y)(h-y) \cdot \left(\frac{h-y}{2} + y \right) \Rightarrow \frac{b}{2h} (h-y)^2 (2y+h)$$

$$I = \frac{bh^3}{12} \times 2 \Rightarrow \frac{bh^3}{6}$$

$$\tau = \frac{V \frac{b}{6h} (h-y)^2 (2y+h)}{\frac{bh^3}{6} \cdot \frac{b}{h} (h-y)} = \frac{V}{bh^3} (h-y) (2y+h)$$

$$\tau = \frac{V}{bh^3} (h-y) (2y+h)$$

$$\text{for } \tau_{max}, \quad \frac{d\tau}{dy} = 0 \quad , \quad \frac{d}{dy} \left[\frac{V}{bh^3} (h-y) (2y+h) \right] = 0$$

$$\frac{d}{dy} \left\{ \frac{V}{bh^3} (2yh + h^2 - 2y^2 - yh) \right\} = 0$$

$$\frac{V}{bh^3} (h + 0 - 4y) = 0.$$

$$y = \frac{h}{4}$$

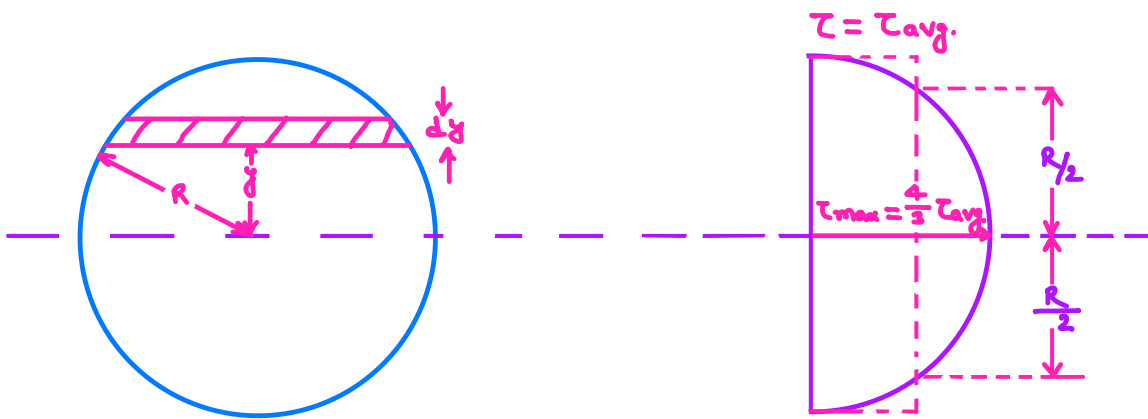
$$\tau_{\max} = \frac{V}{bh^3} \left[\left(h - \frac{h}{4} \right) \left(2 \cdot \frac{h}{4} + h \right) \right] \Rightarrow \frac{V}{bh^3} \left(\frac{3h}{4} \right) \left(\frac{3h}{2} \right)$$

$$\tau_{\max} = \frac{9}{8} \frac{V}{bh} = \frac{9}{8} \tau_{\text{avg.}}$$

$$\text{At } y=0, \quad \tau = \frac{V}{bh^3} (h-0) (2(0)+h) = \frac{V}{bh} = \tau_{\text{avg}}$$

$$\tau = \tau_{\text{avg.}}$$

Shear Stress Distribution in circular section



$$\tau = \frac{VA\bar{y}}{I \cdot b}$$

$$A\bar{y} = \int dA \bar{y} = \int_y^R dy \sqrt{R^2 - y^2} \times 2 \cdot y = \int_y^R 2y \sqrt{R^2 - y^2} dy$$

$$A\bar{y} = - \int_{\sqrt{R^2 - y^2}}^0 2x dx \cdot x \Rightarrow \int_0^{\sqrt{R^2 - y^2}} x^2 dx$$

$$A\bar{y} = 2 \left| \frac{x^3}{3} \right|_0^{\sqrt{R^2 - y^2}} \Rightarrow \frac{2}{3} \left[(R^2 - y^2)^{3/2} - 0 \right]$$

$$A\bar{y} = \frac{2}{3} (R^2 - y^2)^{3/2}$$

$$\text{let } \sqrt{R^2 - y^2} = x$$

$$R^2 - y^2 = x^2$$

$$0 - 2y dy = 2x dx$$

$$-y dy = x dx$$

$$\text{when } y=y \Rightarrow x = \sqrt{R^2 - y^2}$$

$$y=R \Rightarrow x=0$$

$$I = \frac{\pi D^4}{64} = \frac{\pi R^4}{4}$$

$$\tau = \frac{V \cdot \frac{2}{3} (R^2 - y^2)^{3/2}}{\frac{\pi R^4}{4} \cdot 2x (R^2 - y^2)^{1/2}} = \frac{4}{3} \frac{V (R^2 - y^2)}{\pi R^4}$$

$$\tau = \frac{4}{3} \frac{V}{\pi R^2} \left(1 - \frac{y^2}{R^2}\right) \Rightarrow \tau = \frac{4}{3} \tau_{avg} \left(1 - \frac{y^2}{R^2}\right)$$

Here maximum Shear Stress occurs at $y=0$, i.e at N.A

$$\tau_{max} = \frac{4}{3} \tau_{avg.}$$

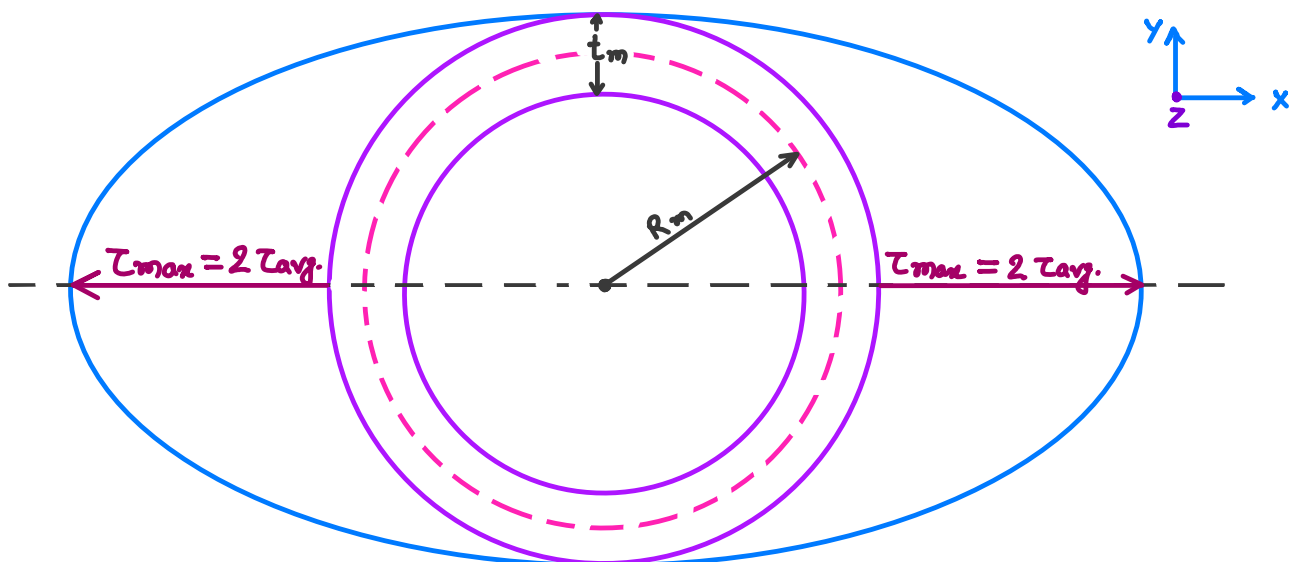
Also, for $\tau = \tau_{avg.}$

$$\tau_{avg.} = \frac{4}{3} \tau_{avg.} \left(1 - \frac{y^2}{R^2}\right) \Rightarrow 1 = \frac{4}{3} \left(1 - \frac{y^2}{R^2}\right)$$

$$\frac{3}{4} = 1 - \frac{y^2}{R^2} \Rightarrow \frac{y^2}{R^2} = 1 - \frac{3}{4} = \frac{1}{4}$$

$$y = \pm \frac{R}{2}$$

Shear Stress Distribution for thin walled Section :-



$$\text{Area} = 2\pi R_m \cdot t_m$$

$$I_{xx} + I_{yy} = I_{zz}$$

$$I_{xx} = I_{yy} \rightarrow (\text{Symmetrical})$$

$$I_{zz} = A R_m^2$$

$$2I_{xx} = A R_m^2$$

$$I_{xx} = \frac{A R_m^2}{2} = \frac{2\pi R_m t_m R_m^2}{2}$$

$$I_{xx} = \pi R_m^3 t_m$$

$$A\bar{y} = \frac{\pi \left(R_m + \frac{t_m}{2}\right)^2}{2} \times \frac{4\left(R_m + \frac{t_m}{2}\right)}{3\pi} - \frac{\pi \left(R_m - \frac{t_m}{2}\right)^2}{2} \times \frac{4\left(R_m - \frac{t_m}{2}\right)}{3\pi}$$

$$A\bar{y} = \frac{2}{3} \left[\left(R_m + \frac{t_m}{2}\right)^3 - \left(R_m - \frac{t_m}{2}\right)^3 \right]$$

$$A\bar{y} = \frac{2}{3} R_m \left[\left(1 + \frac{t_m}{2R_m}\right)^3 - \left(1 - \frac{t_m}{2R_m}\right)^3 \right]$$

$$= \frac{2}{3} R_m^3 \left[\left(1 + \frac{3t_m}{2R_m} + \dots\right) - \left(1 - \frac{3t_m}{2R_m} + \dots\right) \right]$$

$$= \frac{2}{3} R_m^3 \left[1 + \frac{3t_m}{2R_m} - 1 + \frac{3t_m}{2R_m} \right]$$

$$= \frac{2}{3} R_m^3 \times \frac{3t_m}{R_m}$$

$$A\bar{y} = 2 R_m^2 t_m$$

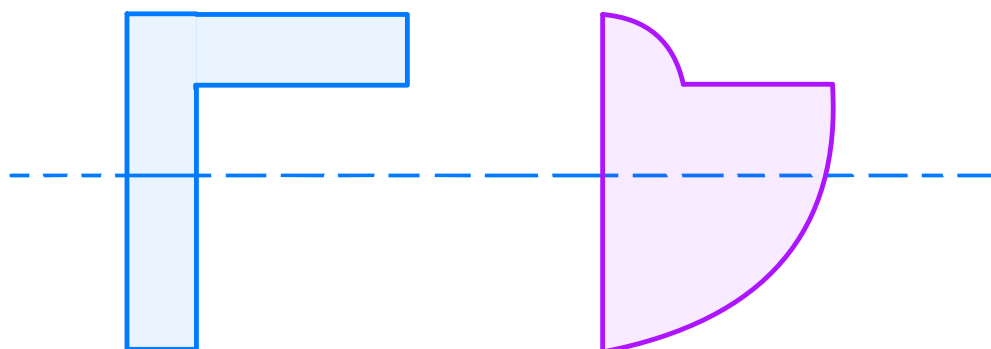
$$\tau = \frac{V A \bar{y}}{I \cdot b} = \frac{V (2 R_m^2 \cdot t_m)}{\pi R_m^3 t_m (2 t_m)} = \boxed{\frac{V}{\pi \cdot R_m t_m} = \tau}$$

$\therefore$ this is the Shear stress at N.A., so it is maximum.

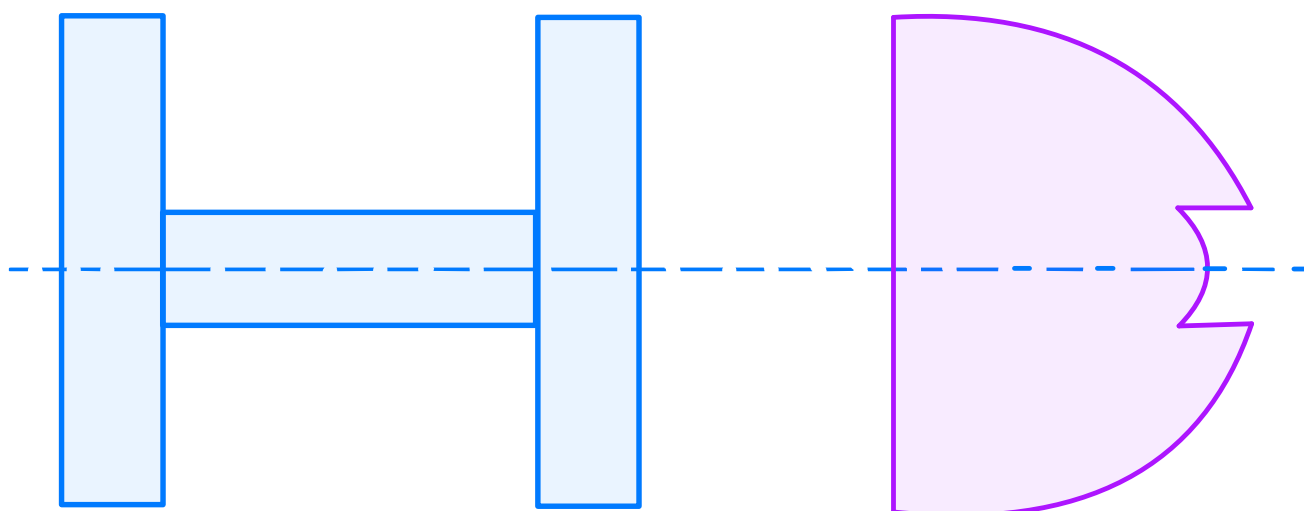
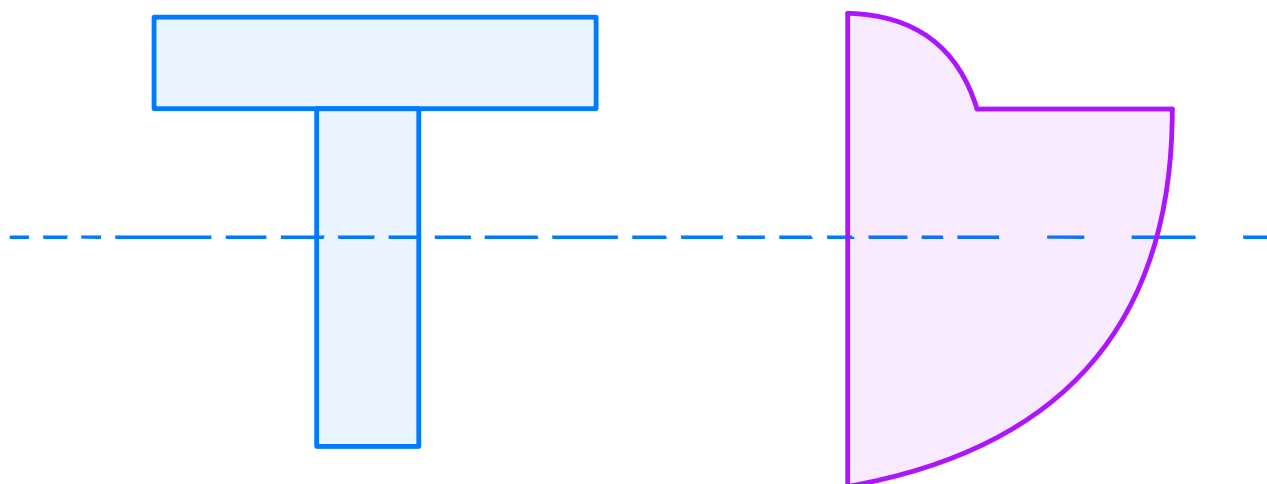
$$\tau_{\max} = \frac{V}{\pi} \cdot \frac{1}{R_m t_m} = \frac{2V}{2\pi R_m t_m} = 2 \left(\frac{V}{2\pi R_m t_m} \right)$$

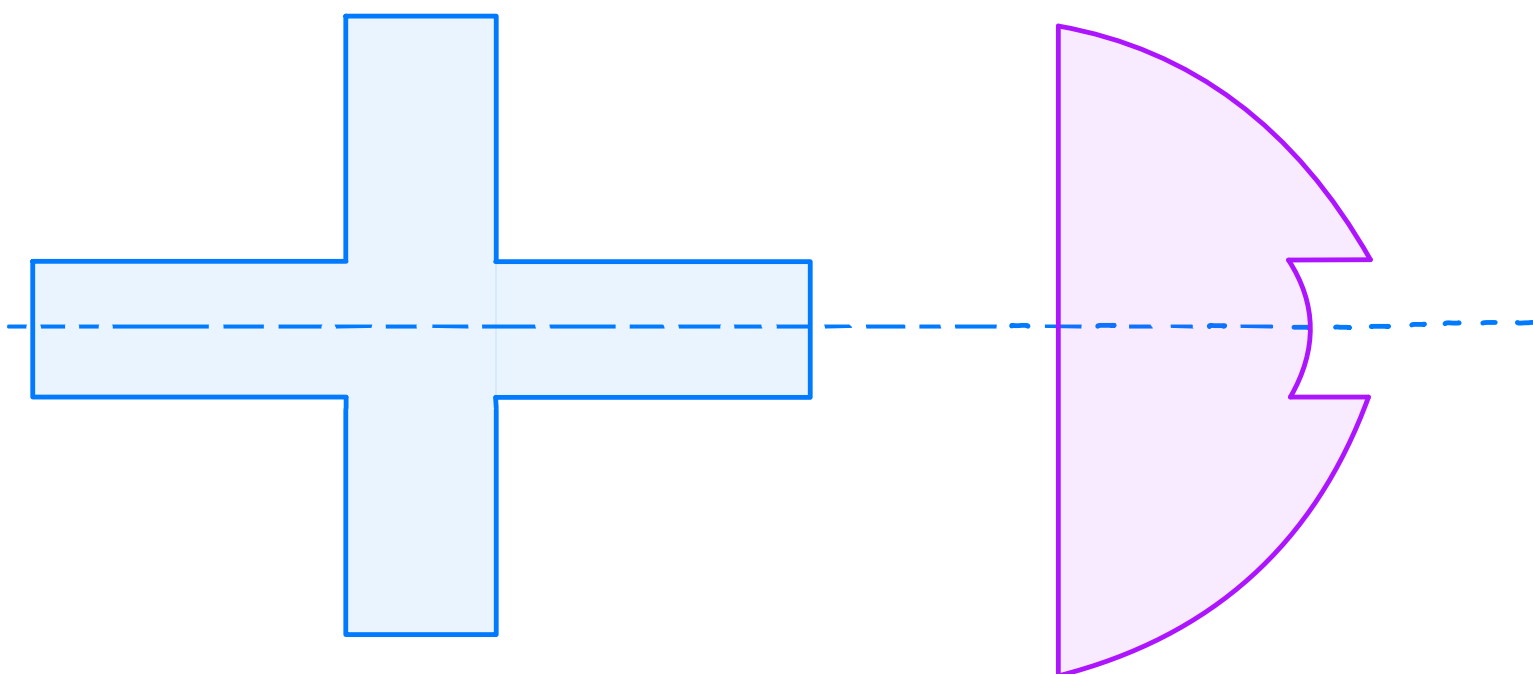
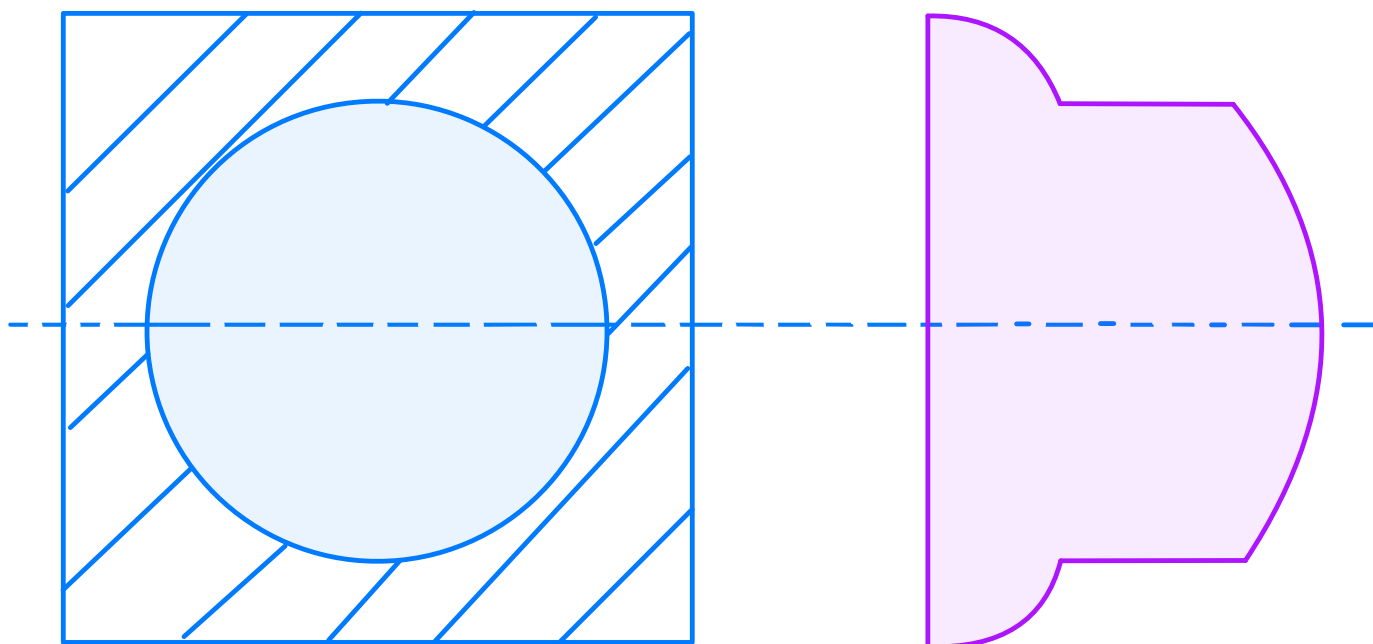
$$\boxed{\tau_{\max} = 2 \tau_{\text{avg.}}}$$

Transverse Shear Stress Distribution Diagram



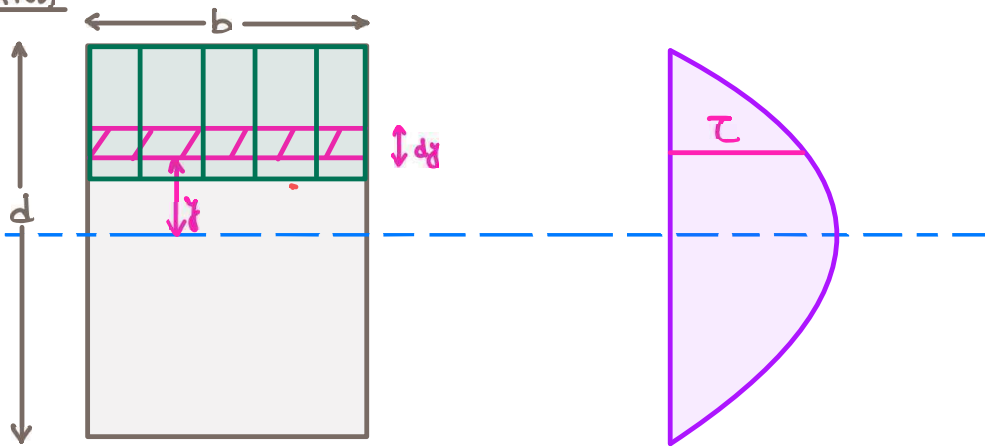
$$\text{Shear Stress} \propto \frac{1}{\text{width}}$$





Q. > find the shear force resisted by top $\frac{1}{3}$ rd of a rectangular beam if shear force on section is "V."

Solution



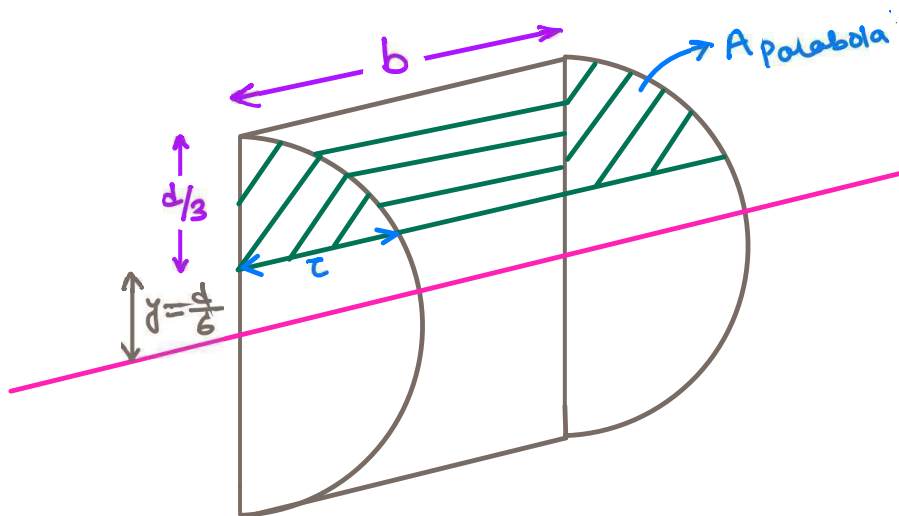
$$dF = \tau dA = \frac{V}{2I} \left(\frac{d^2}{4} - y^2 \right) b dy$$

$$F = \int_{d/6}^{d/2} dF = \frac{Vb}{2I} \left(\frac{d^2}{4} - y^2 \right) dy$$

$$F = \frac{Vb}{2I} \left[\frac{d^2}{4} y - \frac{y^3}{3} \right]_{d/6}^{d/2}$$

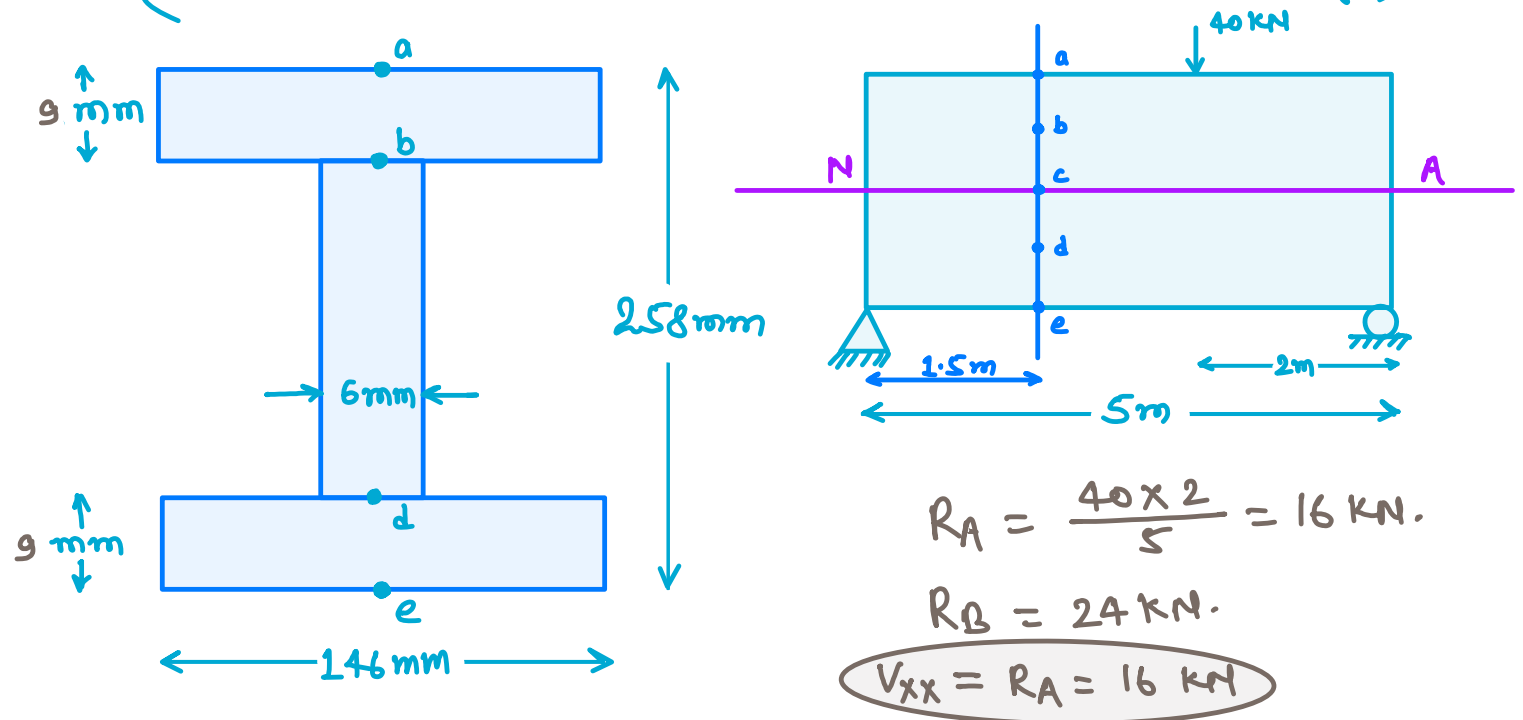
$$F = \frac{7}{27} V$$

or



$$F = A \cdot b \Rightarrow \int dA \cdot b \Rightarrow F = \frac{7}{27} V$$

Q. > A simply supported beam having I-cross-section of length 5m in which transverse load of 40 kN is acting as shown. Find the shearing stress at points a, b, c, d, e (point b is within web & point d is in flange).



$$R_A = \frac{40 \times 2}{5} = 16 \text{ kN.}$$

$$R_B = 24 \text{ kN.}$$

$$V_{xx} = R_A = 16 \text{ kN}$$

Solution

$$I = \frac{146 \times 258^3}{12} - \frac{(146 - 6)(258 - 2 \times 9)^3}{12} = 4.76 \times 10^7 \text{ mm}^4.$$

a) point A, $A\bar{y} = 0$
 $\tau = \frac{VA\bar{y}}{Ib} = 0$

b) point b, $A\bar{y} = 146 \times 9 \times \left(\frac{258-9}{2}\right) = 163593 \text{ mm}^3$
 $\tau = \frac{VA\bar{y}}{Ib} = \frac{16 \times 10^3 \times 163593}{4.76 \times 10^7 \times 6} = 9.165 \text{ N/mm}^2.$

c) point c,
 $A\bar{y} = 146 \times 9 \times \left(\frac{258-9}{2}\right) + \frac{(258-18)}{2} \times 6 \times \left(\frac{258-18}{4}\right)$
 $A\bar{y} = 206793 \text{ mm}^3$
 $\tau = \frac{VA\bar{y}}{Ib} = \frac{16 \times 10^3 \times 206793}{4.76 \times 10^7 \times 6} = 11.57 \text{ N/mm}^2$

d.) At point d,

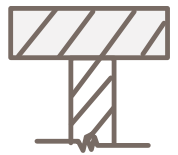
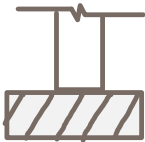
$$A\bar{y} = 146 \times 9 \left(\frac{258-9}{2} \right) = 163593 \text{ mm}^3$$

OR,

$$A\bar{y} = 146 \times 9 \times \left(\frac{258-9}{2} \right) + (258-18) \times 6 \times 9$$

$$= 163593 \text{ mm}^3.$$

$$\tau = \frac{VA\bar{y}}{I \cdot b} = \frac{16 \times 10^3 \times 163593}{4.76 \times 10^7 \times 146} = 0.376 \text{ N/mm}^2$$

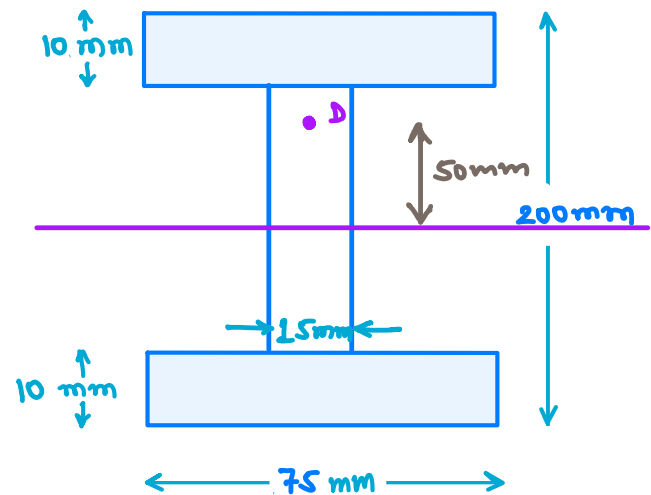
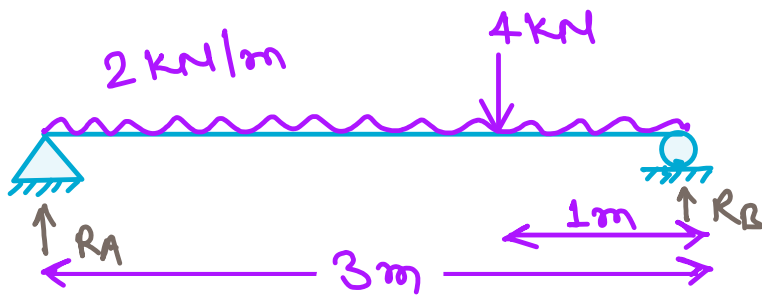


e.) point e,

$$A\bar{y} = 0$$

$$\tau = 0.$$

Q. A Simply Supported beam of 3m span is subjected to the load as shown. The beam is of I-section & all dimensions are shown in mm. determine the principal stress at point 'd' in the web. The section is located at a distance of 1m from right support of the beam.



Solution

$$\sum f_y = 0$$

$$R_A + R_B = 2 \times 3 + 4 = 10 \text{ kN} \quad \dots (i)$$

$$R_A \times 3 = 2 \times 3 \times \frac{3}{2} + 4 \times 1 \Rightarrow R_A = 4.33 \text{ kN}$$

$$R_B = 5.67 \text{ kN.}$$

$$V_{xx} \text{ left side} = R_A - 2 \times 2 = 4.33 - 4 = 0.33 \text{ kN}$$

$$V_{xx} \text{ on right side} = R_A - 2 \times 2 - 4 = -3.67 \text{ kN.}$$

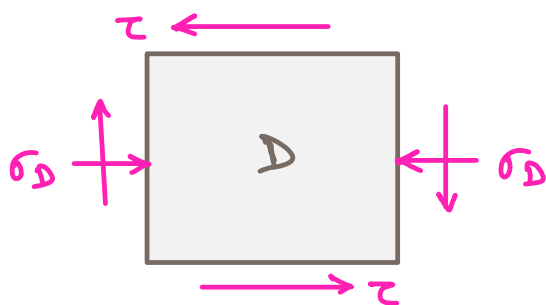
$$\begin{aligned} Bm_{xx} &= R_A \times 2 - 2 \times 2 \times 1 \\ &= 4.33 \times 2 - 4 = 4.67 \text{ kN-m.} \end{aligned}$$

$$\begin{aligned} I &= \frac{75 \times 200^3}{12} - \frac{60 \times 180^3}{12} \\ &= 2.084 \times 10^7 \text{ mm}^4 \end{aligned}$$

$$\sigma_D = \frac{M}{I} y = \frac{4.67 \times 10^6 \times 50}{2.084 \times 10^7} = 11.2 \text{ N/mm}^2 \text{ (compressive)}$$

$$\tau = \frac{VA\bar{y}}{I \cdot b} = \frac{-3.67 \times 10^3 \left[75 \times 10 \times (100 - 5) + (100 - 10 - 50) \times 15 \times \left(\frac{40}{2} + 50 \right) \right]}{2.084 \times 10^7 \times 15}$$

$$\tau = 1.337 \text{ N/mm}^2 \text{ (Anticlockwise)}$$



$$\sigma_x = -11.2 \text{ N/mm}^2$$

$$\sigma_y = 0$$

$$\tau_{xy} = -1.337 \text{ N/mm}^2$$

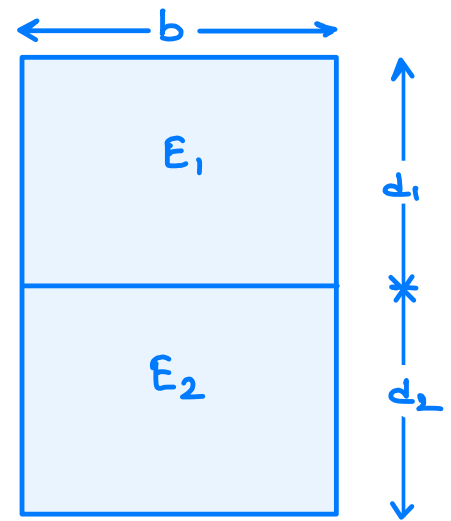
$$\begin{aligned} \sigma_{p1}/\sigma_{p2} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ &= \frac{-11.2 + 0}{2} \pm \sqrt{\left(\frac{-11.2 - 0}{2}\right)^2 + (-1.337)^2} \end{aligned}$$

$$\sigma_{p1} = -11.35 \text{ N/mm}^2$$

$$\sigma_{p2} = 0.15 \text{ N/mm}^2$$

Composite Beam

- A beam made up of more than one material is termed as composite beam.
- Shear stress in these beams is computed as follows :-



- Find the transform section.
- Find the N.A. of transformed section.
- Find shear force per unit length (shear flow at given section)

$$q = \frac{VA\bar{y}}{I}$$

$A\bar{y} \rightarrow$ moment of transformed area above given point.

$I \rightarrow$ moment of inertia of transformed section about N.A.

- Shear stress at any point ,

$$\tau = \frac{VA\bar{y}}{I \cdot b} \times 1$$

$b \rightarrow$ width of Actual Section.

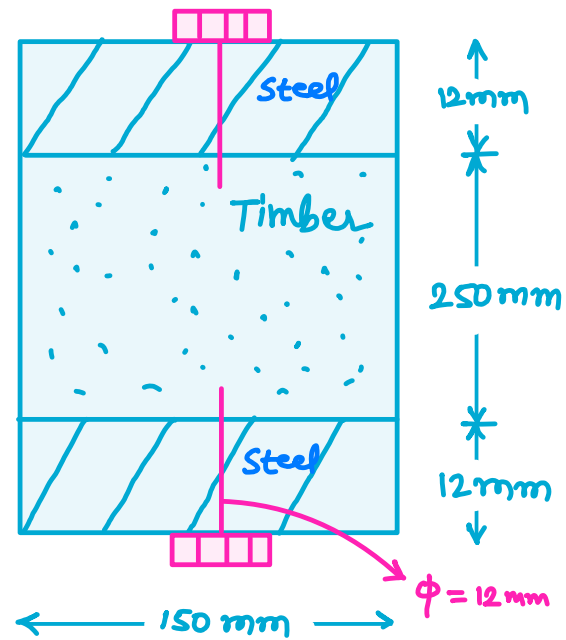
Q.7 A Composite beam is made by attaching the timber and Steel portions shown with bolt of 12 mm dia. Spaced longitudinally every 200 mm.

$$E_{\text{Timber}} = 10^4 \text{ N/mm}^2$$

$$E_{\text{Steel}} = 2 \times 10^5 \text{ N/mm}^2$$

Find the avg shear stress in bolt.

Assume, $V = 4 \text{ kN}$



Solution

$$m = \frac{E_s}{E_t} = \frac{2 \times 10^5}{10^4} = 20$$

transforming timber into Steel,

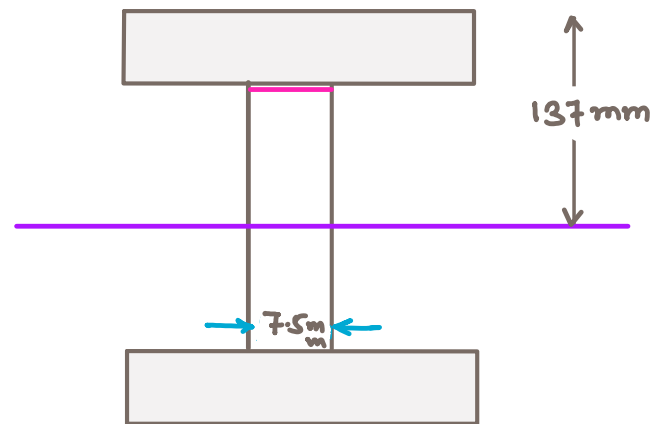
$$b_{eq} = \frac{150}{m} = \frac{150}{20} = 7.5 \text{ mm}$$

$$I = 150 \frac{(250 + 24)^2}{12} - \frac{142.5 \times 250^3}{12}$$

$$I = 7.16 \times 10^7 \text{ mm}^4$$

$$q = \frac{VA\bar{y}}{I} = \frac{4 \times 10^3 [150 \times 12 \times (137 - 6)]}{7.16 \times 10^7}$$

$$q = 13.17 \text{ N/mm}$$

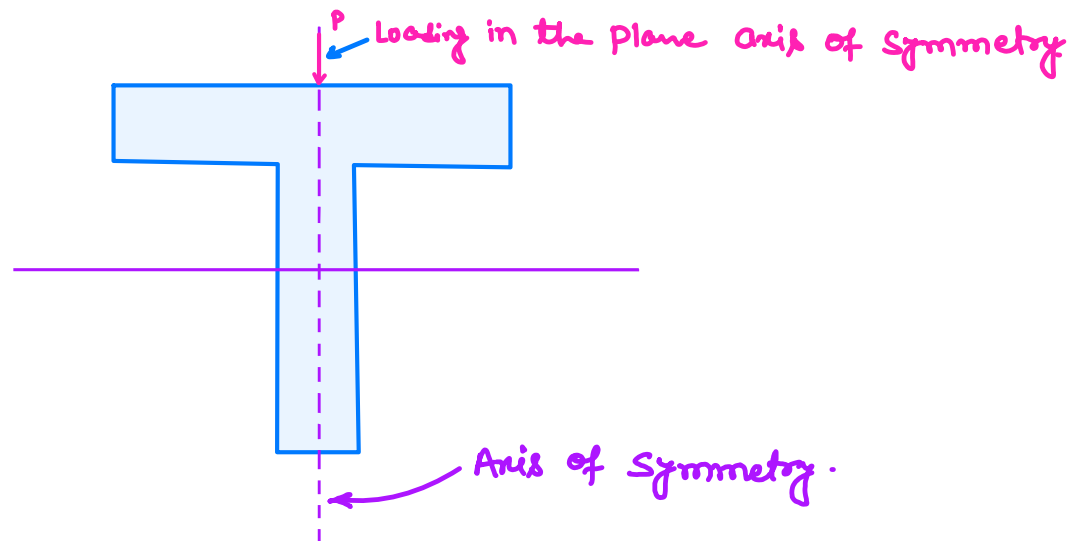


$$\text{Shear force} = q p = 13.17 \times 200 = 2634.63 \text{ N}$$

$$\text{Average shear} = \frac{2634.63}{\frac{\pi (12)^2}{4}} = 23.29 \text{ N/mm}^2$$

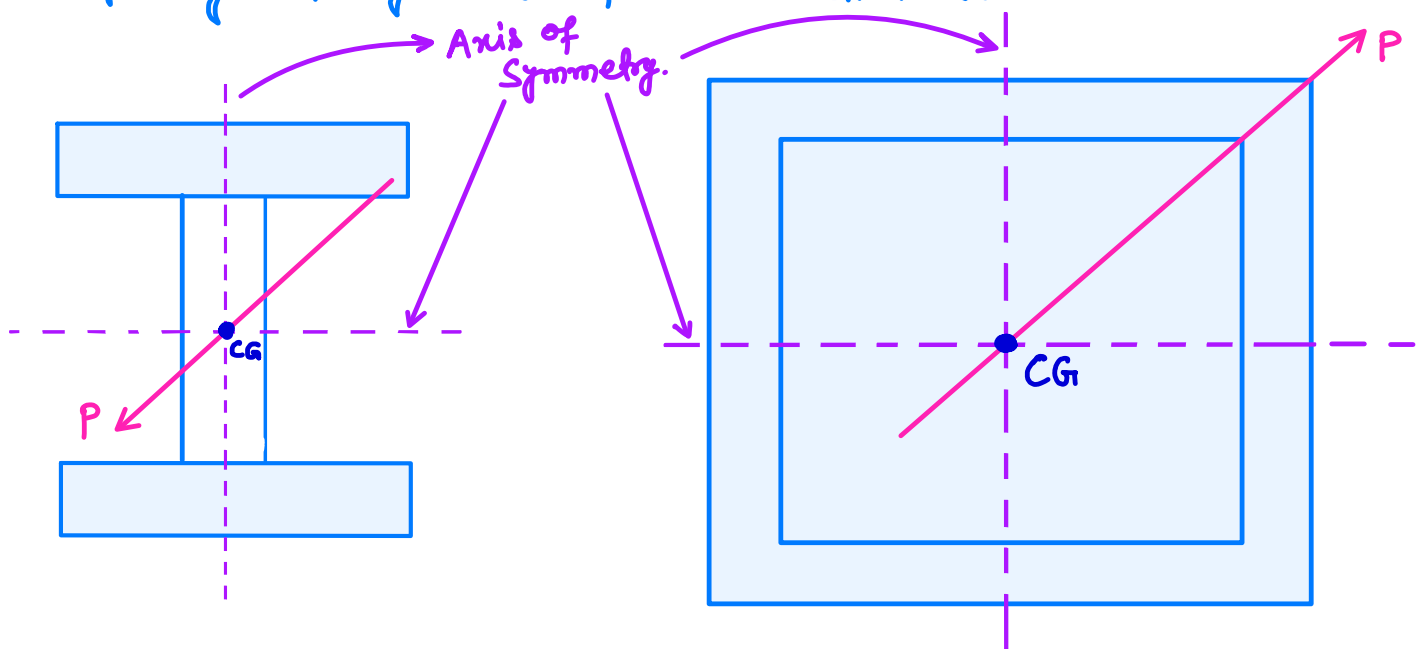
- The Shear Stress $\tau = \frac{VA\bar{y}}{I \cdot b}$ is applicable at point in cross-section only when there is no twisting of the section.
- Hence, we must know when the section would have no twisting as follows:-

(a) when the section have a plane of symmetry and loading is in that plane only, there is no twisting of section.

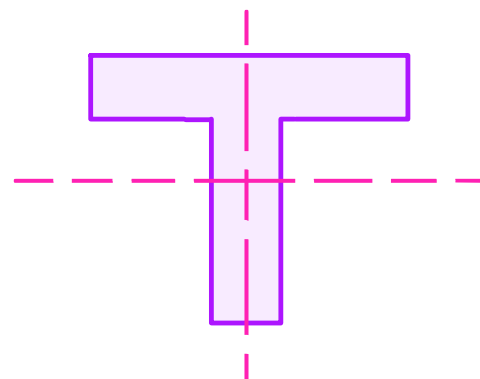
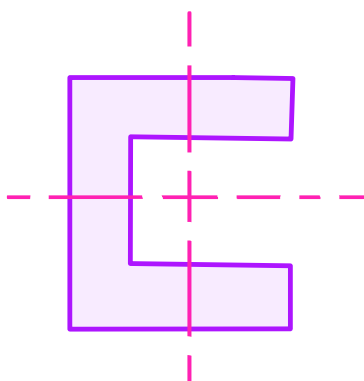
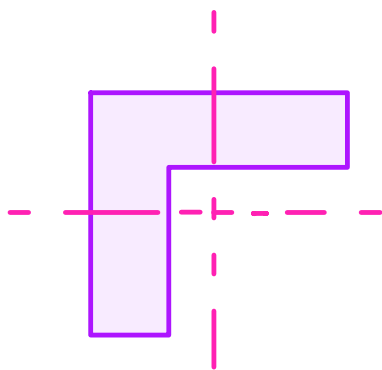
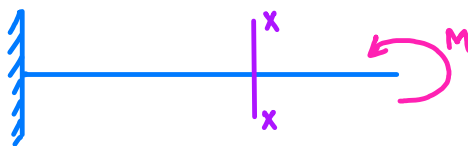


MCQ

(b) when a doubly symmetric section has skew (inclined) loading passing through C.G. of the Section it will have no twisting.

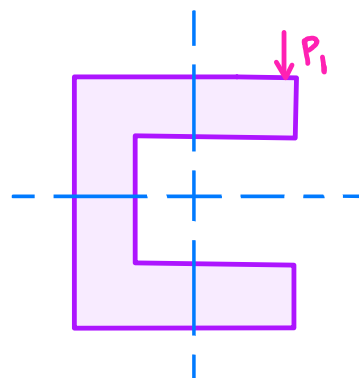
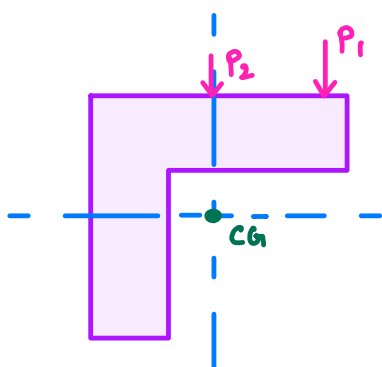
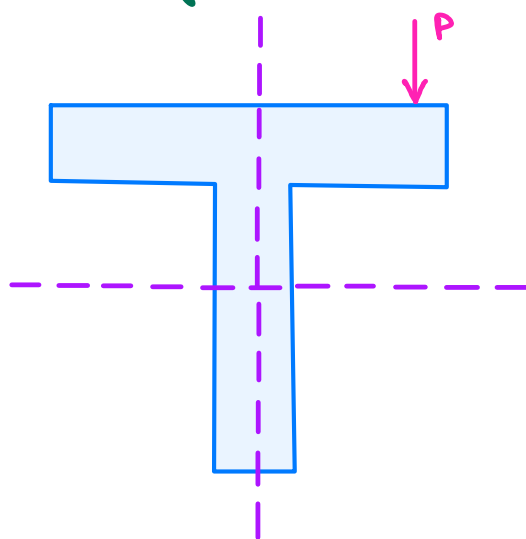
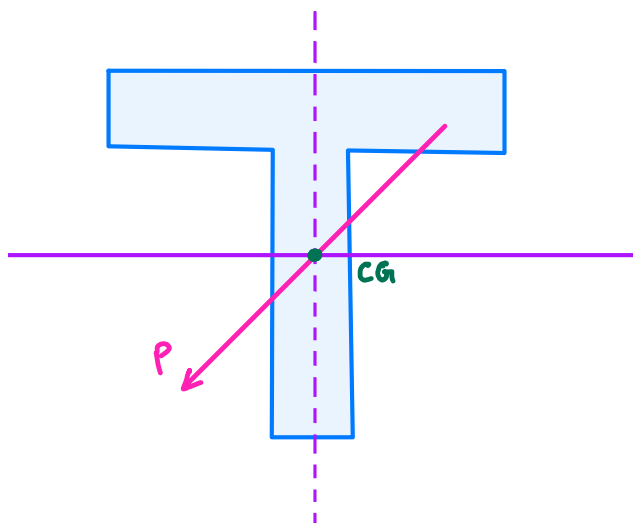


(c). when section is unsymmetrical or section has one plane of symmetry and is subjected to pure bending, there would be no twisting.

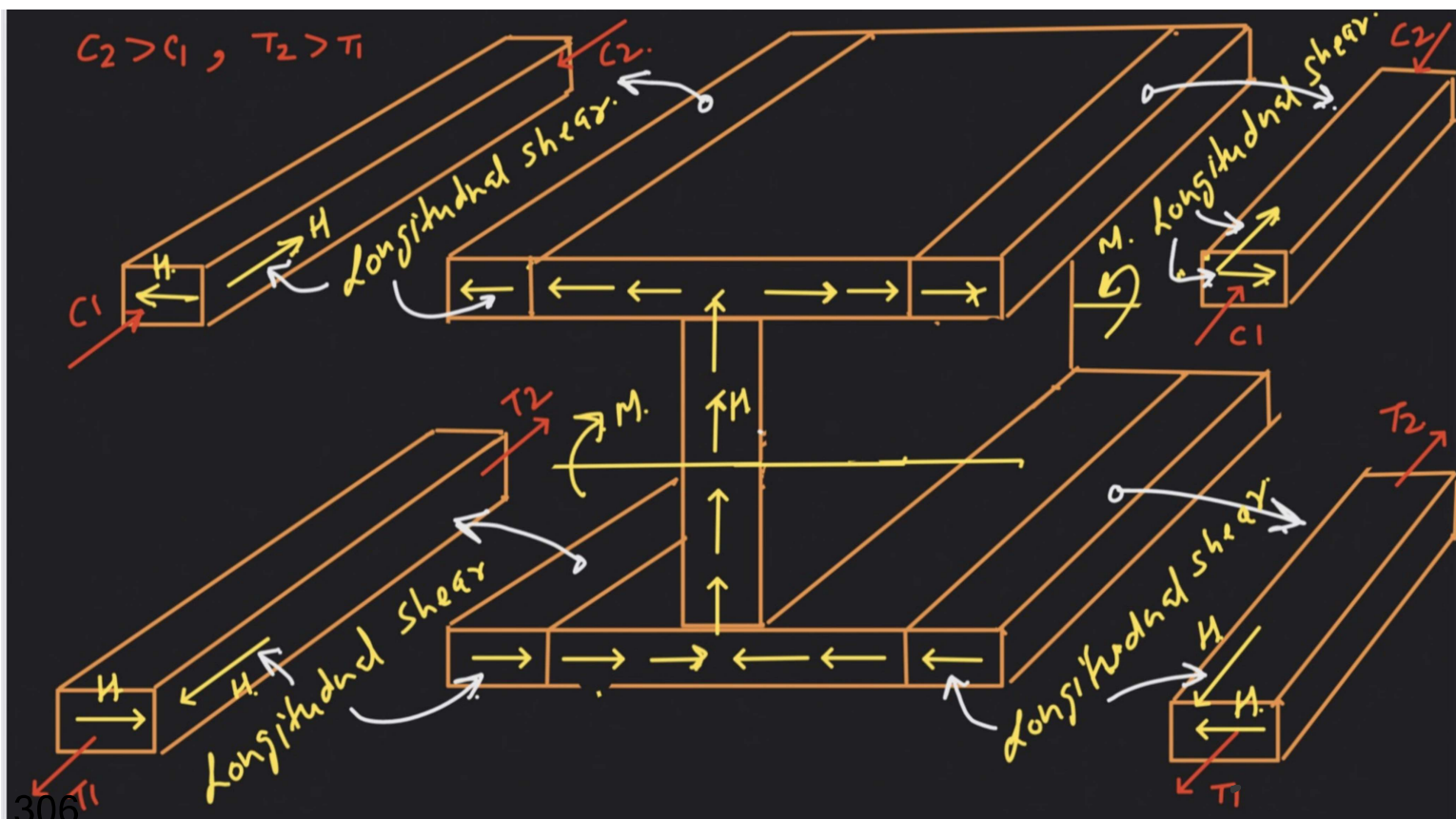
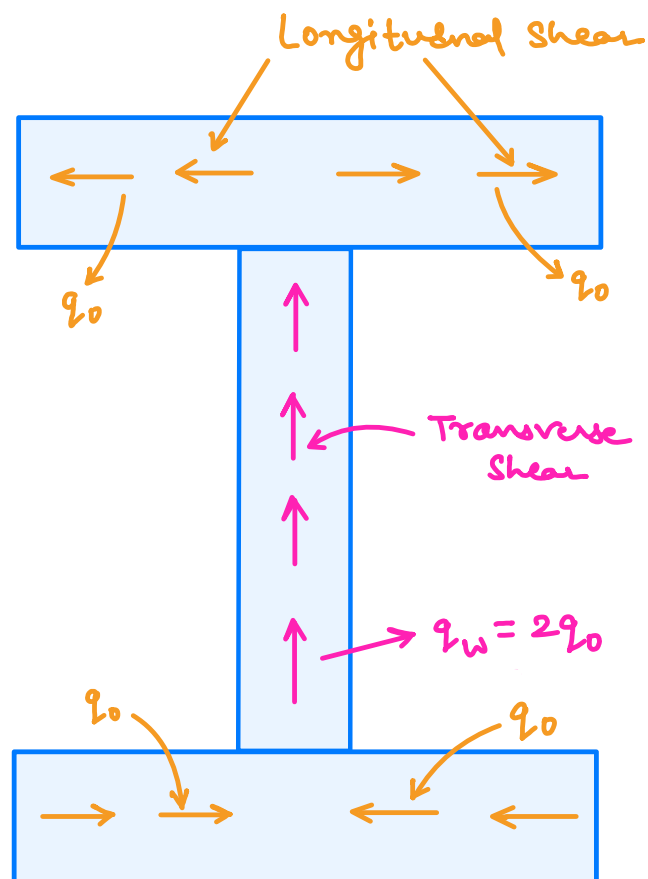
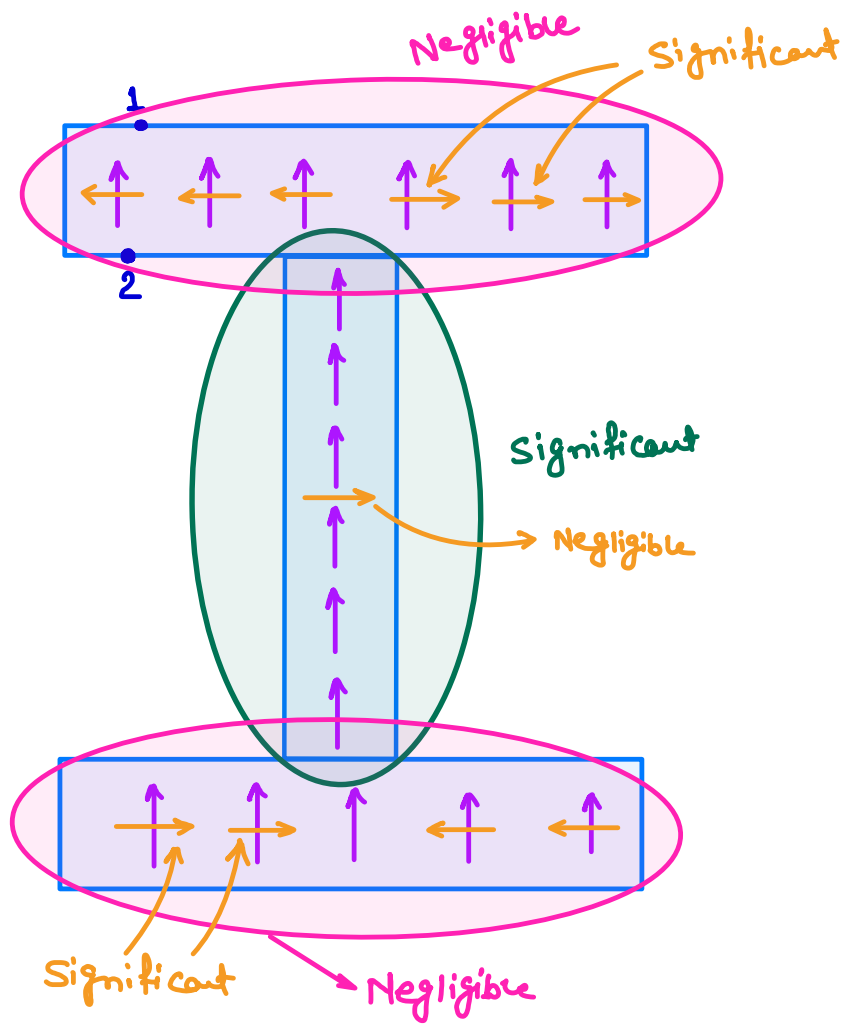


Note :-

Twisting would be observed in following cases:-

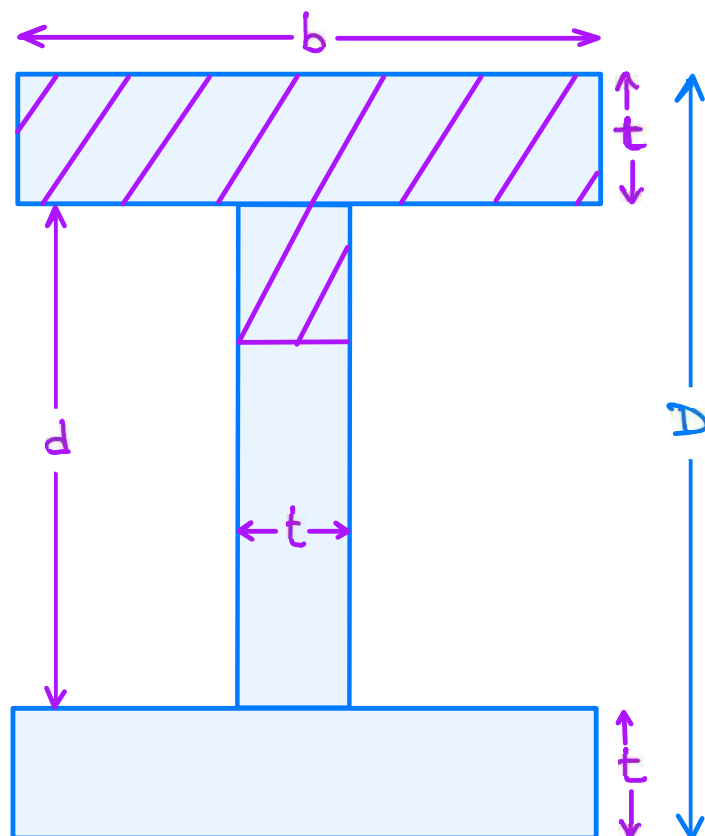


SHEAR FLOW CONCEPT



- The representation of the shear shown appears like a flow hence it is termed as Shear flow in walled sections.
- Since shear flow behaves like flow of liquid or electrons in which continuity equation is valid, here also it is valid.

Variation of Shear Flow :-



$$q_{web} = \frac{V(D^2 - d^2)B}{8I} + \frac{V}{2I} \left(\frac{d^2}{4} - y^2 \right) t$$

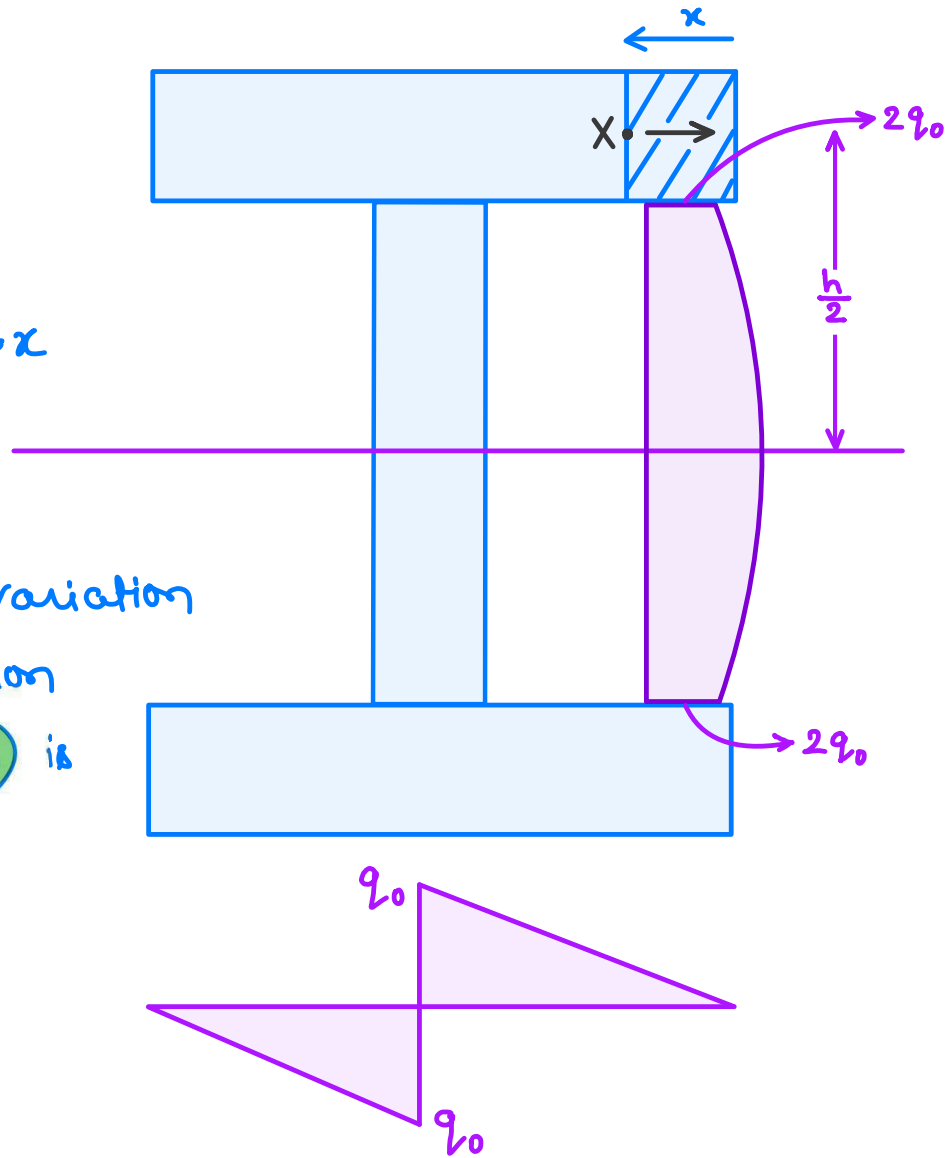
→ Hence, shear flow variation in vertical direction is **parabolic**.

$$q = \frac{VA\bar{y}}{I}$$

$$A\bar{y} = (tx) \frac{h}{2}$$

$$q = \frac{vt\pi h}{2I} = \frac{vt h}{2I} \cdot x$$

→ Hence, Shear flow variation in horizontal direction (longitudinal shear flow) is linear.



Note:-

- At Junction of Flange & web Shear flow from one side be q_0 in flange.

$$q_0 = \frac{Vth(b/2)}{2I} = \frac{Vthb}{4I} = \frac{V}{4I} \left(\frac{D-d}{2} \right) b(D-t)$$

$$q_0 = \frac{V}{4I} \left(\frac{D-d}{2} \right) b \left(D - \frac{D-d}{2} \right) = \frac{V}{4I} \left(\frac{D-d}{2} \right) b \left(\frac{D+d}{2} \right)$$

$$q_0 = \frac{V(D^2-d^2)b}{16I}$$

$$\text{Shear flow from both sides} = 2q_0 = \frac{V(D^2-d^2)b}{8I}$$

Now Shear flow in web at Junction $(y = \frac{d}{2})$

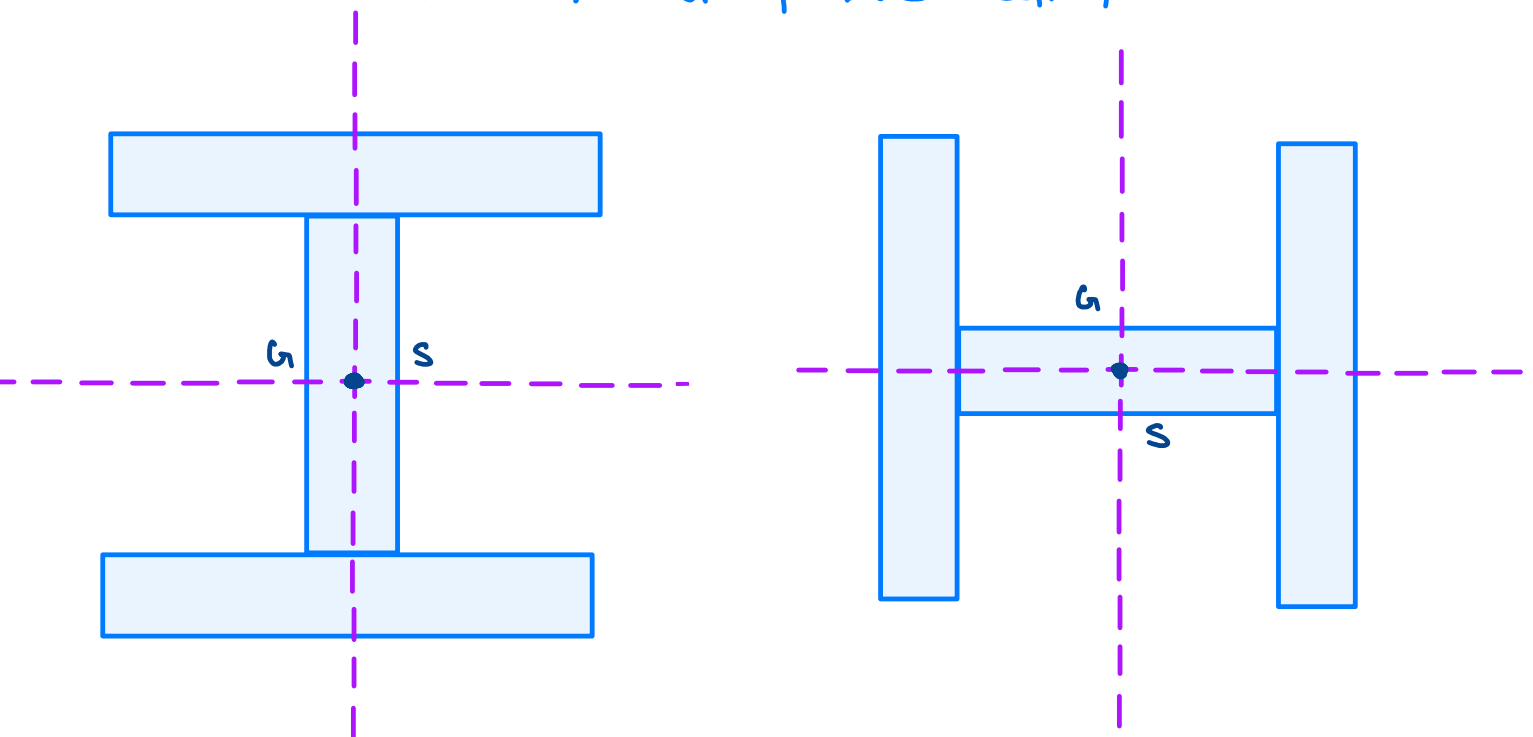
$$q_{web} = \frac{V(D^2 - d^2)b}{8I}$$

Hence, $2q_o = q_{web}$

Continuity equation is valid

Unsymmetrical loading of thin walled member (Shear centre)

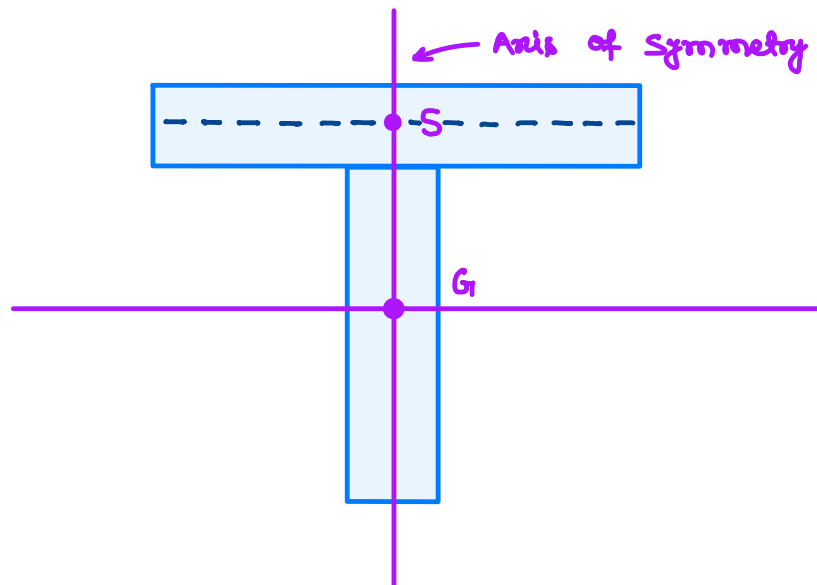
- In case of unsymmetrical loading if load does not pass through Shear centre the section will twist however if the load passes through Shear centre section will have only bending & No twisting.
- In case of section having 2 axis of symmetry the shear centre will lie at C.G. of the section.



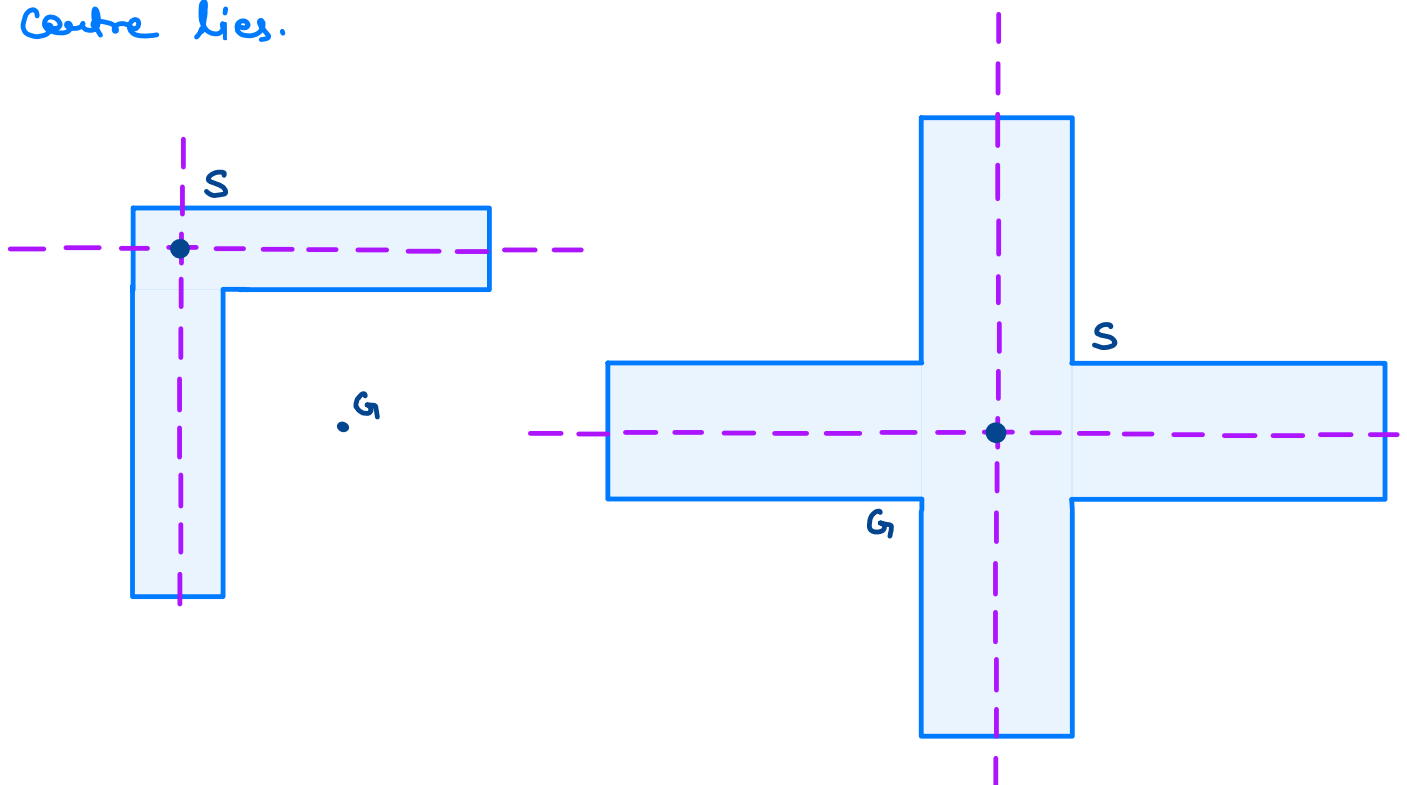
$G \rightarrow$ Centre of gravity.

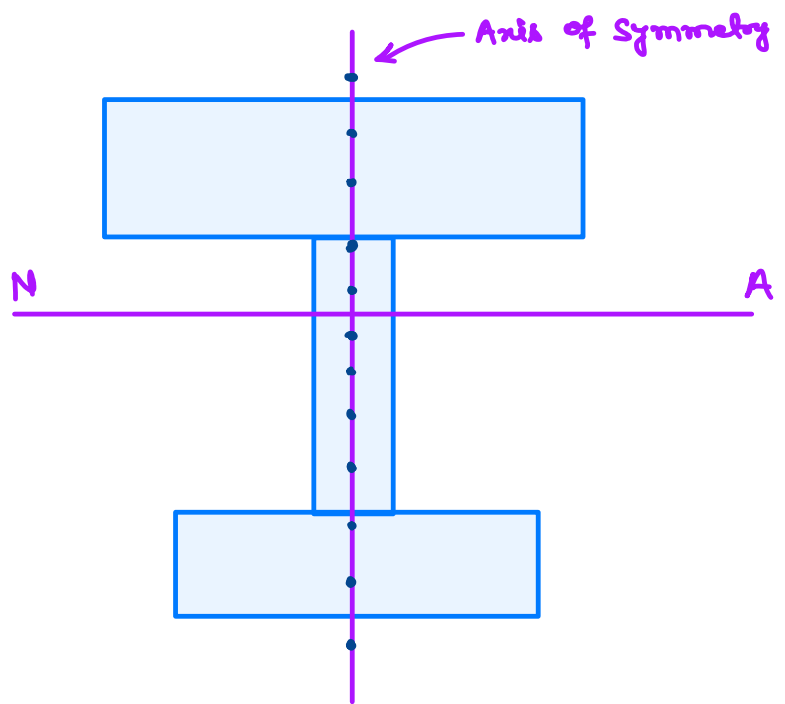
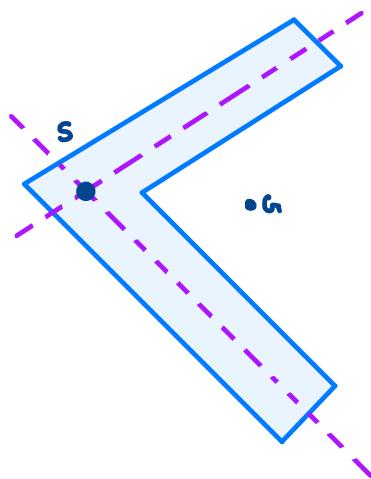
$S \rightarrow$ Shear centre

- If section has only one axis of Symmetry then Shear Centre would lie on the axis of Symmetry.



- Shear Centre location is the function of Shape of the Cross-Section & is independent of loading.
- If the thin section is composed of 2 rectangular sections Shear Centre will lie at the intersection of their Centre lies.

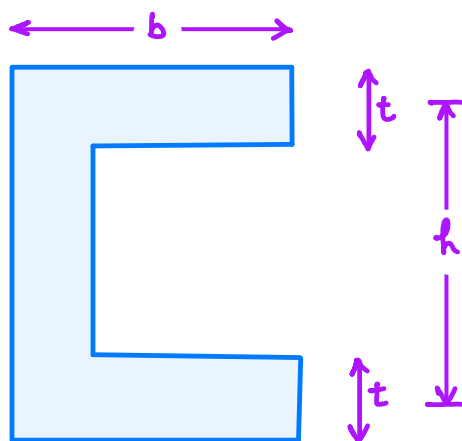




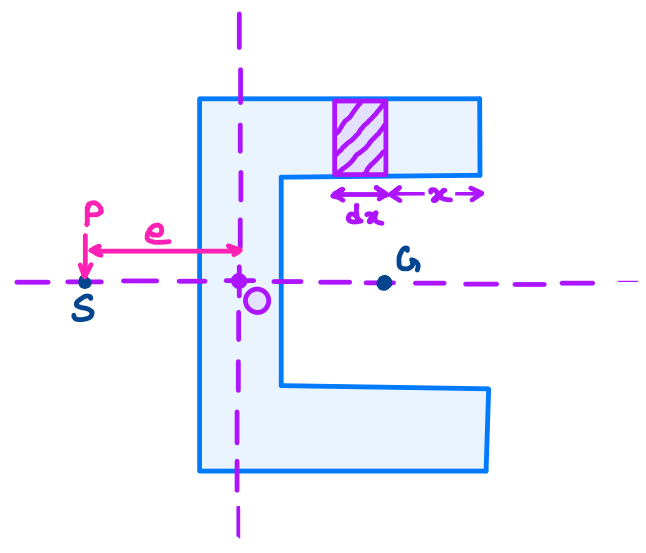
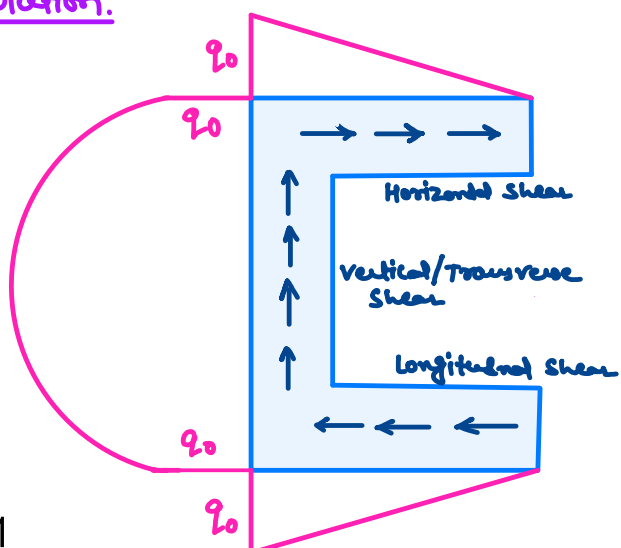
Eg:-

Compute the Shear Centre for the given cases

a.)



Solution.



for No twisting, $\Sigma M_o = 0$

$$Pe - H \frac{h}{2} - H \frac{h}{2} = 0.$$

$$Pe = Hh$$

$$e = \frac{Hh}{P}$$

$$\text{Now, } \tau = \frac{VA\bar{y}}{I \cdot b} = \frac{P(xt) \frac{h}{2}}{It} = \frac{Phx}{2I}$$

$$dH = \tau dA = \frac{Phx}{2I} t \cdot dx$$

$$H = \int dH = \int \frac{Phx \cdot t}{2I} dx$$

$$H = \frac{Ph t}{2I} \int_0^b x dx = \frac{Ph t}{2I} \left| \frac{x^2}{2} \right|_0^b$$

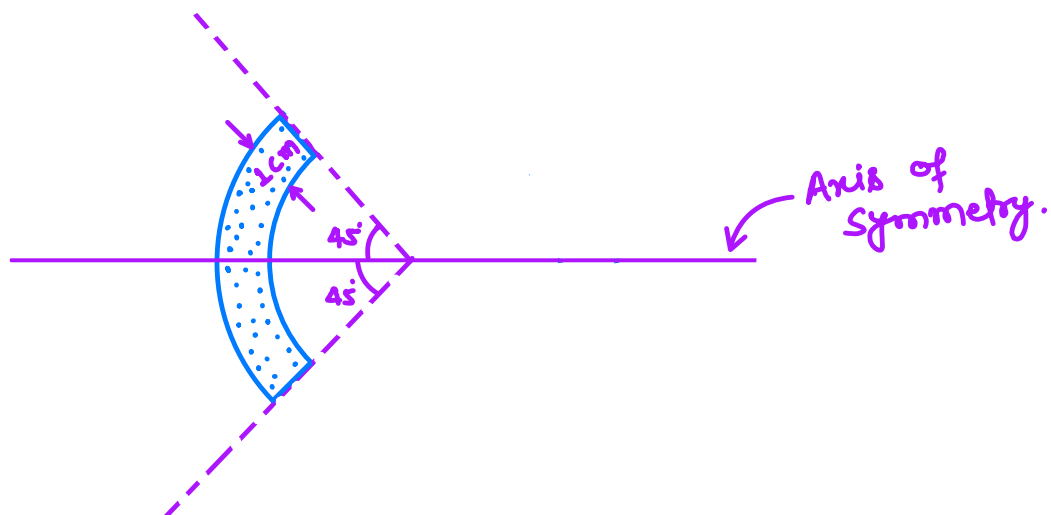
$$H = \frac{Ph t b^2}{4I}$$

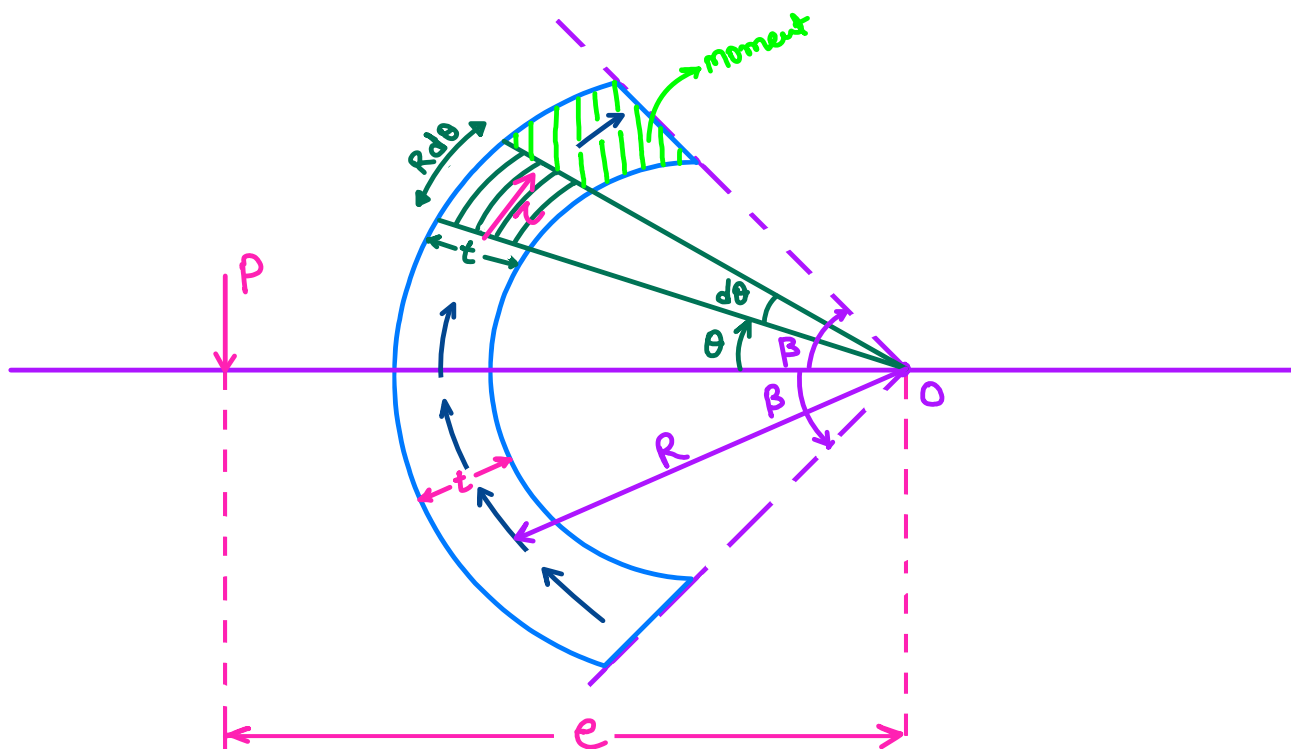
$$e = \frac{H \cdot h}{P}$$

$$= \frac{Ph t b^2}{4I} \cdot \frac{h}{P}$$

$$e = \frac{h^2 b^2 t}{4I}$$

b. >





for No twisting, $\Sigma M_O = 0$

$$\int_{-\beta}^{+\beta} [\tau \cdot dA] R - P e = 0$$

$$e = \frac{1}{V} \int_{-\beta}^{\beta} \tau dA R = \frac{1}{V} \int_{-\beta}^{\beta} \frac{V A \bar{y}}{I \cdot t} t R^2 d\theta$$

$$A \bar{y} = \int y dA = \int_{\theta}^{\beta} R \sin \theta R d\theta t = \int_{\theta}^{\beta} R^2 t \sin \theta d\theta$$

$$A \bar{y} = R^2 t [\cos \theta]_{\beta}^{\theta}$$

$$\underline{A \bar{y}} = R^2 t (\cos \theta - \cos \beta) \text{ --- (1)}$$

$$I = \int y^2 dA = \int_{-\beta}^{\beta} R^2 \sin^2 \theta R d\theta t = 2 \int_0^{\beta} R^2 \sin^2 \theta R d\theta t$$

$$I = R^3 t \int_0^{\beta} 2 \sin^2 \theta d\theta = R^3 t \int_0^{\beta} (1 - \cos 2\theta) d\theta = R^3 t \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\beta}$$

$$I = R^3 t \left[\left(\beta - \frac{\sin 2\beta}{2} \right) - (0 - 0) \right]$$

$$313 \quad I = R^3 t \left[\beta - \sin \beta \cos \beta \right] \text{ --- (ii)}$$

$$e = \int_{-\beta}^{\beta} \frac{R^2 t [\cos \theta - \cos \beta] R^2 d\theta}{R^3 t [\beta - \sin \beta \cos \beta]}$$

$$e = \frac{R}{\beta - \sin \beta \cos \beta} \int_{-\beta}^{\beta} (\cos \theta - \cos \beta) d\theta$$

$$e = \frac{R}{\beta - \sin \beta \cos \beta} \left| \sin \theta - \theta \cos \beta \right|_{-\beta}^{\beta}$$

$$= \frac{R}{\beta - \sin \beta \cos \beta} \left[(\sin \beta - \beta \cos \beta) - (-\sin \beta + \beta \cos \beta) \right]$$

$$e = \frac{2R (\sin \beta - \beta \cos \beta)}{\beta - \sin \beta \cos \beta}$$

If $\beta = 45^\circ$

$$e = \frac{2R \left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}} \right)}{\left(\frac{\pi}{4} - \frac{1}{2} \right)} = 1.063 R = e$$

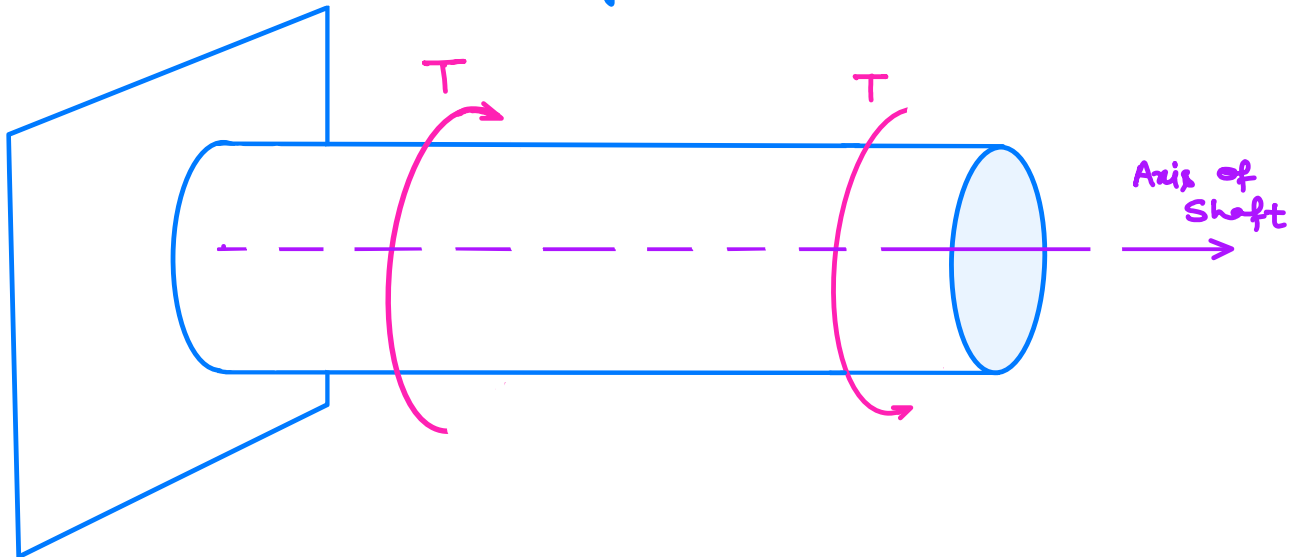
Due to transverse load $\rightarrow$ Shear stress

transverse load $\rightarrow$ Bending stress

Torsion $\rightarrow$ Shear stress

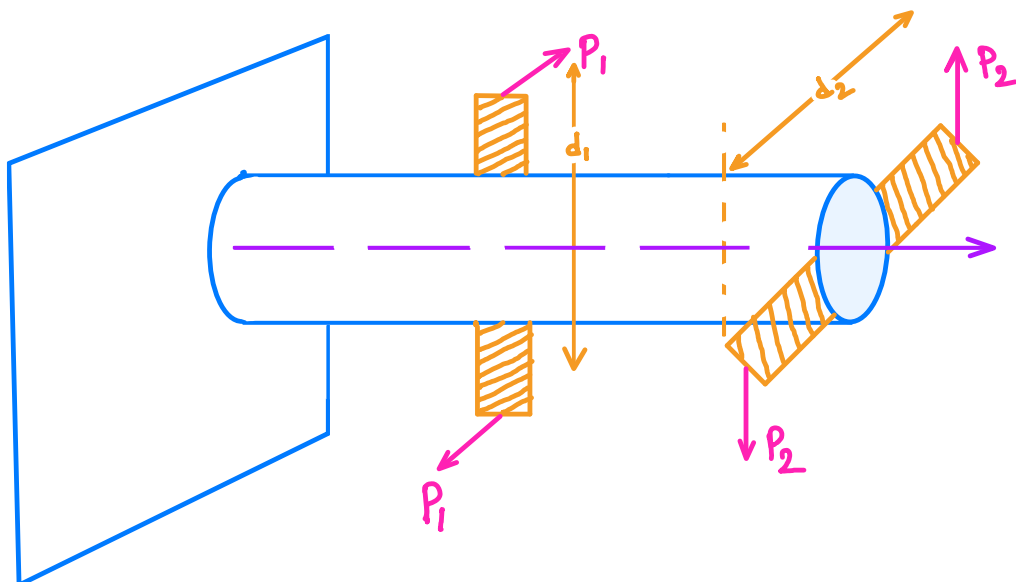
Torsion of circular shaft

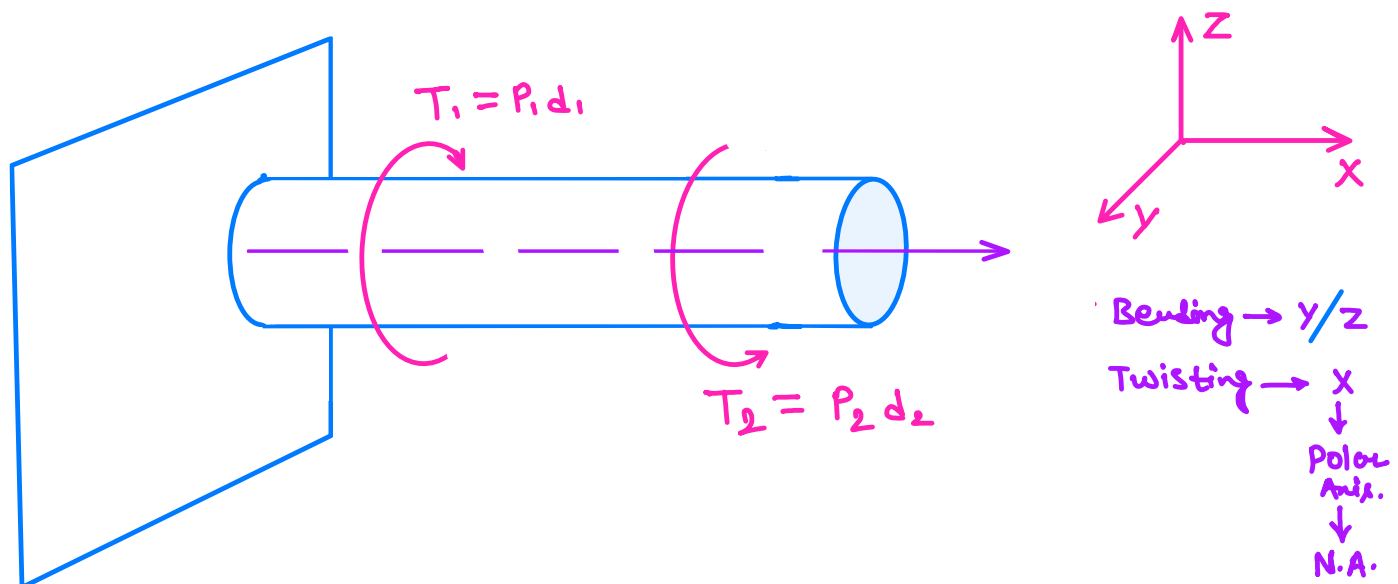
- Torsion refers to the twisting of straight bar when it is loaded by torque that tends to produce rotation about the longitudinal axis of bar.



Note:-

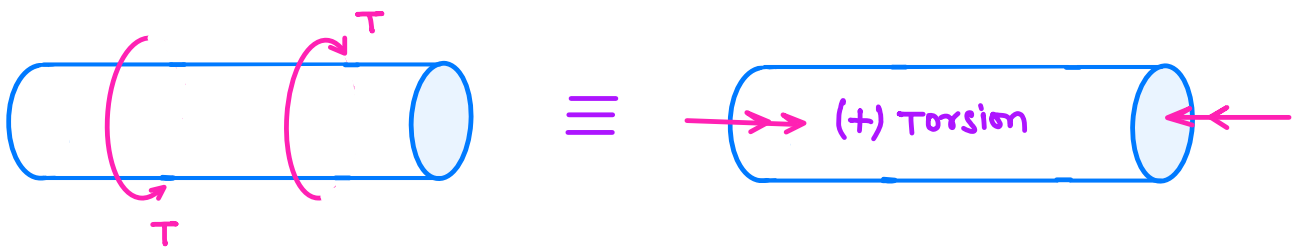
Pure Torsion → A member is said to be in pure Torsion when its cross-sections are subjected to only torsional moment and not accompanied by axial force or Bending moment.



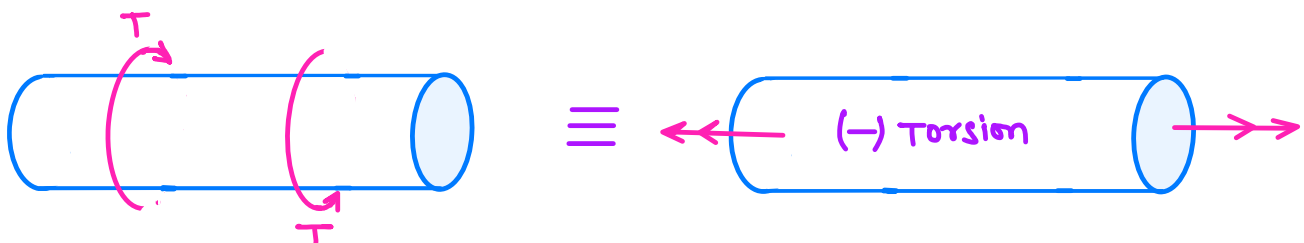


Note - 2

Sign Convention :- Direction of Torsion vector is given by right hand thumb rule.

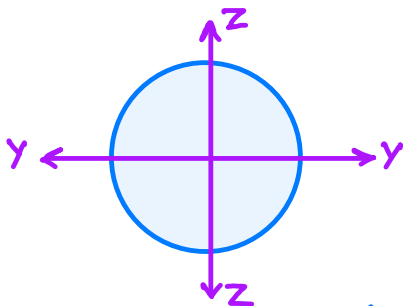


- when the direction of torsion vector points towards the section it is positive (+) and away from section is Negative (-ve).



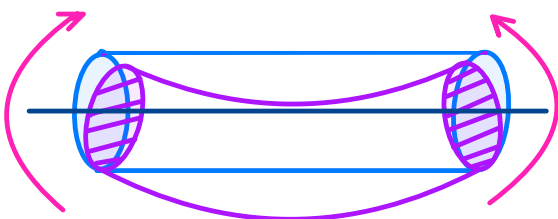
Note - 3Comparison of Bending and twisting moment .-Bending Moment

- a) Here either load is in transverse direction and eccentricity is in longitudinal direction or vice-versa.
- b) In bending cross-section rotates about transverse axis hence it is N.A.

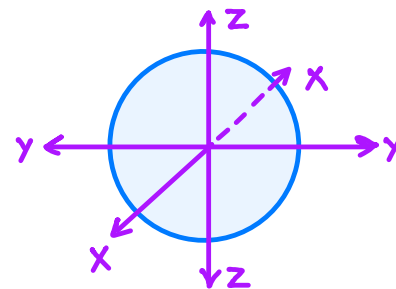


$$I_{xx} = I_{zz} = \frac{\pi D^4}{64}$$

- c) A plane & vertical section remains plane but inclined after bending.

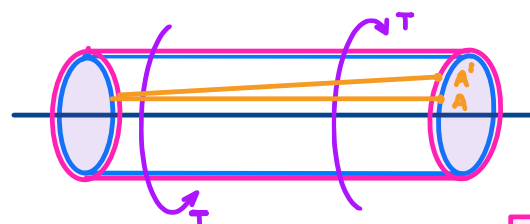
Twisting moment

- a) Direction of load and eccentricity is in transverse direction.
- b) In twisting cross-section rotates about polar/longitudinal axis, hence it is N.A.



$$I_{xx} = I_{yy} + I_{zz} = \frac{\pi D^4}{32}$$

- c) A plane and vertical section remains plane & vertical after twisting.

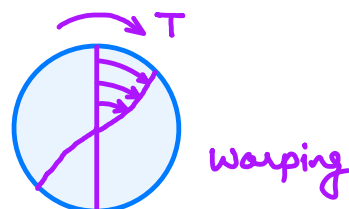
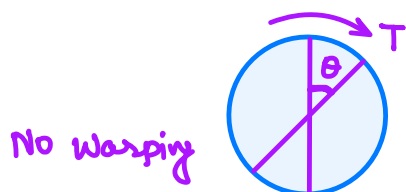


overlapping

Torsional Formula

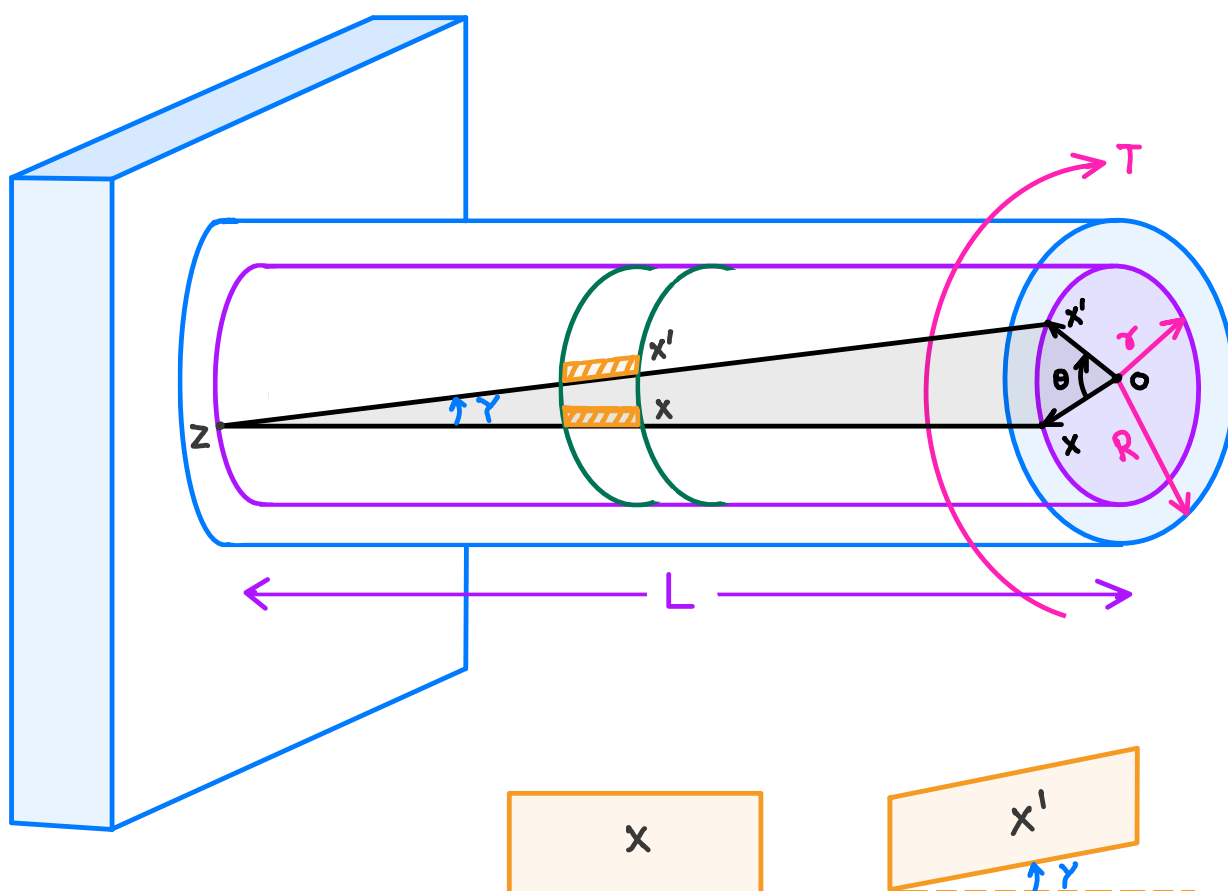
ASSUMPTIONS :-

- a) Circular Section remains circular.
- b) Plane sections remains plane & do not warp.

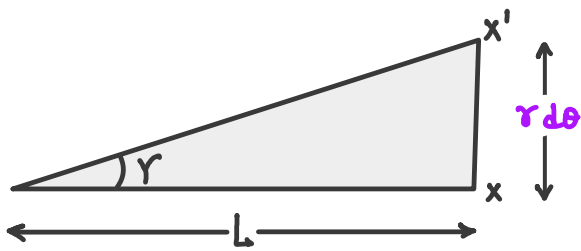


i.e. angle of twist remains constant after twisting

- c) Shaft is loaded by twisting couples in planes that are perpendicular to the axis of shaft.
- d) Stresses do not exceed proportional limit.



$\gamma \rightarrow$ Shear Strain.



$$\tan \gamma = \frac{r\theta}{L}$$

If α is very small, $\alpha \cong \tan \alpha$

$$\gamma = \frac{r\theta}{L} \quad (\gamma \propto \tau)$$

Note:-

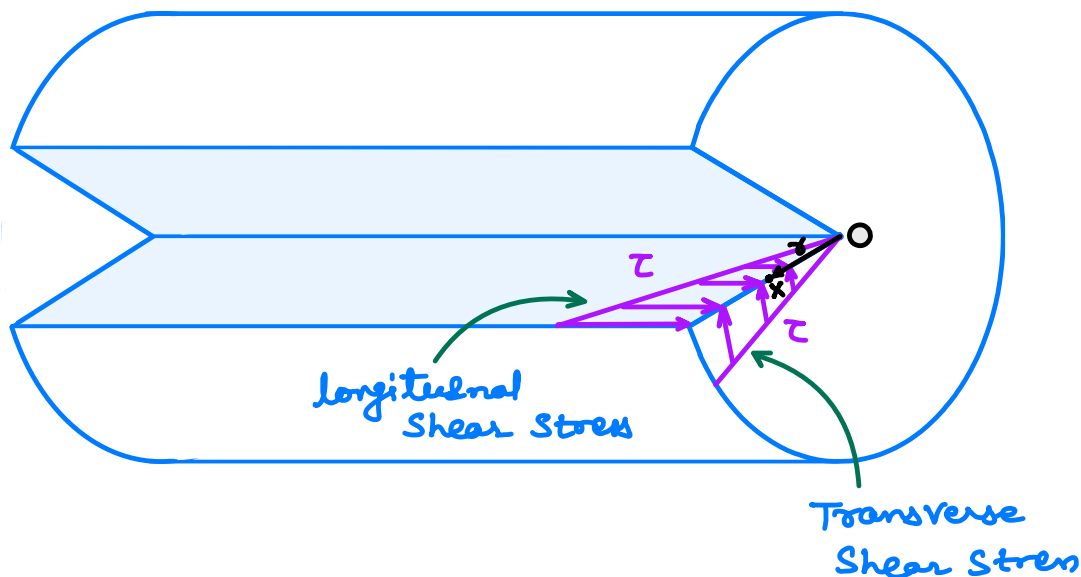
Hence Strain variation is linear over cross-section.

from Hooke's law,

$$G = \frac{\tau}{\gamma} \Rightarrow \tau = G\gamma = G \frac{r\theta}{L} \quad (\tau \propto \tau)$$

Note:-

Hence, Shear stress variation is also linear over cross-section



Now,

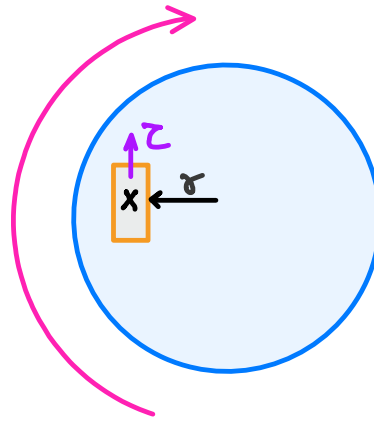
$$\boxed{\frac{\tau}{r} = \frac{G\theta}{L}} \quad \text{--- (i)}$$

Torque, $T = \int_A (\tau \cdot dA) r$

$$T = \int_A \frac{G\theta r}{L} \cdot r dA$$

$$T = \frac{G\theta}{L} \int r^2 dA$$

$$T = \frac{G\theta J}{L}$$



Note:-

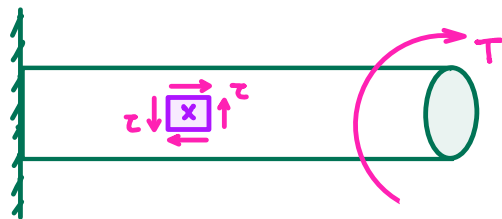
$$\int_A r^2 dA = J = \text{Polar moment of inertia.}$$

Hence, $\boxed{\frac{T}{J} = \frac{G\theta}{L}} \text{----- (ii)}$

from (i) & (ii)

$$\boxed{\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}} \text{***}$$

Note:-



$$\begin{aligned}\sigma_x &= 0 \\ \sigma_y &= 0 \\ \tau_{xy} &= \tau\end{aligned}$$

$$\sigma_{p_1}/\sigma_{p_2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{p_1} = \tau \text{ (Tension)}$$

$$\sigma_{p_2} = -\tau \text{ (Compression)}$$

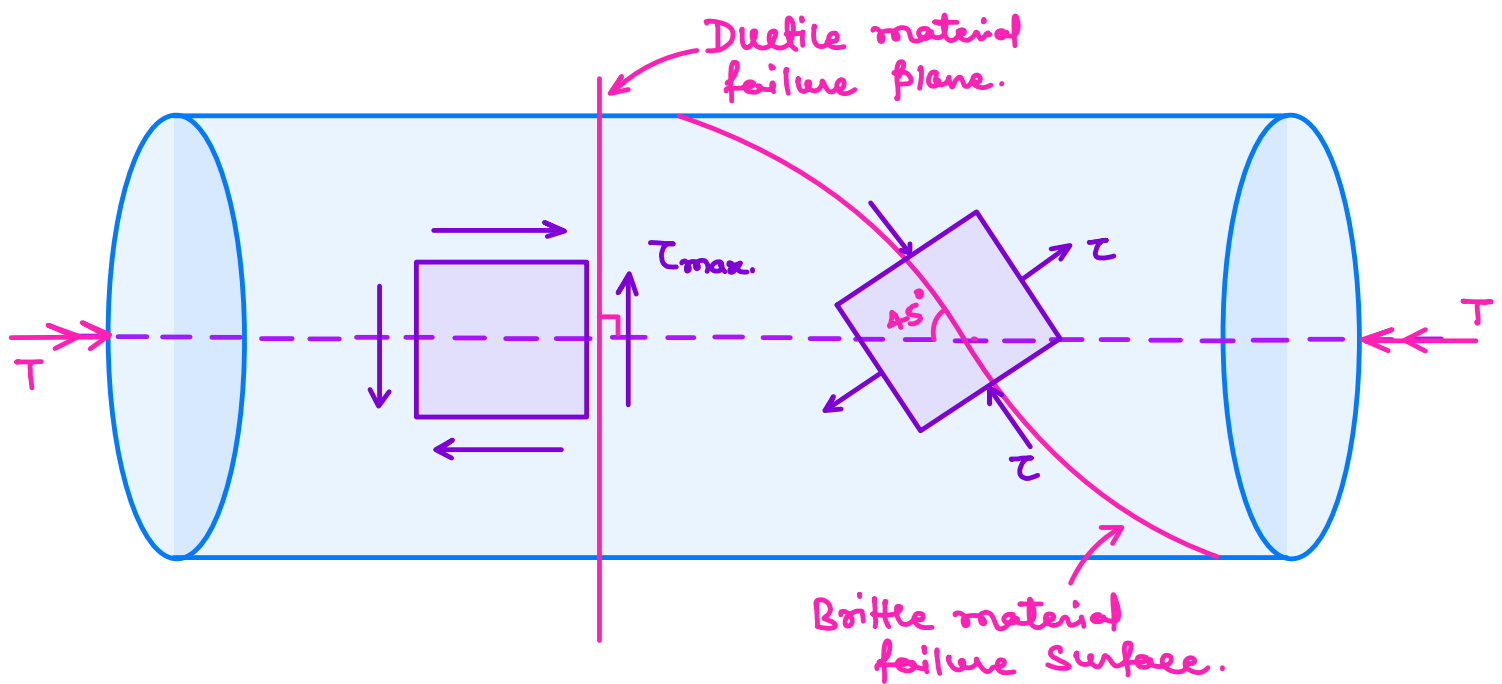
$$\theta_{p_1} = 45^\circ \quad \theta_{p_2} = 135^\circ$$

Strength

Ductile $\rightarrow C > T > S$

Brittle $\rightarrow T > C > S$

- If a torsion bar is made of a material that is weaker in tension than in shear (i.e. brittle material failure) will occur in tension along a helix inclined at 45° to axis of bar.
- However for ductile material, failure is due to shear hence these material under torsion will fail on a section subjected to maximum shear stress i.e. on section perpendicular to the axis of shaft.



$$\begin{aligned}\epsilon_{p_1} &= \frac{\sigma_{p_1}}{E} - \mu \frac{\sigma_{p_2}}{E} \\ &= \frac{\tau}{E} - \mu \frac{(-\tau)}{E}\end{aligned}$$

$$\boxed{\epsilon_{p_1} = \frac{\tau}{E} (1 + \mu)} \quad \text{--- (i)}$$

$$\epsilon_{p_1} = \frac{\epsilon_x + \epsilon_y}{2} + \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\boxed{\epsilon_{p_1} = \frac{\gamma_{xy}}{2}} \quad \text{--- (ii)}$$

- Under pure torsion, maximum Normal strain is half of Shear strain.

⇒ from (i) & (ii),

$$\frac{\tau}{E}(1+\mu) = \frac{\gamma_{xy}}{2}$$

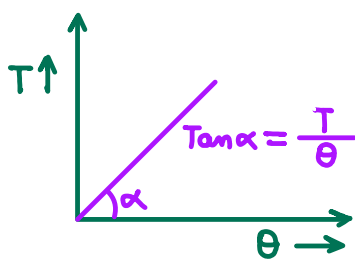
$$\frac{\tau}{\gamma_{xy}} = \frac{E}{2(1+\mu)}$$

$$G = \frac{E}{2(1+\mu)}$$

$$E = 2G(1+\mu)$$

Note :-

- Torsion test can be performed to find the value of G .
- In this test we apply the torsion over the bar and Note the corresponding angle of twist (θ) to find the value of G .



- using torsional formula :- $\frac{T}{J} = \frac{G\theta}{L}$

$$G = \frac{T}{\theta} \frac{L}{J} = \tan \alpha \frac{L}{J}$$



$$\theta_{A/B} = \theta_A - \theta_B = \theta_{A/C} + \theta_{C/D} + \theta_{DE} + \theta_{EB}$$

$$\theta_A - \theta_B = \frac{-T_A L_1}{G J_1} + \frac{(T_C - T_A) L_2}{G J_2} + \frac{(T_D + T_C - T_A) L_3}{G J_3} + \frac{T_B L_4}{G J_4}$$

$$\frac{T}{J} = \frac{G\theta}{L}$$

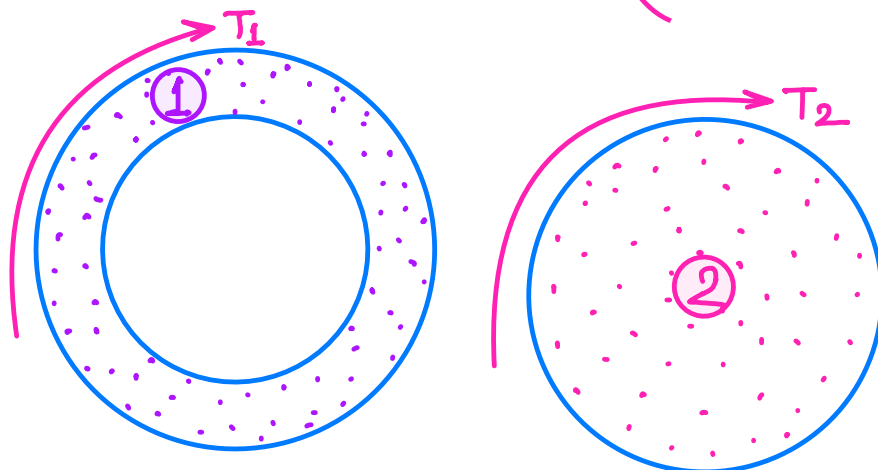
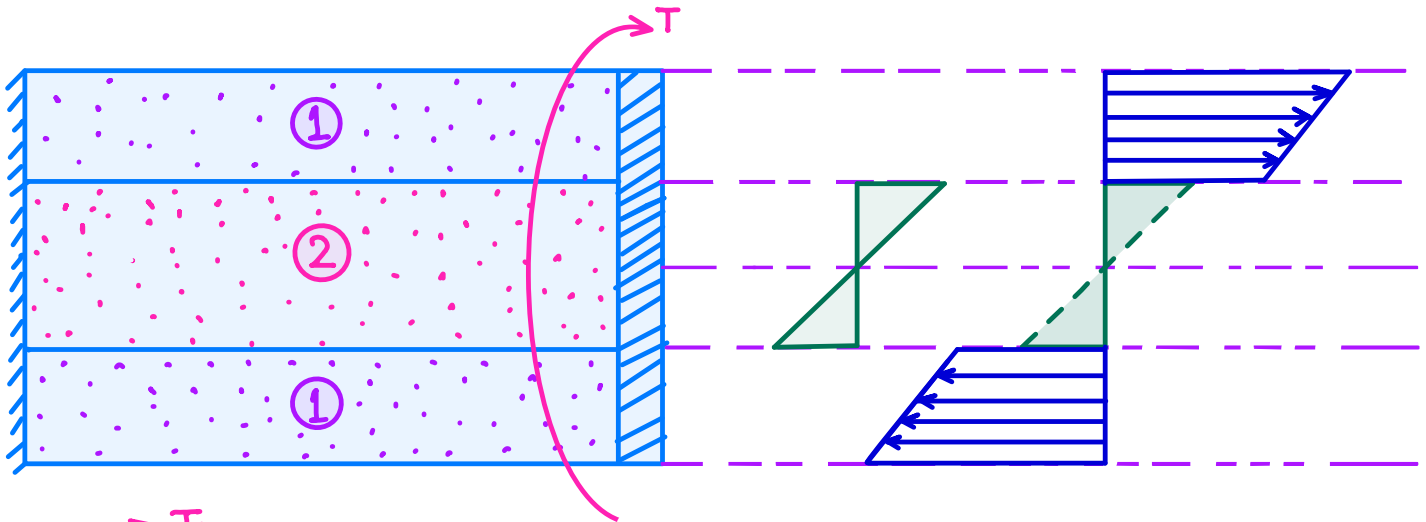
$$\theta = \frac{TL}{GJ}$$

Bars in Parallel.

$$T = T_1 + T_2 \quad \text{--- (i)}$$

$$\theta = \theta_1 = \theta_2$$

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \quad \text{--- (ii)}$$



$$\frac{T}{J} = \frac{\tau}{r}$$

$$\tau = \frac{T r}{J}$$

⇒ from (i) & (ii)

$$T_1 = \frac{G_1 J_1 \cdot T}{G_1 J_1 + G_2 J_2}$$

$$T_2 = \frac{G_2 J_2 \cdot T}{G_1 J_1 + G_2 J_2}$$

$$\theta = \frac{TL}{G_1 J_1 + G_2 J_2}$$

Note :-

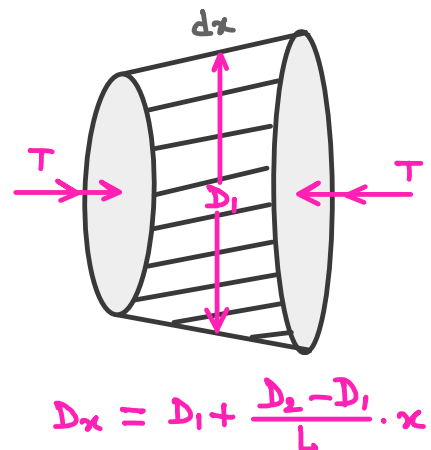
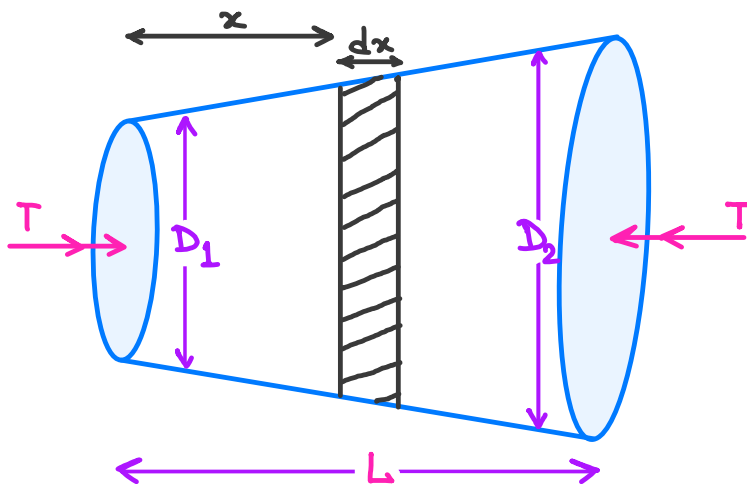
GJ = Torsional Rigidity

GA = Shear Rigidity

EI = Flexural rigidity

EA = Axial rigidity.

Tapered Section :-



$$d\theta = \frac{T dx}{G J_x} = \frac{T \cdot dx}{G \frac{\pi}{32} \left(D_1 + \frac{D_2 - D_1}{L} \cdot x \right)^4}$$

$$\begin{aligned} \theta &= \int d\theta = \int_0^L \frac{T \cdot dx}{G \frac{\pi}{32} \left(D_1 + \frac{D_2 - D_1}{L} \cdot x \right)^4} \\ &= \frac{32T}{\pi G} \int_0^L \frac{dx}{\left(D_1 + \frac{D_2 - D_1}{L} \cdot x \right)^4} \end{aligned}$$

$$\theta = \frac{32T}{\pi G} \cdot \frac{L}{D_2 - D_1} \int_{D_1}^{D_2} \frac{1}{y^4} dy$$

$$\theta = \frac{32TL}{3\pi G D_1 D_2} \quad **$$

$$\underline{\text{Let}}, D_1 + \frac{D_2 - D_1}{L} \cdot x = y$$

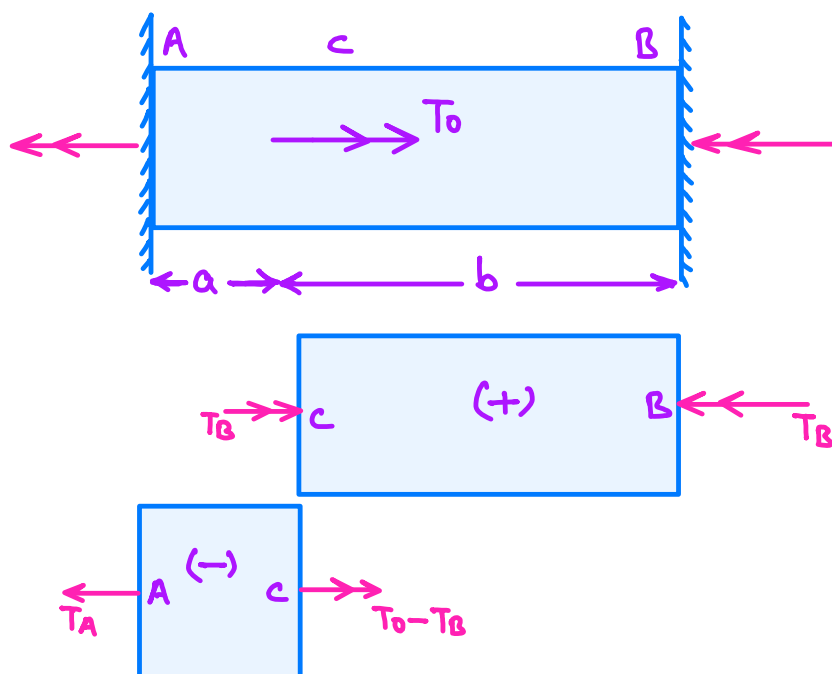
$$\left(\frac{D_2 - D_1}{L} \right) dx = dy$$

$$x=0, \quad y=D_1$$

$$x=L, \quad y=D_2$$

Statically Indeterminate Case

find T_A & T_B ?



from static equilibrium condition :

$$T_A + T_B = T_0 \quad \text{--- (i)}$$

$$\theta_{B/A} = 0 \quad \text{--- (ii)} \quad (\text{from compatibility relationship})$$

$$\theta_{BA} = \theta_{Bc} + \theta_{cA} = 0$$

$$\frac{T_B b}{GJ} + \frac{\{-(T_0 - T_B)\} a}{GJ} = 0$$

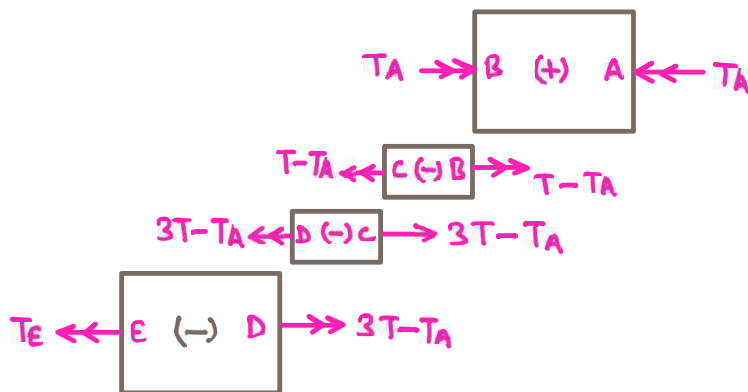
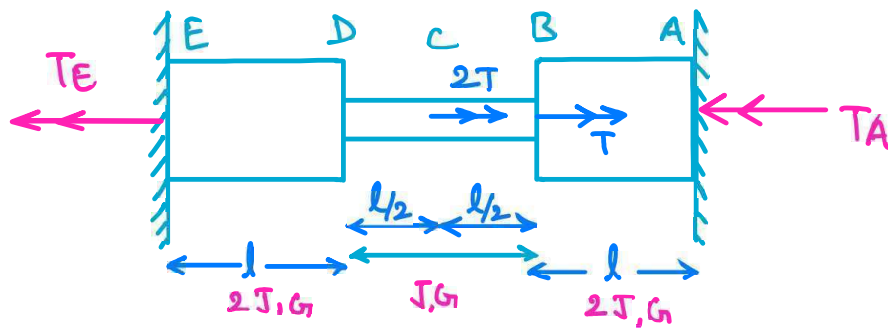
$$T_B (a+b) = T_0 a \Rightarrow$$

$$T_B = \frac{T_0 a}{(a+b)}$$

and

$$T_A = \frac{T_0 b}{a+b}$$

Q.7 find T_A & T_D and also compute θ_B and θ_C .



a) from static equilibrium

$$T_A + T_E = 3T \quad \text{--- (i)}$$

$$\theta_{AE} = 0 \quad \text{--- (ii)}$$

$$\begin{aligned} \theta_{AE} &= \theta_{AB} + \theta_{BC} + \theta_{CD} + \theta_{DE} = 0 \\ &= \frac{T_A L}{2GJ} + \frac{\{-(T-T_A)\} \frac{l}{2}}{GJ} + \frac{\{-(3T-T_A)\} \frac{l}{2}}{GJ} + \frac{\{-(3T-T_A)\} l}{2GJ} = 0 \end{aligned}$$

$$T_A = \frac{7}{4} T$$

$$T_B = \frac{5T}{4}$$

$$\begin{aligned} b) \quad \theta_{AB} &= \theta_A - \theta_B = \frac{T_A L}{2GJ} \\ &= 0 - \theta_B = \frac{7/4 T L}{2GJ} \Rightarrow \theta_B = \frac{-7 T L}{8 GJ} \end{aligned}$$

$$\begin{aligned} \theta_{DE} &= \theta_D - \theta_E = \frac{-(3T-T_A)l}{2GJ} \\ &= \theta_D - 0 = \frac{(\frac{7}{4}T - 3T)l}{2GJ} \Rightarrow \theta_D = \frac{-5 T L}{8 GJ} \end{aligned}$$

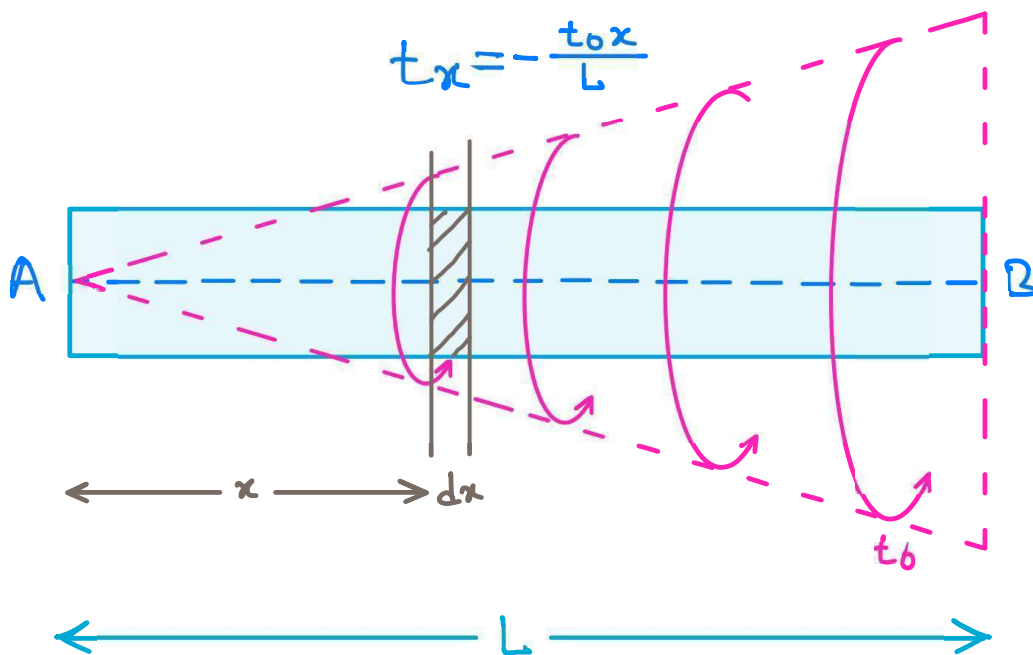
Q.7

An elastic circular bar AB of constant GJ (Torsional rigidity) is subjected to uniformly varying Torque (t_x) as shown.

Determine the rotation of the bar along its length and the reaction at end A & B if

a) End A is free & End B is fixed.

b) Both Ends A & B are fixed.



Solution:-

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \Rightarrow d\theta = \frac{T \cdot dx}{GJ}$$

$$\Rightarrow \frac{d\theta}{dx} = \frac{T}{GJ}$$

$$\text{Also, } \frac{d}{dx} \left[\frac{d\theta}{dx} \right] = \frac{dT}{dx} \cdot \frac{1}{GJ}$$

$$\frac{d^2\theta}{dx^2} = \frac{t_x}{GJ} = \frac{-t_0 x}{GJL}$$

$$GJ \frac{d^2\theta}{dx^2} = \frac{-t_0 x}{L} \Rightarrow \text{on integration}$$

$$GJ \frac{d\theta}{dx} = \frac{-t_0 x^2}{2L} + C_1$$

$$T = \frac{-t_0 x^2}{2L} + C_1$$

Also on further integration :-

$$\theta = \frac{t_0 x^3}{6GJL} + C_1 x + C_2$$

(a) A is free & B is fixed.

At $x = 0$, Torque, $T = 0$ & $x = L$, $\theta = 0$

$$0 = \frac{-t_0 \cdot 0^2}{2L} + C_1 \quad \& \quad 0 = \frac{-t_0 L^2}{6GJL} + 0 \cdot L + C_2$$

$$C_1 = 0$$

$$C_2 = \frac{t_0 L^2}{6GJ}$$

$$\theta = \frac{-t_0 x^3}{6GJL} + \frac{t_0 L^2}{6GJ}$$

(b) Both A & B are fixed.

at $x = 0$, $\theta = 0$ and $x = L$, $\theta = 0$

$$0 = \frac{-t_0 \cdot 0^2}{6GJL} + C_1 \cdot 0 + C_2$$

$$0 = \frac{-t_0 \cdot L^2}{6GJL} + C_1 L + 0$$

$$C_2 = 0$$

$$C_1 = \frac{t_0 L}{6GJ}$$

$$\theta = \frac{-t_0 x^3}{6GJL} + \frac{t_0 L}{6GJ} x$$

Torsional Strain Energy

a> $U = \frac{1}{2} T \cdot \theta$ [Area under T- θ curve]

b> $U = \frac{1}{2} T \cdot \frac{TL}{GJ} = \frac{T^2 L}{2GJ}$ (if T, G and J are constant)

or $U = \int_0^L \frac{T^2 dx}{2GJ}$

c> $\bar{U}$ (strain energy per unit volume) = $\frac{1}{2} \tau \cdot \gamma$

Note:-

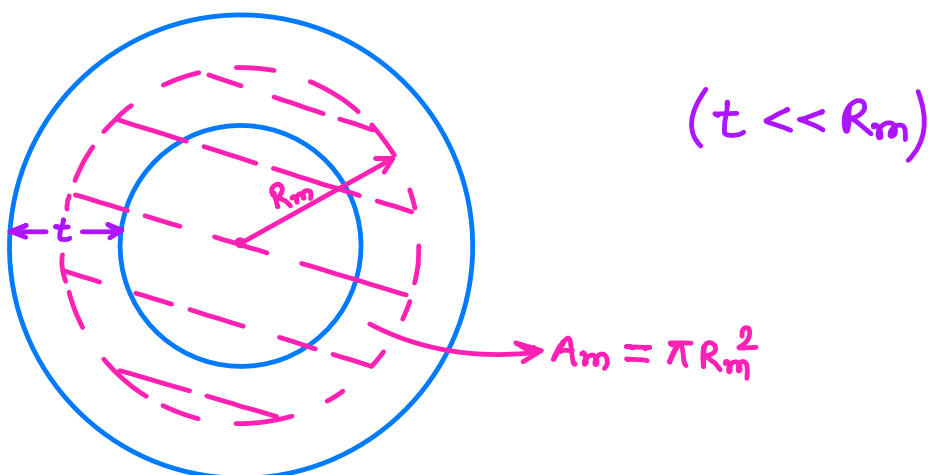
Member subjected to torque are used to transfer the power in general which is given by

$$P = T \cdot \omega$$

$$\omega = \frac{2\pi N}{60}$$

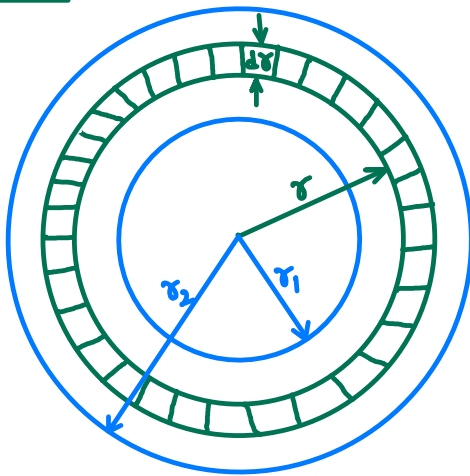
Thin Walled Hollow Shaft

- For thin walled section, maximum shear stress is taken corresponding to mean radius.



$$\frac{T}{J} = \frac{\tau_{\max}}{R_m} \Rightarrow \tau_{\max} = \frac{TR_m}{J}$$

Note :-



$$J = \int dJ = \int dA \cdot r^2 = \int_{r_1}^{r_2} 2\pi r dr \cdot r^2 = \int_{r_1}^{r_2} 2\pi r^3 dr$$

$$J = \left| 2\pi \frac{r^4}{4} \right|_{r_1}^{r_2} = \frac{\pi}{2} [r_2^4 - r_1^4]$$

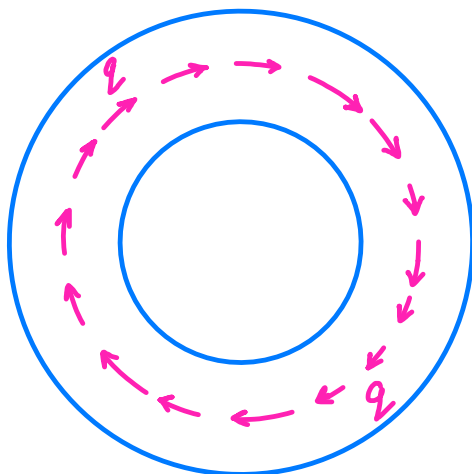
But in our case,

$$J = \int dJ = \int dA \cdot R_m^2 = \int (2\pi R_m t) R_m^2$$

$$J = 2\pi R_m^3 t$$

$$\tau_{\max} = \frac{TR_m}{2\pi R_m^3 t} = \frac{T}{2\pi R_m^2 t}$$

$$\tau_{\max} = \frac{T}{2\pi R_m^2 t}$$



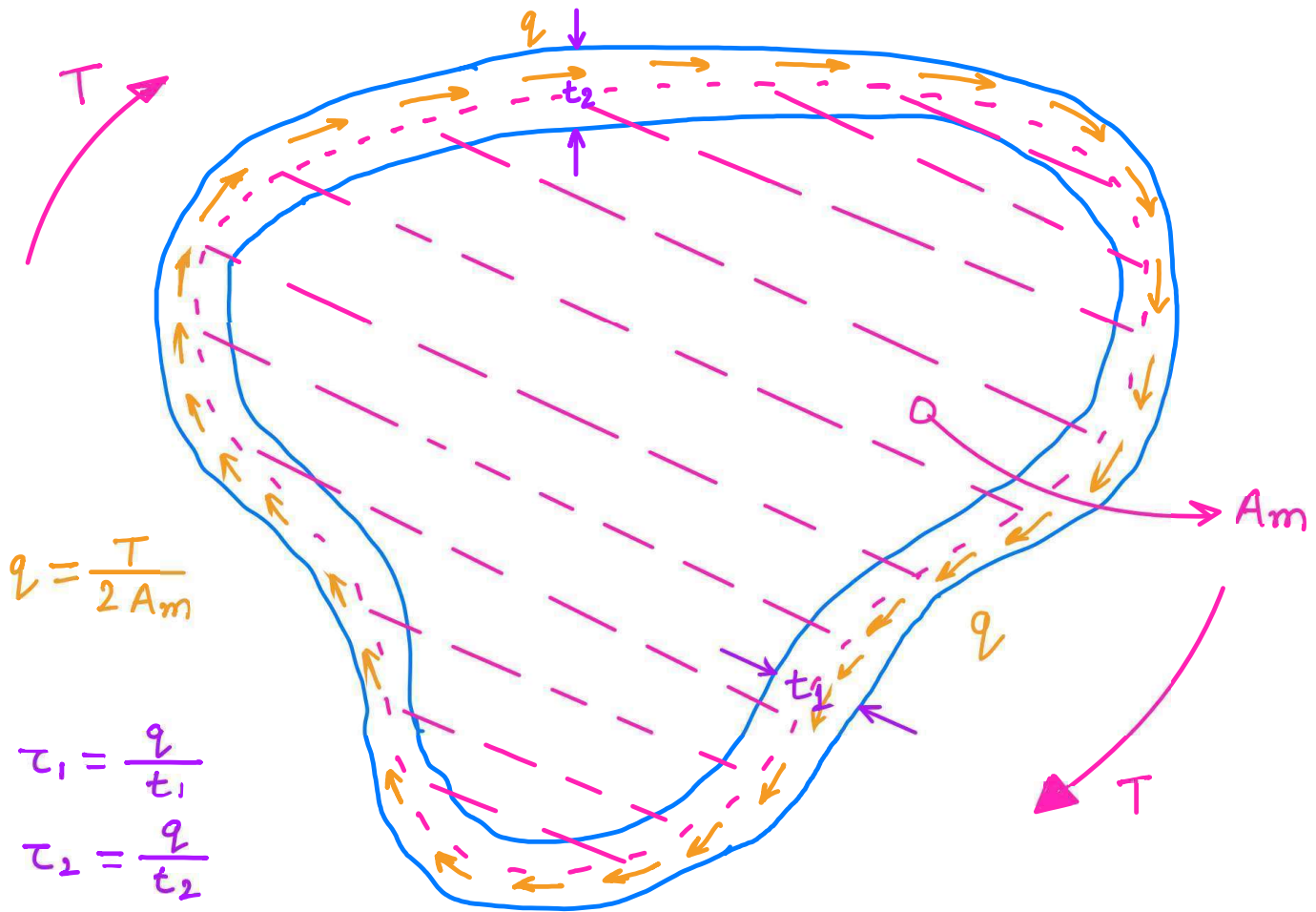
$$\tau_{\max} \cdot t = \frac{T}{2\pi R_m^2}$$

$$\tau_{\max} \cdot t = q = \text{Shear flow.}$$

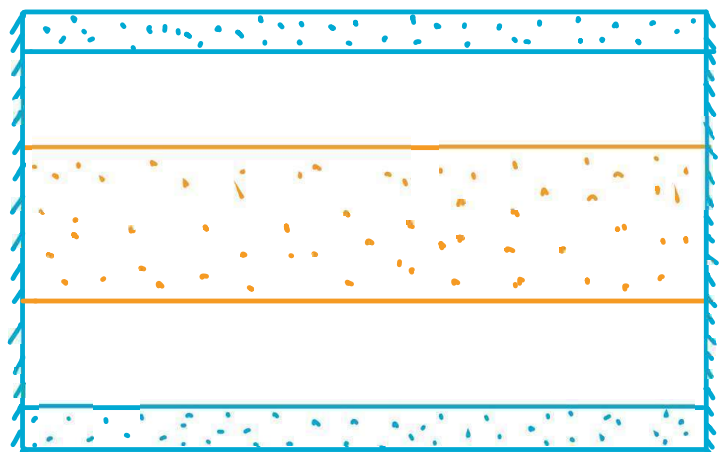
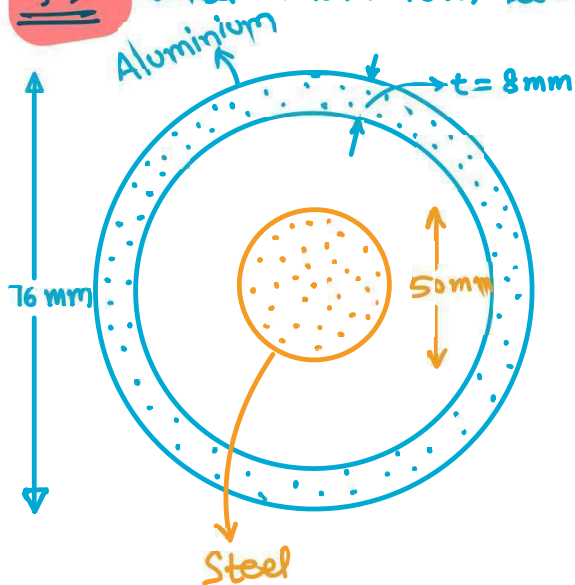
Hence,

$$q = \frac{T}{2\pi R_m^2} = \frac{T}{2A_m}$$

- The above expression for shear flow i.e. $q = \frac{T}{2A_m}$ is derived for a circular thin walled section but it is applicable for any shape of the walled section.



Q What maximum torque can be applied on the section??



$$\tau_{\text{per, Al}} = 70 \text{ MPa} \quad , \quad G_{\text{Al}} = 27 \text{ GPa}$$

$$\tau_{\text{per, St}} = 120 \text{ MPa} \quad , \quad G_{\text{St}} = 77 \text{ GPa}$$

Soln $T_o = T_{St} + T_{Al} \dots (i)$

$$\theta_{St} = \theta_{Al}$$

$$\frac{T_{St} L}{G_{St} J_{St}} = \frac{T_{Al} L}{G_{Al} J_{Al}} \dots (ii)$$

from (i) & (ii)

$$T_{St} = \frac{G_{St} J_{St} \cdot T_o}{G_{St} J_{St} + G_{Al} J_{Al}} = \frac{T_o}{1 + \frac{27}{77} \cdot \frac{\frac{\pi}{32} (76^4 - 60^4)}{\frac{\pi}{32} (50^4)}}$$

$$T_{St} = 0.466 T_o$$

$$T_{Al} = \frac{G_{Al} J_{Al} \cdot T_o}{G_{St} J_{St} + G_{Al} J_{Al}} \quad \textcircled{or} \quad T_o - T_{St}$$

$$T_{Al} = 0.533 T_o$$

$$\tau_{per.St} = \frac{T_{St} \cdot R_{St}}{J_{St}} = \frac{0.466 T_o \times 25}{\frac{\pi}{32} (50)^4} = 120 \text{ mpa}$$

$$T_o = 6.32 \text{ KN-m.}$$

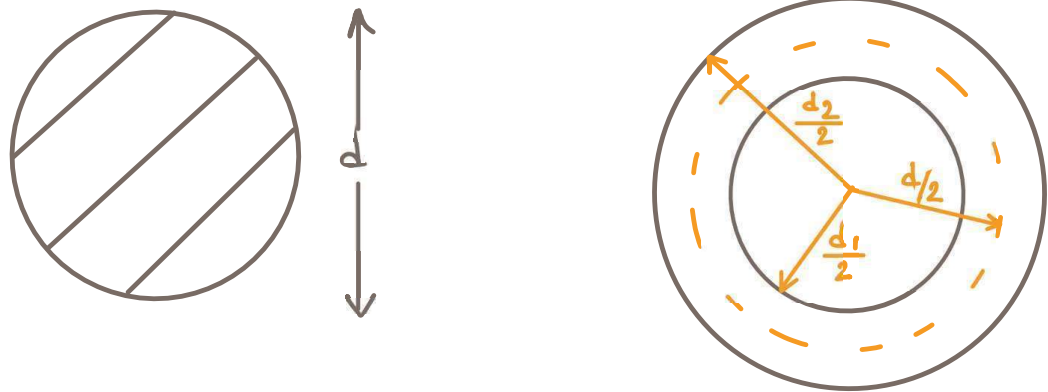
$$\tau_{per.Al} = \frac{T_{Al} \cdot R_{Al}}{J_{Al}} = \frac{0.533 T_o \times 38}{\frac{\pi}{32} (76^4 - 60^4)} = 70 \text{ mpa}$$

$$T_o = 6.909 \text{ KN-m}$$

$$\begin{aligned} \text{Maximum Torque that can be applied} &= (6.32, 6.909)_{\min} \\ &= \underline{\underline{6.32 \text{ KN-m.}}} \end{aligned}$$

Q. > A Solid circular shaft of diameter "d" has a Same weight as the hollow circular shaft of mean diameter d, assuming the Same maximum shear stress in both the cases determine the ratio of torque transmitted by two shafts. Also compare the Angle of twist per unit length in the two shafts and comment upon the strength & stiffness of the two shafts.

Solution



$$d = \frac{d_1 + d_2}{2} \quad \dots (i)$$

for Same weight

$$W_s = W_H$$

$$\gamma V_s = \gamma V_H$$

$$A_s L = A_H L \Rightarrow A_s = A_H$$

$$\frac{\pi d^2}{4} = \frac{\pi}{4} (d_2^2 - d_1^2) \Rightarrow d^2 = (d_2 - d_1)(d_2 + d_1)$$

$$d^2 = (d_2 - d_1) 2d$$

$$d_2 - d_1 = \frac{d}{2} \quad \dots (ii)$$

from (i) & (ii),

$$d_1 = \frac{3d}{4}$$

$$d_2 = \frac{5d}{4}$$

$$\tau_{\max S} = \tau_{\max H}$$

$$\frac{T_S \frac{d}{2}}{\frac{\pi d^4}{32}} = \frac{T_H \frac{d_2}{2}}{\frac{\pi}{32} (d_2^4 - d_1^4)}$$

$$\frac{T_S}{d^3} = \frac{T_H \cdot 5d}{4 \left\{ \left(\frac{5}{4}d \right)^4 - \left(\frac{3}{4}d \right)^4 \right\}} \Rightarrow \frac{T_H}{T_S} = 1.7$$

(b) For same weight of the material hollow shaft is stronger as compared to solid shaft because hollow shaft has greater torque resistance.

(c) Angle of twist (θ).

$$\frac{\theta_S}{\theta_H} = \frac{\frac{T_S L}{G J_S}}{\frac{T_H L}{G J_H}} = \frac{T_S}{T_H} \cdot \frac{J_H}{J_S} = \frac{1}{1.7} \frac{(d_2^4 - d_1^4)}{d^4} = 1.25$$

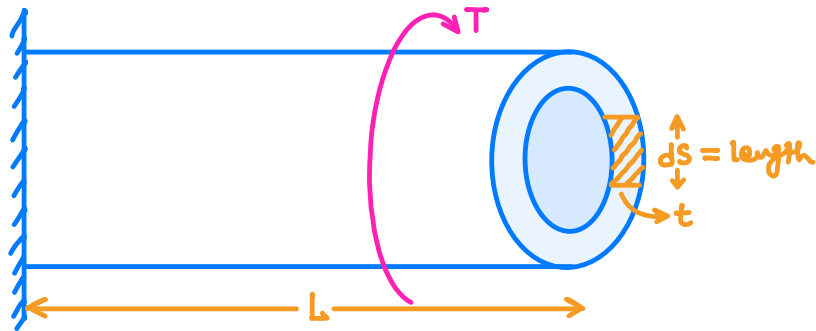
$$\frac{\theta_S}{\theta_H} = 1.25$$

d. > Torsional stiffness = $\frac{GJ}{L}$

$$\frac{K_H}{K_S} = \frac{G J_H L}{L J_S G} = \frac{J_H}{J_S} = 2.125$$

$$\frac{K_H}{K_S} = 2.125$$

Angle of twist in thin tube



$$U_{\text{strip}} = \frac{1}{2} \tau \cdot \gamma \cdot \text{volume} = \frac{\tau^2}{2G} \cdot ds \cdot t \cdot L$$

$$\text{Now, } \tau t = \frac{T}{2A_m}$$

$$U_{\text{strip}} = \frac{1}{2G} ds \cdot L \left(\frac{T}{2A_m} \right)^2 \cdot \frac{1}{t}$$

$$U_{\text{strip}} = \frac{T^2 \cdot L}{8G A_m^2 t} ds$$

$$U_{\text{total}} = \int U_{\text{strip}} = \int \frac{T^2 L \cdot ds}{8G A_m^2 t}$$

$$U_{\text{total}} = \frac{T^2 L}{8G A_m^2} \int \frac{ds}{t}$$

Now,

$$\theta = \frac{\partial U_{\text{Total}}}{\partial T} = \frac{2TL}{8G A_m^2} \int \frac{ds}{t}$$

$$\theta = \frac{TL}{4G A_m^2} \int \frac{ds}{t}$$

Q. >

for given case compute shear stress in tube & θ at free end

$$R_o = 100 \text{ mm}$$

$$E = 2 \times 10^5 \text{ MPa}$$

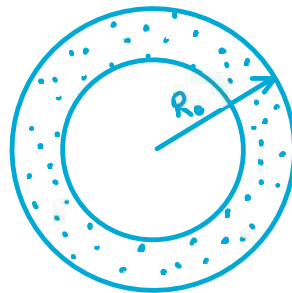
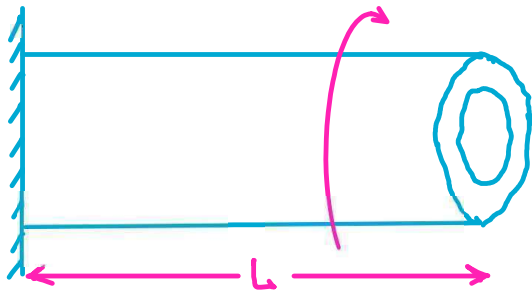
$$t = 2 \text{ mm.}$$

$$\mu = 0.25$$

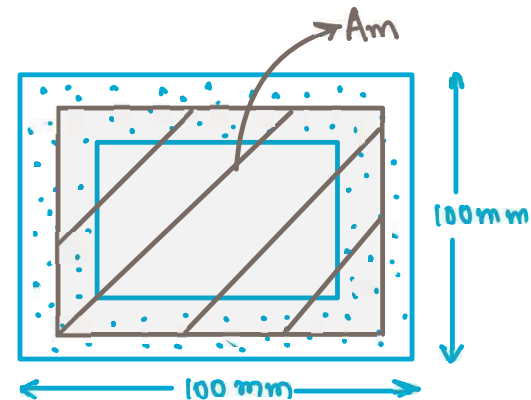
$$L = 1000 \text{ mm}$$

$$X-X \text{ area} = 100 \times 100 \text{ mm}^2$$

$$T = 1 \text{ KN-m}$$



(a)



(b)

Solution

$$a) \quad \frac{T}{J} = \frac{\tau}{R_m} \Rightarrow \tau_{\text{tube}} = \frac{T R_m}{J} = \frac{T R_m}{2\pi R_m t R_m^2}$$

$$\tau_{\text{tube}} = \frac{T}{2\pi R_m^2 t} = \frac{1 \times 10^6}{2 \times 3.14 \times (100-1)^2 \times 2} = 8.12 \text{ MPa.}$$

$$\theta = \frac{TL}{GJ} = \frac{1 \times 10^6 \times 10^3}{\frac{2 \times 10^5}{2(1+0.25)} \times 2\pi \times 99^3 \times 2}$$

$$\{E = 2G(1+\mu)\}$$

$$= 1.025 \times 10^{-3} \text{ rad}$$

$$b) \quad \tau_t = \frac{T}{2A_m t}$$

$$\tau = \frac{T}{2A_m t} = \frac{1 \times 10^6}{2 \times 98 \times 98 \times 2} = 26 \text{ MPa.}$$

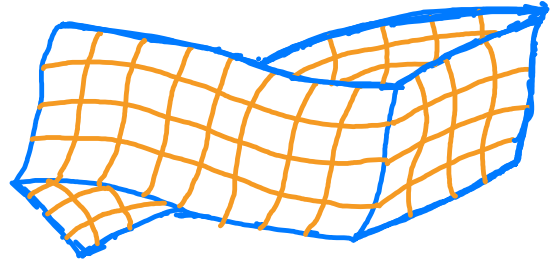
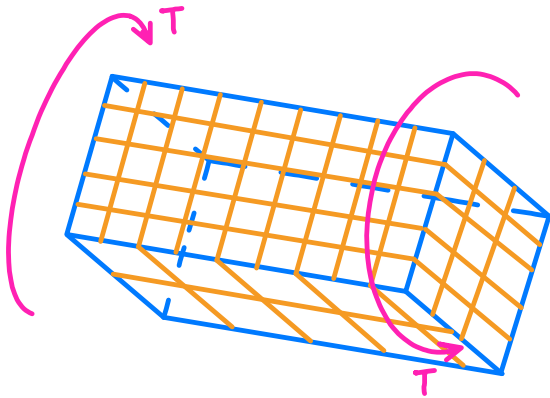
$$\theta = \frac{TL}{4A_m^2 G} \sum \frac{ds}{t}$$

$$= \frac{1 \times 10^6 \times 10^3}{4 \times (98 \times 98)^2 \times 8 \times 10^4} \times \frac{4 \times 98}{2}$$

$$\theta = 6.64 \times 10^{-3} \text{ Radians.}$$

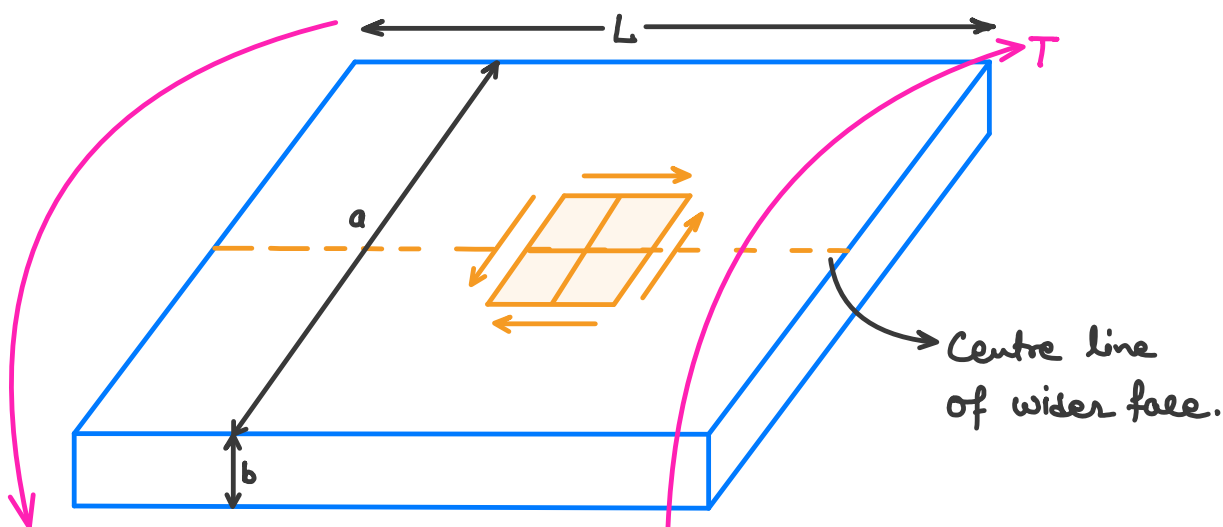
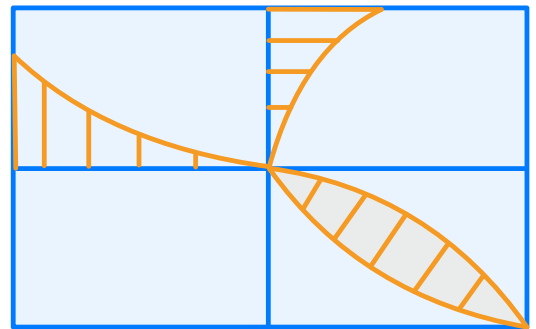
Torsion of Non-circular Members

- Torsion of Non-circular section results in warping of the section.



* For Rectangular Section

- a.) Maximum Shear Stress occurs along the centre line of wider face.
- b.) Shear stress at corner is zero.



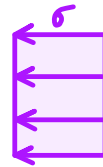
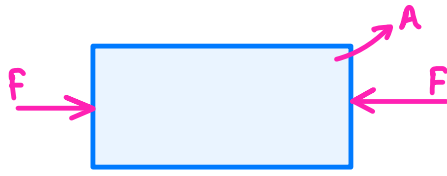
$$\tau_{max} = \frac{T}{c_1 a b^2} \quad , \quad \theta = \frac{T L}{c_2 a b^3 G}$$

c_1 & c_2 are constant, depends upon a & b .

Combined Stresses

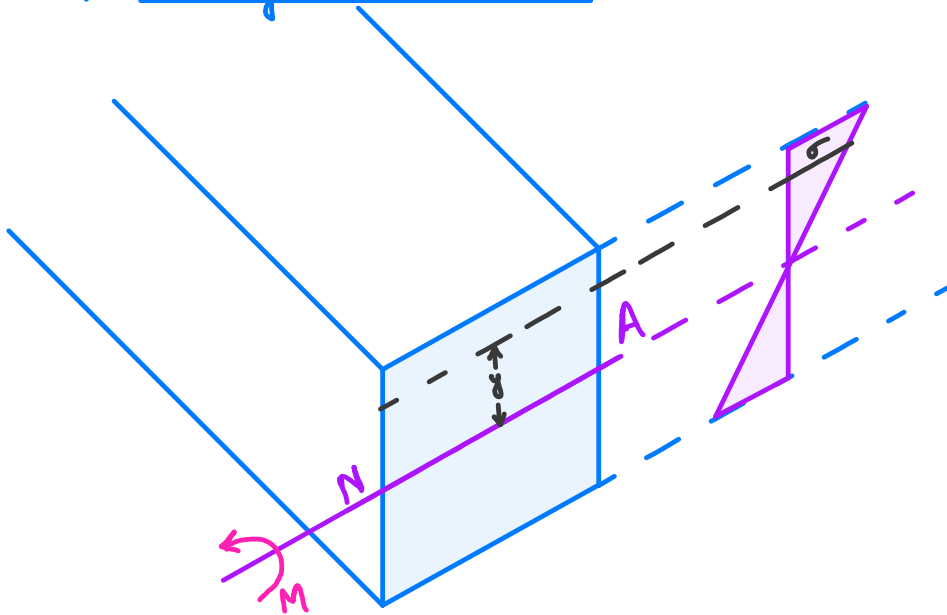
— Stresses to which member may be subjected includes

a> Direct Normal Stress (Axial Stress)



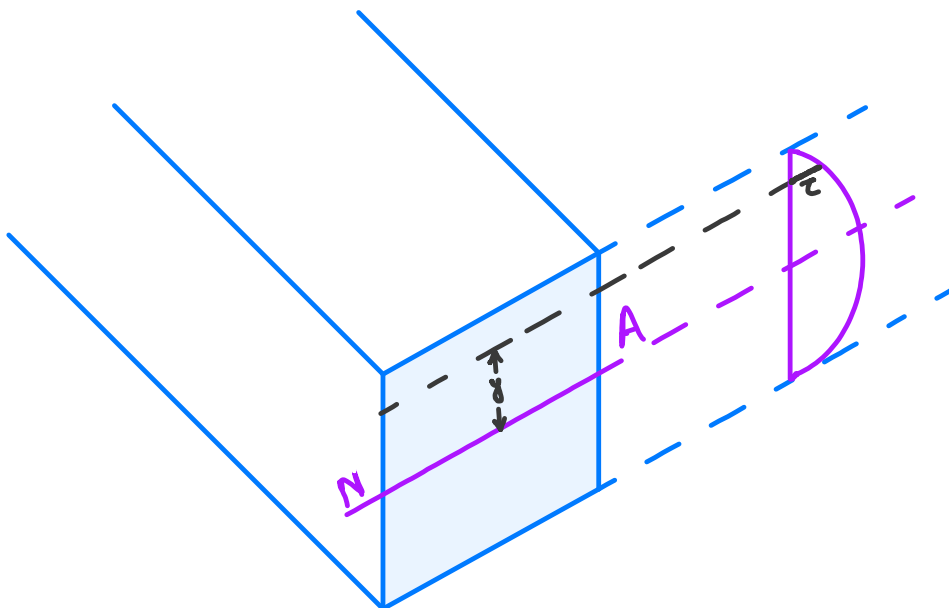
$$\sigma = \frac{F}{A}$$

b> Bending Normal Stress



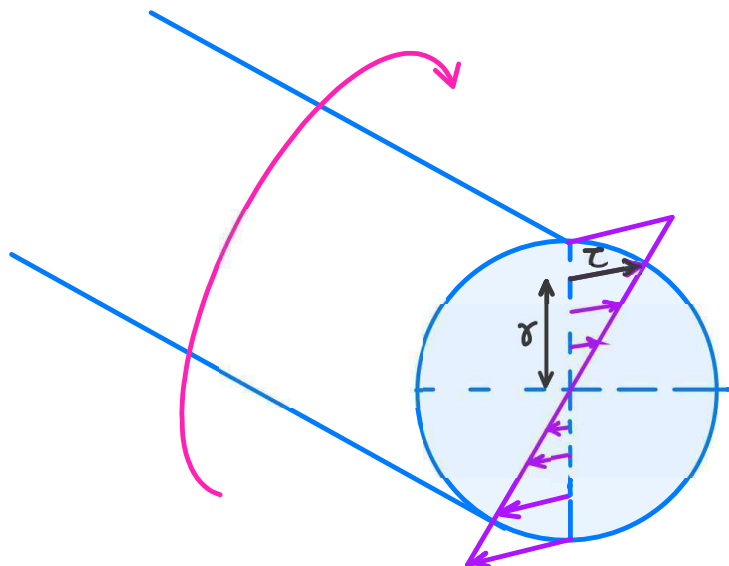
$$\sigma = \frac{My}{I}$$

c> Transverse Shear Stress



$$\tau = \frac{VA\bar{y}}{I \cdot b}$$

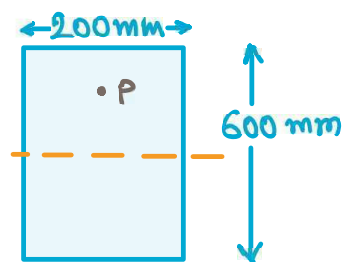
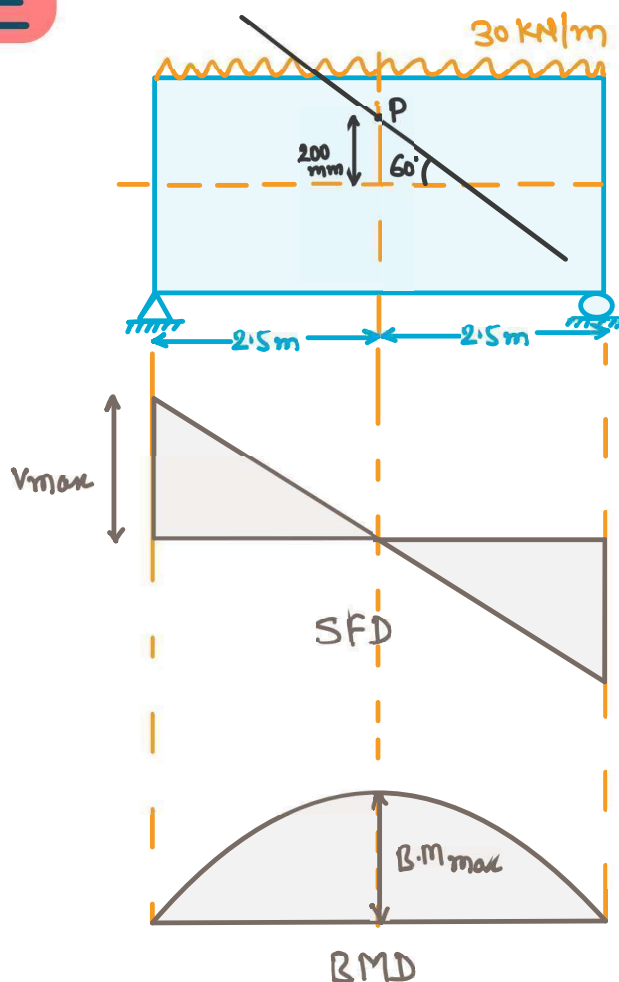
d> Torsional Shear Stress



$$\tau = \frac{T r}{J}$$



Find the normal & shear stress at point "P" on inclined plane.

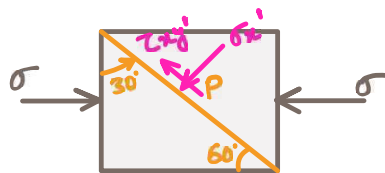


At point P, $V = 0$

$$\tau = \frac{V A \bar{y}}{I \cdot b} = 0$$

$$BM_{max} = M = \frac{w l^2}{8} = \frac{30 \times 5^2}{8} = 93.75 \text{ kNm}$$

$$\sigma = \frac{M y}{I} = \frac{93.75 \times 10^6 \times 200}{\frac{200 \times 600^3}{12}} = 5.2 \text{ MPa}$$



$$\sigma_x = -\sigma = -5.2 \text{ MPa.}$$

$$\sigma_y = 0$$

$$\tau_{xy} = 0$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{x'} = \frac{-5.2}{2} + \left(\frac{-5.2}{2} \right) \cos(2 \times 30') + 0$$

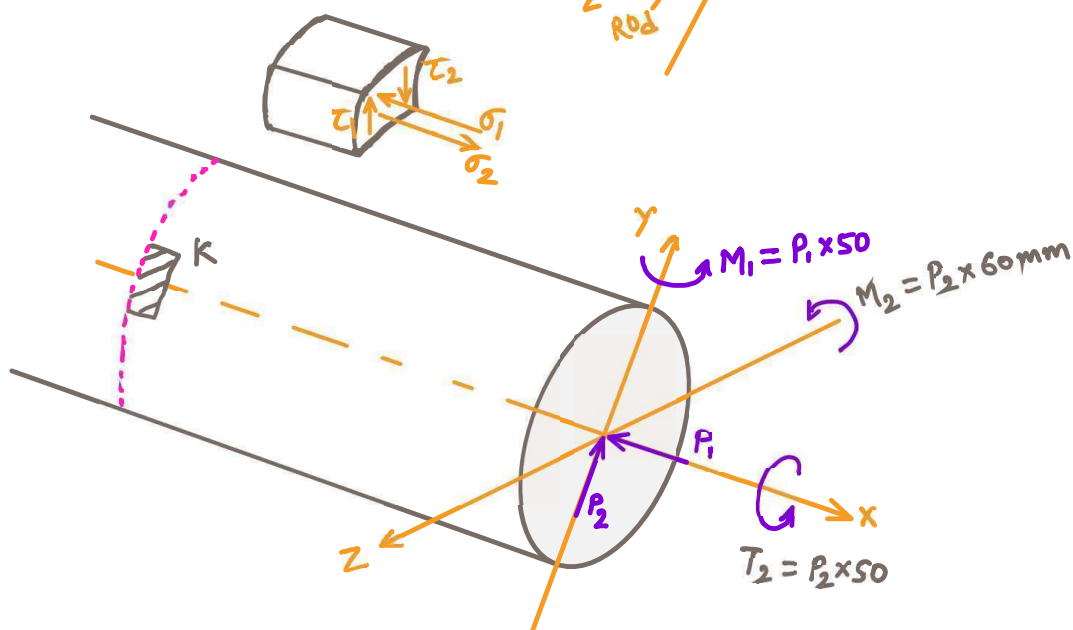
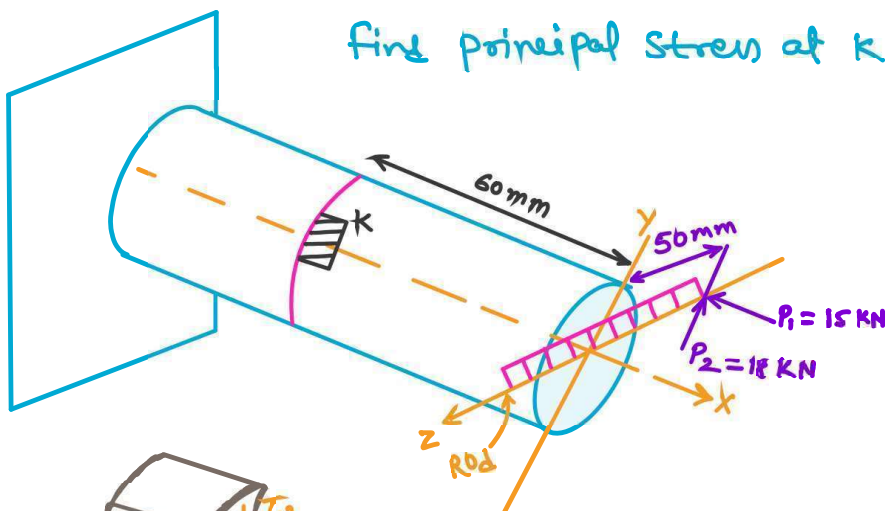
$$\sigma_{x'} = -3.9 \text{ MPa.}$$

$$\tau_{xy'} = -\left(\frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$= -\left(\frac{-5.2}{2} \right) \sin 2 \times 30' + 0 = 2.25 \text{ MPa.}$$

Q>

find principal stress at K, Dia of shaft = 40mm



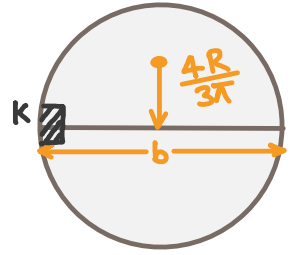
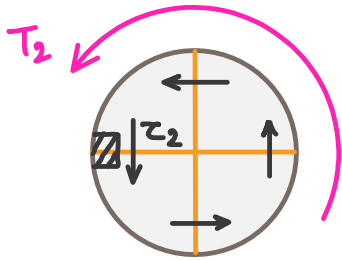
$$\begin{aligned} \sigma_1 &\rightarrow P_1 \\ \tau_1 &\rightarrow P_2 \\ \tau_2 &\rightarrow T_2 \\ \sigma_2 &\rightarrow M_1 \end{aligned}$$

a) Effect of P_1 : $\sigma_1 = \frac{P_1}{\frac{\pi d^2}{4}} = \frac{15 \times 10^3}{\left(\frac{\pi \times (40)^2}{4}\right)} = 11.93 \text{ MPa.}$

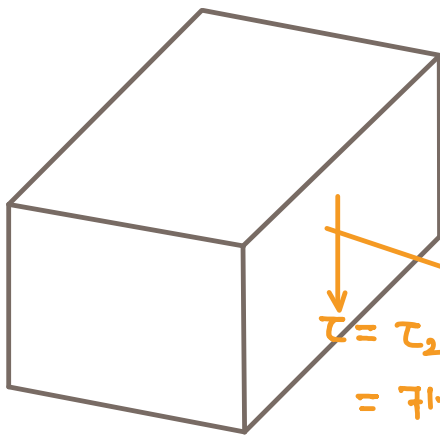
b) Effect of P_2 : $\tau_1 = \frac{V A \bar{y}}{I \cdot b} = \frac{18 \times 10^3 \times \frac{\pi (40)^2}{8} \times 4 \times \left(\frac{40}{2}\right) \cdot \frac{1}{3\pi}}{\frac{\pi (40)^4}{64} \times 40}$

$$\tau_1 = 19.09 \text{ MPa}$$

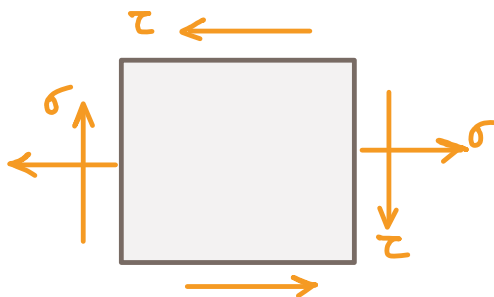
c) Effect of T_2 : $\tau_2 = \frac{T_2 \cdot r}{J} = \frac{18 \times 10^3 \times 50 \times 20}{\frac{\pi (40)^4}{32}} = 71.62 \text{ MPa}$



d) Effect of M_1 : $\sigma_2 = \frac{M \bar{y}}{I} = \frac{15 \times 10^3 \times 50 \times 20}{\frac{\pi (40)^4}{64}} = 119.37 \text{ MPa.}$



$$\begin{aligned} \tau &= \tau_2 - \tau_1 & \sigma &= \sigma_2 - \sigma_1 = 119.36 - 11.93 = 107.43 \text{ MPa} \\ &= 71.62 - 19.09 \\ &= 52.53 \text{ MPa.} \end{aligned}$$



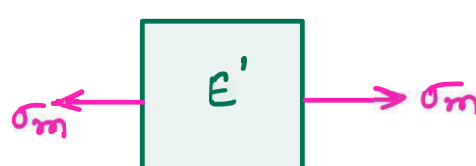
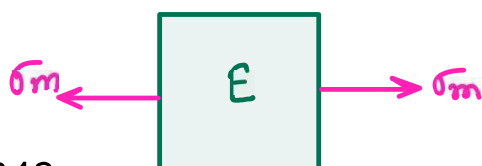
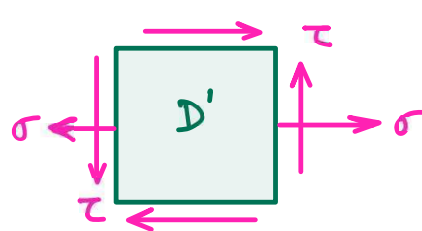
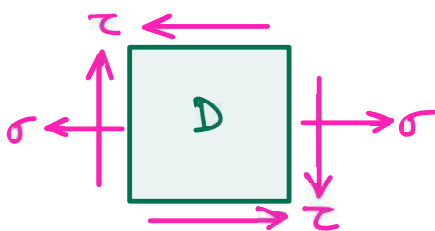
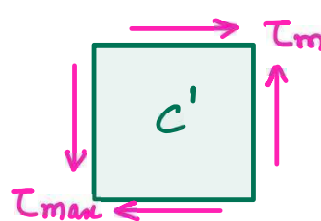
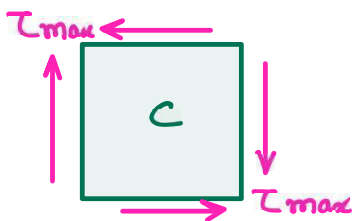
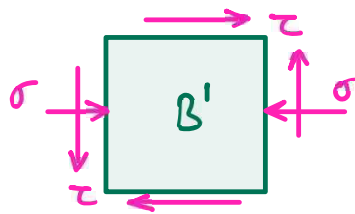
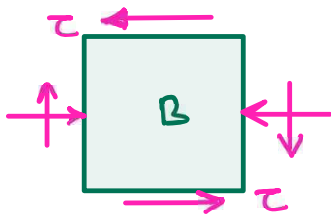
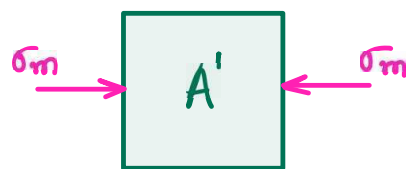
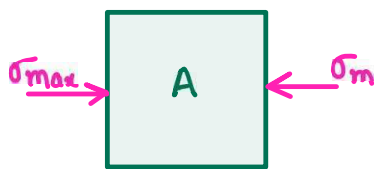
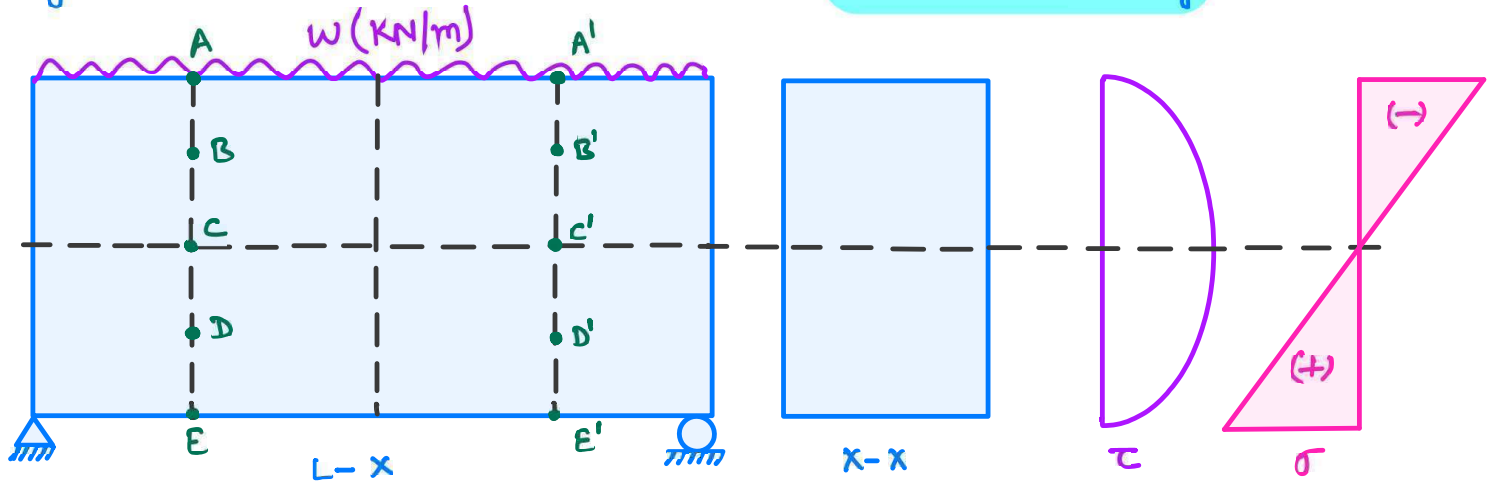
$$\begin{aligned} \sigma_{p_1}/\sigma_{p_2} &= \frac{\sigma}{2} \pm \sqrt{\left(\frac{\sigma}{2}\right)^2 + (-\tau)^2} \\ &= \frac{107.43}{2} \pm \sqrt{\left(\frac{107.43}{2}\right)^2 + (-52.53)^2} \end{aligned}$$

$$\sigma_{p_1} = 128.84 \text{ MPa}$$

$$\sigma_{p_2} = -21.41 \text{ MPa}$$

Analysis of Combined Bending & Transverse Shear stress

- If a beam is loaded longitudinally, any given point on it at any given section would be subjected to "Shear and Bending".



free face
 ↓ ↓
 $\Rightarrow$ Shear Stress = 0
 but
 Shear force $\neq 0$

- Corresponding to these stress elements we can find out maximum principal stress & maximum shear stress by following:-

a> for Rectangular section

- (i) points of maximum bending stress & point of maximum shear stress on the section of maximum Bending moment.
- (ii) point of maximum Bending stress & point of maximum shear stress on the section of maximum shear force.

b> for Flanged Section

- (i) In addition to above points also consider Junction of web and flange.

Q> A prismatic beam ABC is simply supported at A & B.

AB = 20 m , BC = 1 m , C is free end. The entire beam is uniformly loaded with 10 kN/m. The cross-section of the beam is I with the following particulars :-

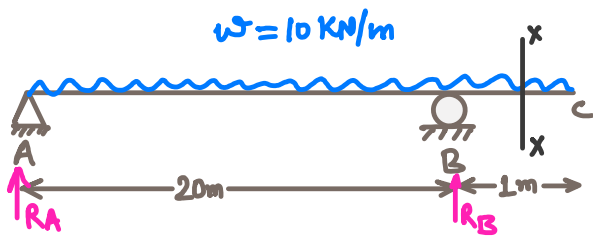
Flange width = 150 mm

thickness = 10 mm

web thickness = 10 mm

overall depth of beam = 300 mm

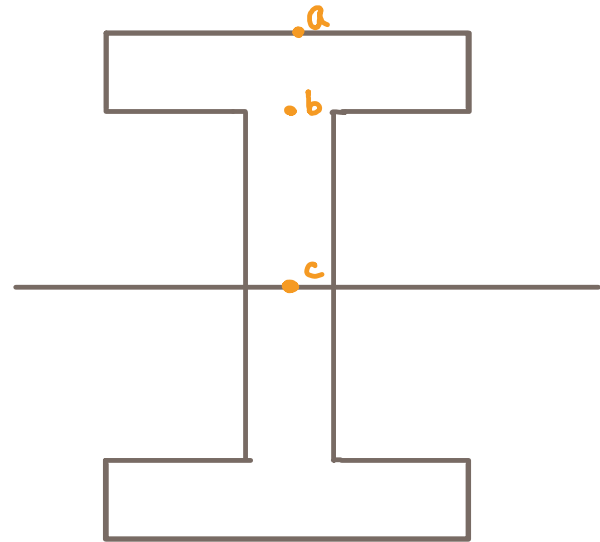
Determine the maximum value of principal stresses occurring anywhere in beam.

Solution

$$R_A + R_B = 210 \text{ kN}$$

$$R_B \times 20 = 10 \times 21 \times \frac{21}{2}$$

$$R_B = 110.25 \text{ kN}, R_A = 99.75 \text{ kN}.$$



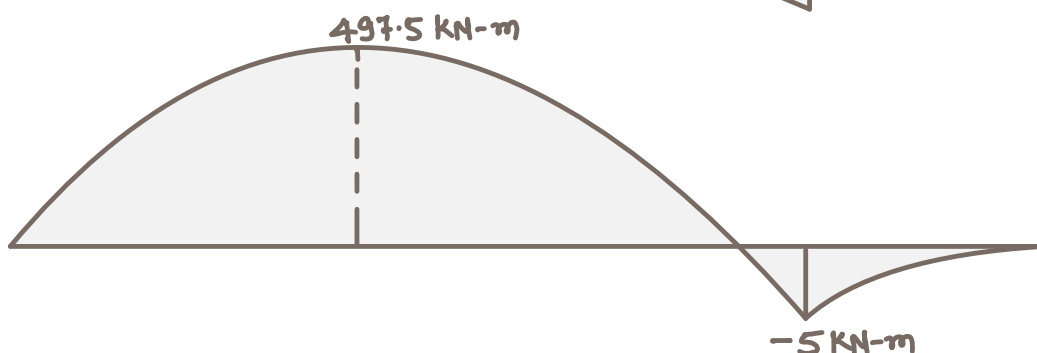
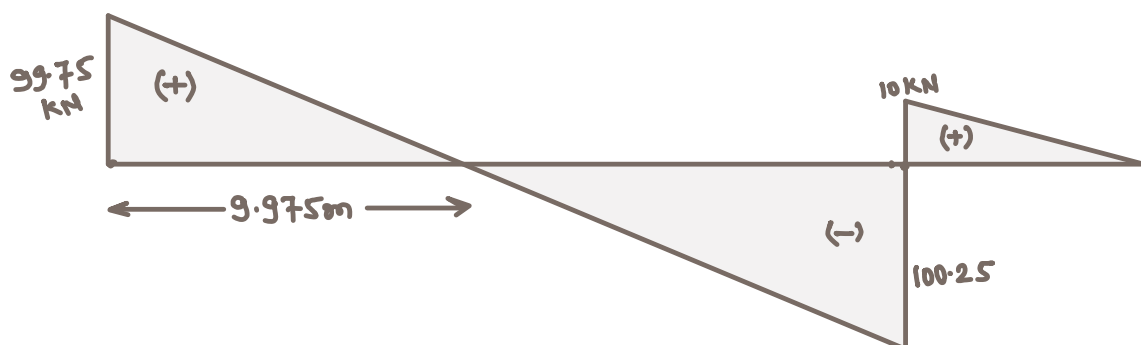
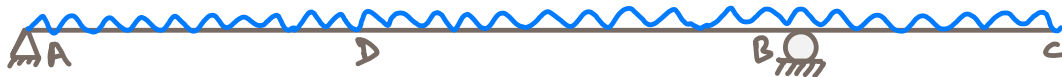
$$SF_A = R_A = 99.75 \text{ kN}$$

$$\Delta SF = SF_B - SF_A = \text{area of loading diagram.}$$

$$SF_B - 99.75 = -200 \Rightarrow SF_B = -200 + 99.75 = -100.25 \text{ kN}.$$

$$\Delta SF = SF_D - SF_A = \text{Area of loading diagram.}$$

$$0 - 99.75 = -10 \times x \Rightarrow x = 9.975.$$



$$\begin{aligned}\Delta BM &= BM_D - BM_A = \text{Area of SFD} \\ &= BM_D - 0 = \frac{1}{2} \times 99.75 \times 99.75 \\ &= (BM_D)_{\max} = 497.5 \text{ kNm.}\end{aligned}$$

$$\begin{aligned}\Delta BM &= BM_C - BM_B = \text{Area of SFD} \\ &= 0 - BM_B = \frac{1}{2} \times 1 \times 10 \\ BM_B &= -5 \text{ kN-m.}\end{aligned}$$

→ At Section D, BM is maximum, SF = 0, So Bending Stress would be maximum at point "a".

$$\sigma_m = \frac{M}{I} y$$



$$y_a = 150 \text{ mm}$$

$$I_{NA} = 150 \times \frac{300^3}{12} - (150-10) \frac{(300-20)^3}{12} = 8.14 \times 10^7 \text{ mm}^4$$

$$\sigma_m = \frac{497.5 \times 10^6 \times 150}{8.14 \times 10^7} = 916.76 \text{ MPa.}$$

$$\sigma_{p1} = \sigma_m = 916.76 \text{ MPa.} \quad \text{————— } \boxed{A}$$

— At Section B, SF is maximum, Bending Stress would be considered at point "c" & "b"

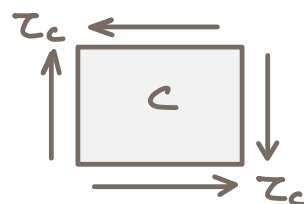
a) At Point "c"

$$\sigma_c = 0$$

$$\tau_{c\max} = \frac{VA\bar{y}}{I \cdot b} = \frac{-100.25 \times [150 \times 10 \times (150-5) + (150-10) \times 10 \times 70]}{8.14 \times 10^7 \times 10} \times 10^3$$

$$\tau_{c\max} = -39 \text{ MPa.}$$

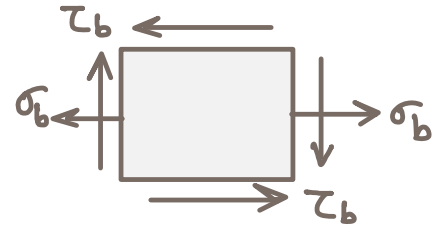
$$\sigma_{p1} = 39 \text{ MPa.} \quad \text{————— } \boxed{B}$$



b.> At point "b"

$$\sigma_b = \frac{M y_b}{I} = \frac{-5 \times 10^6 \times 140}{8.14 \times 10^7} = -8.59 \text{ MPa.}$$

$$\tau_b = \frac{V A \bar{y}}{I \cdot b} = \frac{-100 \cdot 25 \times 10^3 \times [150 \times 10 \times 145]}{8.14 \times 10^7 \times 10} = -26.74 \text{ MPa}$$



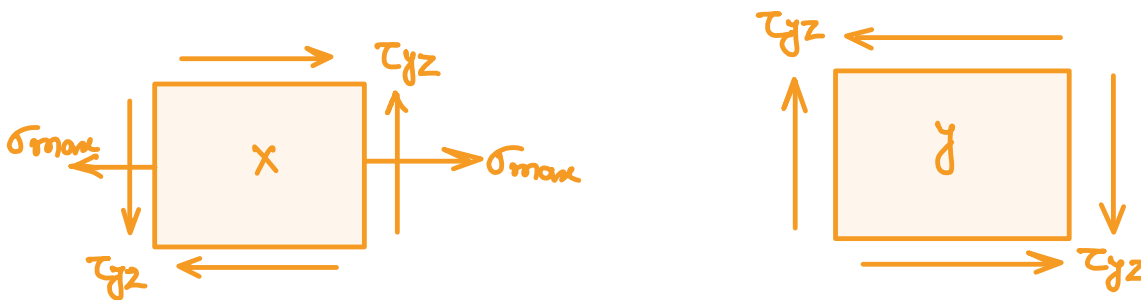
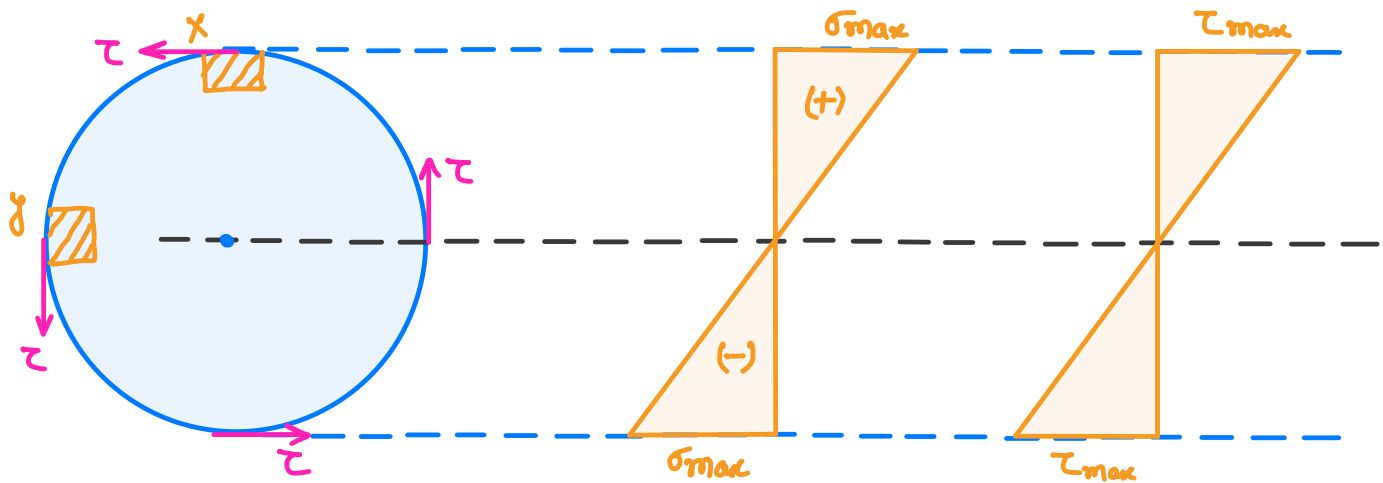
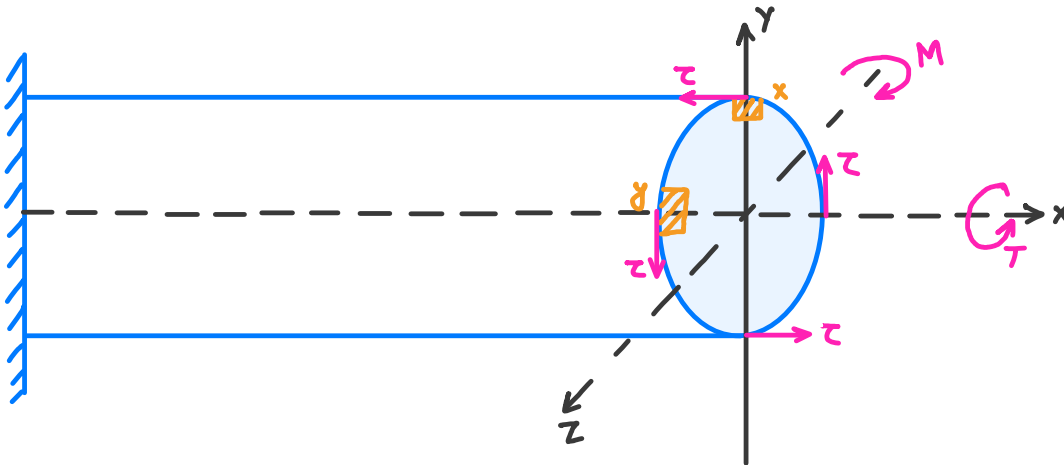
$$\begin{aligned} \sigma_{p_1} &= \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \\ &= \frac{+8.59 + 0}{2} \pm \sqrt{\left(\frac{+8.59 - 0}{2}\right)^2 + (-26.74)^2} \end{aligned}$$

$$\sigma_{p_1} = 31.43 \text{ MPa} \quad \text{--- } \boxed{C}$$

⇒ Hence maximum principal stress is at point "a" for section "D"

$$\sigma_{p_1} = 916.7 \text{ MPa}$$

Analysis of Combined Bending & Torsion



a.) for element "x"

$$\sigma_x = \sigma_{max} = \frac{M y_{max}}{I} = \frac{M D/2}{\pi D^4/64} = \frac{32 M}{\pi D^3}$$

$$\tau_{yz} = \tau_{max} = \frac{T y_{max}}{J} = \frac{T D/2}{\pi D^4/32} = \frac{16 T}{\pi D^3}$$

$$\sigma_{p1}/\sigma_{p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\frac{(\sigma_x - \sigma_y)^2}{2} + \tau_{xy}^2}$$

$$= \frac{32M}{2 \cdot \pi D^3} \pm \sqrt{\left(\frac{16M}{\pi D^3}\right)^2 + \left(\frac{16T}{\pi D^3}\right)^2}$$

$$\sigma_{p1}/\sigma_{p2} = \frac{16}{\pi D^3} \left[M \pm \sqrt{M^2 + T^2} \right]$$

$$\tau_{max} = \frac{\sigma_{p1} - \sigma_{p2}}{2} = \frac{16}{\pi D^3} \sqrt{M^2 + T^2}$$

b.> for element "y"

$$\left. \begin{array}{l} \sigma_x = 0 \\ \tau_{yz} = -\frac{16T}{\pi D^3} \end{array} \right\} \text{Pure Shear Condition}$$

$$\sigma_{p1}/\sigma_{p2} = \pm \frac{16T}{\pi D^3}$$

Note:-

For a particular applied moment and torque maximum principal stress in shaft is given by

$$\sigma_{p1} = \frac{16}{\pi D^3} \left[M + \sqrt{M^2 + T^2} \right]$$

And maximum shear stress anywhere in shaft is

$$\tau_{max} = \frac{16}{\pi D^3} \sqrt{M^2 + T^2}$$

- This can be now used for designing of shaft as follows:-

a.> Maximum principal stress theory.

$$\sigma_{p1} \leq \frac{f_y}{FOS}$$

$$\frac{16}{\pi D^3} \left[M + \sqrt{M^2 + T^2} \right] \leq \frac{f_y}{FOS}$$

Compute D.

b) Maximum Shear stress theory

$$\tau_{\max} \leq \frac{f_s / FOS}{2}$$

$$\frac{16T}{\pi D^3} \sqrt{M^2 + T^2} \leq \frac{f_s}{2 \cdot FOS}$$

Similarly, other theories can also be used as above for design of Shaft.

Equivalent torque and Moment

(a) Equivalent torque (T_e)

- It is a torque which, while acting alone produces maximum shear stress equal to the maximum shear stress due to combined action of Bending & torsion.

$$\frac{16 T_e}{\pi D^3} = \frac{16}{\pi D^3} \sqrt{M^2 + T^2}$$

$$T_e = \sqrt{M^2 + T^2}$$

(b) Equivalent moment (M_e)

- It is that moment which, while acting alone produces maximum normal stress equal to the maximum principal stress due to combined action of bending and Torsion.

$$\frac{32 M_e}{\pi D^3} = \frac{16}{\pi D^3} \left[M + \sqrt{M^2 + T^2} \right]$$

$$M_e = \frac{1}{2} \left[M + \sqrt{M^2 + T^2} \right]$$

So, for designing, $\frac{32 M_e}{\pi D^3} \leq \sigma_{\text{permissible}}$

$$\frac{16 T_e}{\pi D^3} \leq \tau_{\text{permissible}}$$

Q. > A Solid Steel circular shaft is required to carry torque of 40 kN-m and a BM of 20 kNm. Find the size of the shaft using maximum principal stress & maximum shear stress theory.

$$FOS = 2, E = 2 \times 10^5 \text{ N/mm}^2, f_y = 250 \text{ MPa}$$

Solution (a) Maximum principal stress theory

$$\sigma_{p1} \leq \frac{f_y}{FOS}$$

$$\frac{16}{\pi D^3} [M + \sqrt{M^2 + T^2}] \leq \frac{f_y}{FOS}$$

$$\frac{16}{\pi D^3} [20 + \sqrt{20^2 + 40^2}] \times 10^6 \leq \frac{250}{2}$$

$$D \geq 138.15 \text{ mm}$$

(OR) $M_e = \frac{1}{2} [M + \sqrt{M^2 + T^2}] = \frac{1}{2} [20 + \sqrt{20^2 + 40^2}] = 32.36 \text{ kN-m.}$

$$\frac{32 M_e}{\pi D^3} \leq \frac{f_y}{FOS}$$

$$\frac{32 \times 32.36 \times 10^6}{\pi D^3} \leq \frac{250}{2} \Rightarrow \underline{D \geq 138.15 \text{ mm}}$$

(b) Max shear stress theory

$$\tau_{\max} \leq \frac{f_y}{2 FOS}$$

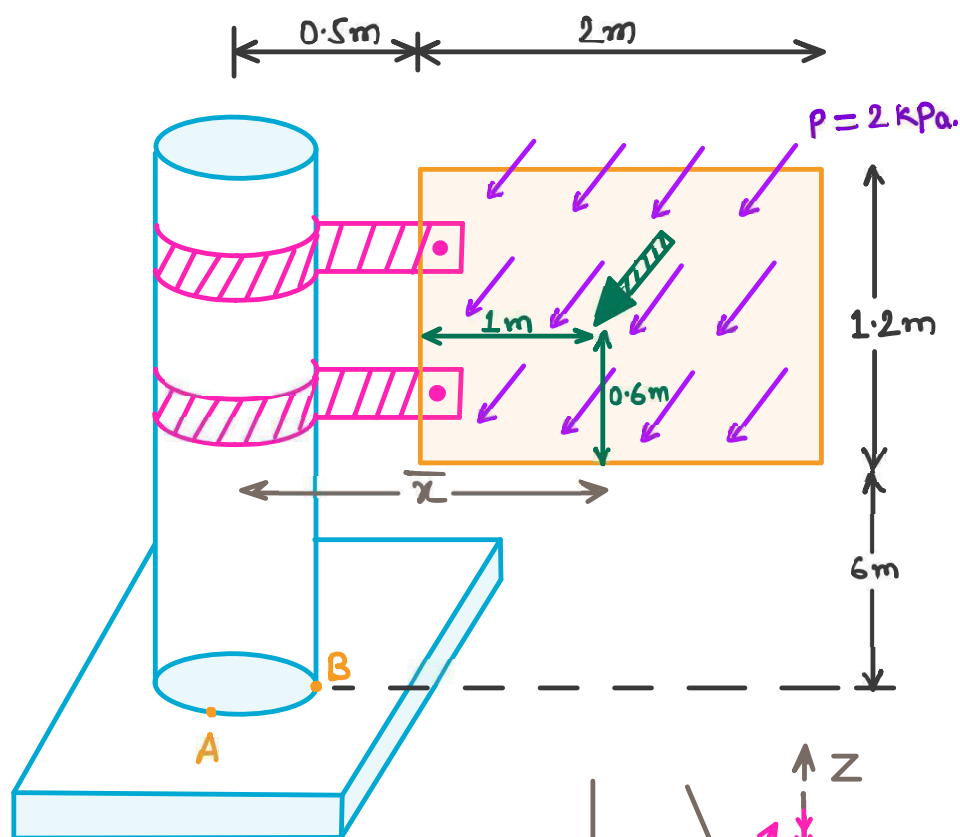
$$\frac{16}{\pi D^3} \sqrt{M^2 + T^2} \leq \frac{250}{2 \times 2}$$

$$\left(\frac{16}{\pi D^3} \sqrt{20^2 + 40^2} \right) \times 10^6 \leq \frac{250}{2 \times 2}$$

$$D \geq 153.9 \text{ mm}$$

Q. >

A Sign board of dimension $2\text{ m} \times 1.2\text{ m}$ is supported by a hollow circular pole having outer diameter of 220 mm and inner diameter of 180 mm . The sign board is offset by 0.5 m from centre line of the pole & its lower edge is 6.0 m above ground level. Compute the principal stresses and maximum shear stress at point A and B at the base of the pole due to wind pressure of 2 kPa against the sign board.



$$F = pA$$

$$= 2 \times 2 \times 1.2 = 4.8 \text{ kN}$$

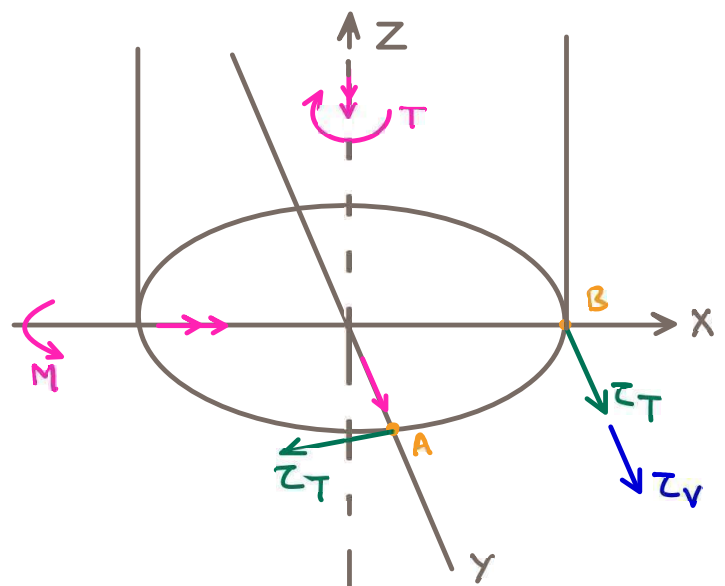
$$T = F \times \bar{x}$$

$$= 4.8 \times 1.5 = 7.2 \text{ kN-m}$$

$$M = F \times \bar{y}$$

$$= 4.8 \times 6.6 = 31.68 \text{ kNm}$$

$$352 \quad V = 4.8 \text{ kN}$$



a.) for point "A"

$$\tau_{zy} = \frac{T d_2/2}{J} = \frac{7.2 \times 10^6 \times \frac{220}{2}}{\frac{\pi}{32}(220^4 - 180^4)} = 6.24 \text{ MPa. (Due to Torque)}$$

$$\sigma_z = \frac{-M y_{\max}}{I} = \frac{-31.68 \times 10^6 \times \frac{220}{2}}{\frac{\pi}{64}(220^4 - 180^4)} = -54.91 \approx -55 \text{ MPa (comp.)}$$

$$\tau_{zy} = 0 \quad (\text{Due to Shear force})$$

$$\begin{aligned} \sigma_{p_1}/\sigma_{p_2} &= \frac{\sigma_z + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_z - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \\ &= \frac{-55 + 0}{2} \pm \sqrt{\left(\frac{-55 + 0}{2}\right)^2 + (6.24)^2} \end{aligned}$$

$$\sigma_{p_1} = -55.69 \text{ MPa}, \quad \sigma_{p_2} = 0.699 \text{ MPa.}$$

$$\tau_{\max} = \frac{\sigma_{p_1} - \sigma_{p_2}}{2} = \frac{-55.69 - 0.699}{2} = -28.2 \text{ MPa}$$

b.) for point B,

$$\tau_{zx} = \tau_{zy} = 6.24 \text{ MPa. (Due to torque)}$$

$$\sigma = 0 \quad (\text{N.A})$$

$$\tau_{zx} = \tau_{\max} = 2 \cdot \tau_{\text{avg}} \quad (\text{Due to Shear})$$

$$= \frac{2V}{A_m} = \frac{2V}{2\pi R_m t}$$

$$= \frac{4.8 \times 10^3}{3.14 \times \left(\frac{220+180}{4}\right) \left(\frac{220-180}{2}\right)}$$

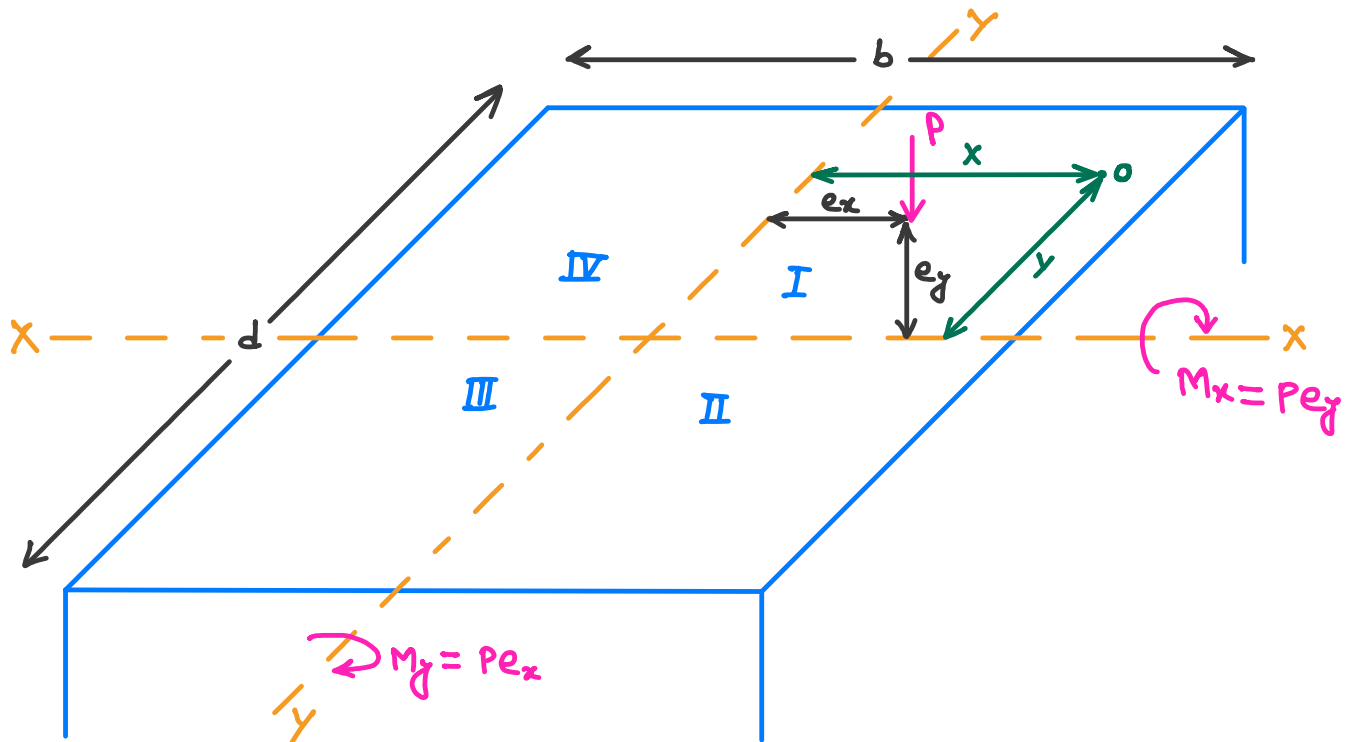
$$\tau_{zx} = 0.7639 \text{ MPa}$$

$$\tau_{zx \text{ Total}} = 6.24 + 0.7639 = 7.004 \text{ MPa.}$$

$$\sigma_{p_1} = \sigma_{p_2} = \tau_{\max} = 7.004 \text{ MPa.}$$

Analysis of Combined Bending and Axial Stress

- Consider a section subjected to combined bending and axial force, normal stress at any location on this section is given by



$\sigma_o = \sigma_{\text{direct}} + \sigma_{\text{bending}} = -\frac{P}{A} - \frac{M_x y}{I_x} - \frac{M_y x}{I_y}$
 $= -\frac{P}{A} - \frac{(P e_y) y}{I_x} - \frac{(P e_x) x}{I_y}$

Combined Stress at point "O".

- By substituting the value of e_x , e_y , x , y with sign we can get combined stresses at concerned point "O".

- In order to find NA or line of zero stress, $\sigma_o = 0$

$$\sigma_o = -\frac{P}{A} - \frac{(P e_y) y}{I_x} - \frac{P (e_x) x}{I_y} = 0$$

- Let $I_x = A r_x^2$, $I_y = A r_y^2$

r_x , r_y are radius of gyration along x & y axis respectively.

$$\frac{P}{A} + \frac{(Pe_y)y}{A r_x^2} + \frac{(Pe_x)x}{A r_y^2} = 0$$

$$\frac{P}{A} \left[1 + \frac{e_y \cdot y}{r_x^2} + \frac{e_x \cdot x}{r_y^2} \right] = 0$$

$$\frac{e_x \cdot x}{r_y^2} + \frac{e_y \cdot y}{r_x^2} = -1 \rightarrow \{ \text{Equation of straight line} \}$$

gn our case,

$$I_x = A r_x^2, \quad I_y = A r_y^2$$

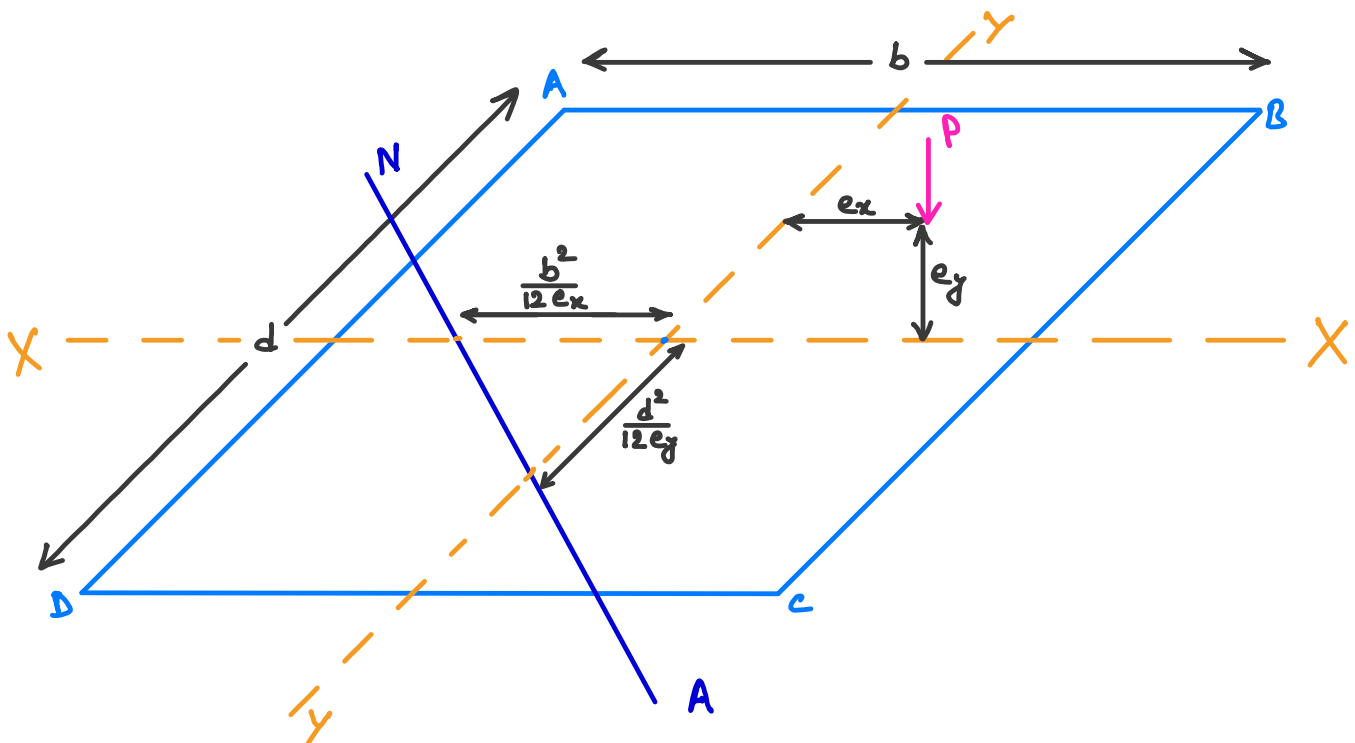
$$\frac{bd^3}{12} = bd r_x^2$$

$$\frac{db^3}{12} = bd r_y^2$$

$$r_x^2 = \frac{d^2}{12}$$

$$r_y^2 = \frac{b^2}{12}$$

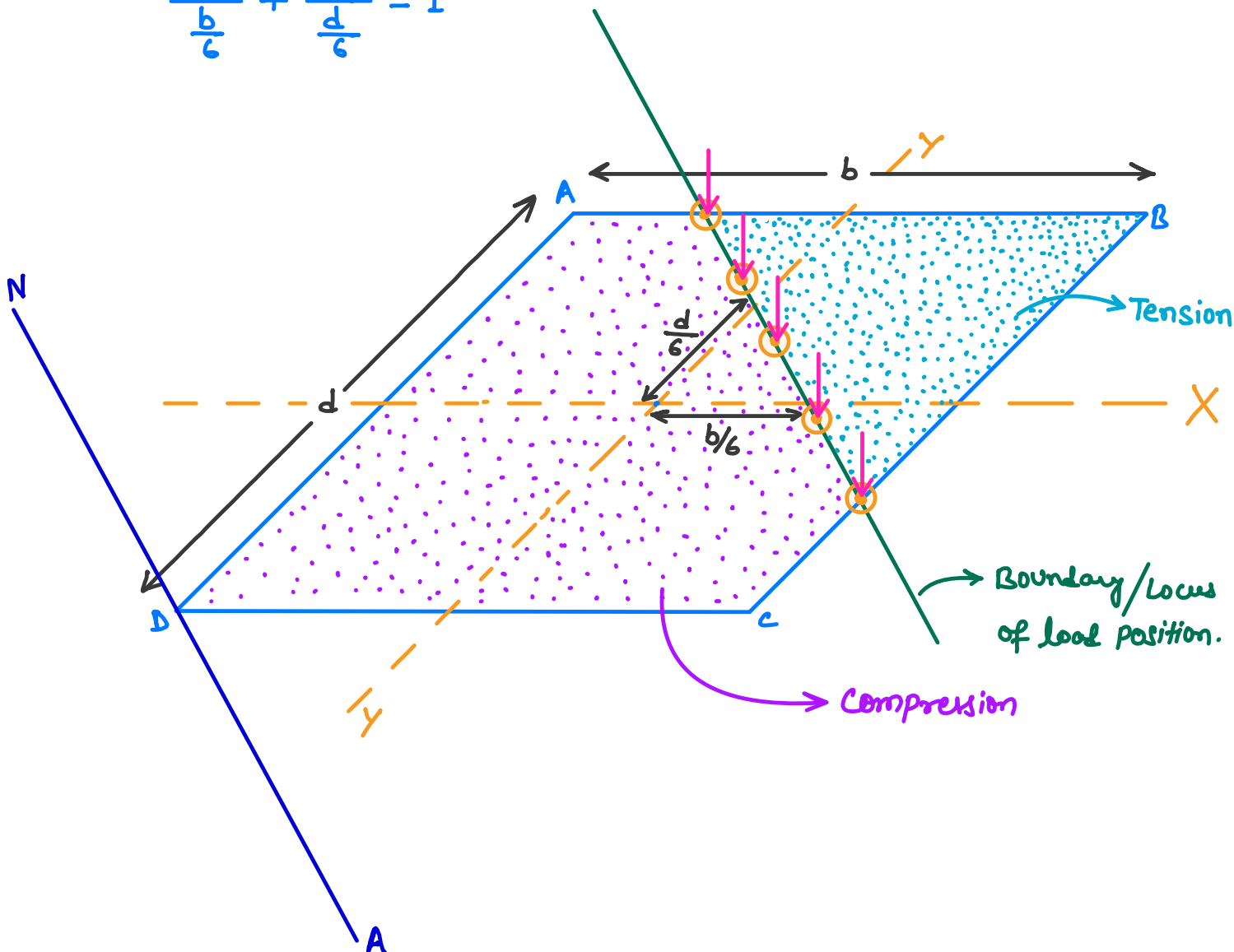
$$\frac{x}{\left(\frac{b^2}{12e_x}\right)} + \frac{y}{\left(\frac{d^2}{12e_y}\right)} = -1$$

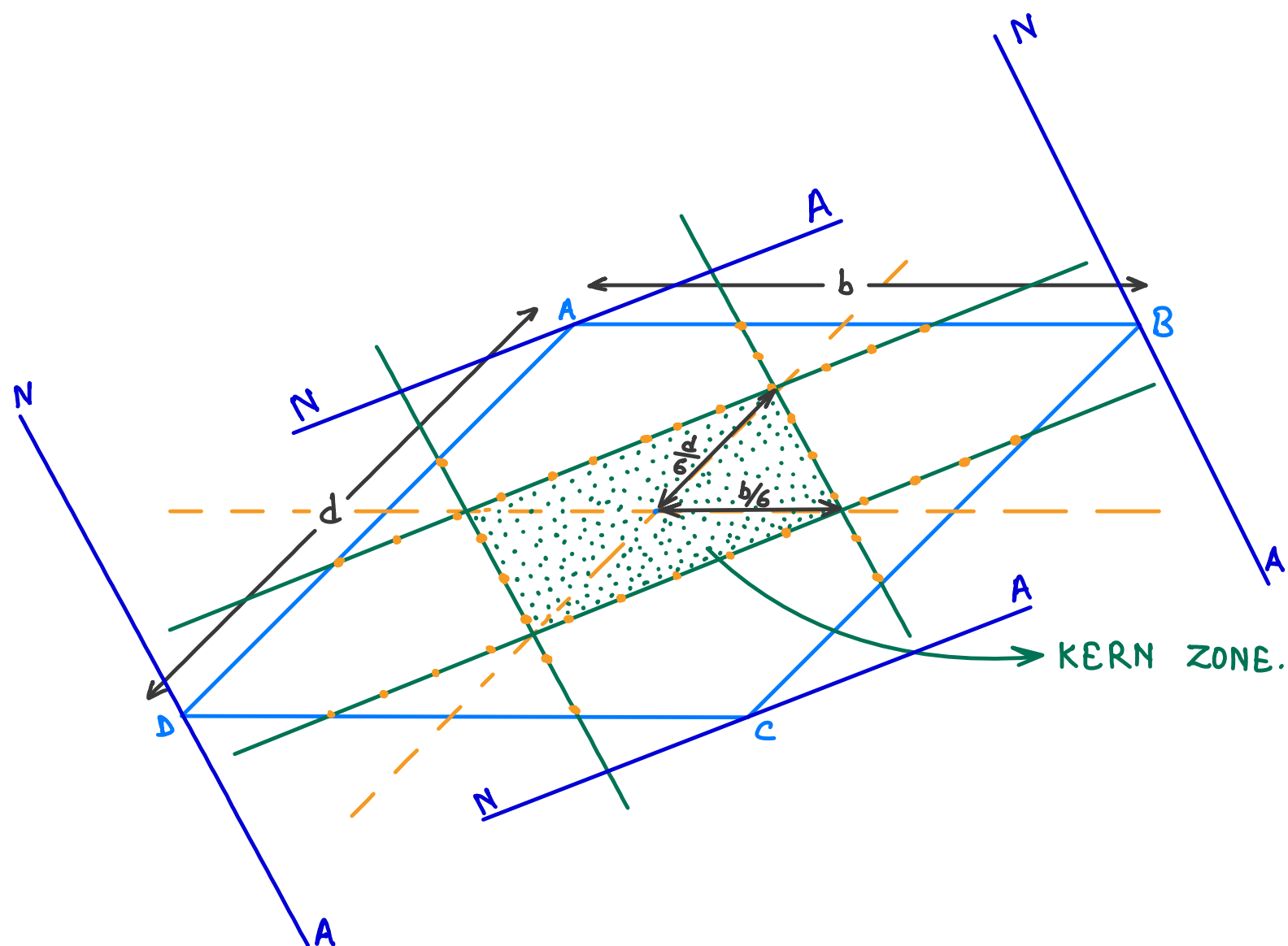


- So, in order to find zone of loading which leads to no tension in the section (**KERN-ZONE**), Pass the N.A from extreme points A, B, C, D
- for N.A to pass from point "D", put $x = -\frac{b}{2}$, $y = -\frac{d}{2}$ in the equation of N.A.

$$\frac{-\frac{b}{2} \cdot e_x}{\left(\frac{b^2}{12}\right)} + \frac{-\frac{d}{2} \cdot e_y}{\left(\frac{d^2}{12}\right)} = -1$$

$$\frac{e_x}{\frac{b}{6}} + \frac{e_y}{\frac{d}{6}} = 1$$





- for Rectangular Section, Kern zone is Rhombus.
- for Square Section, Kern zone is Square.
- for I-Section, Kern zone is Rhombus.

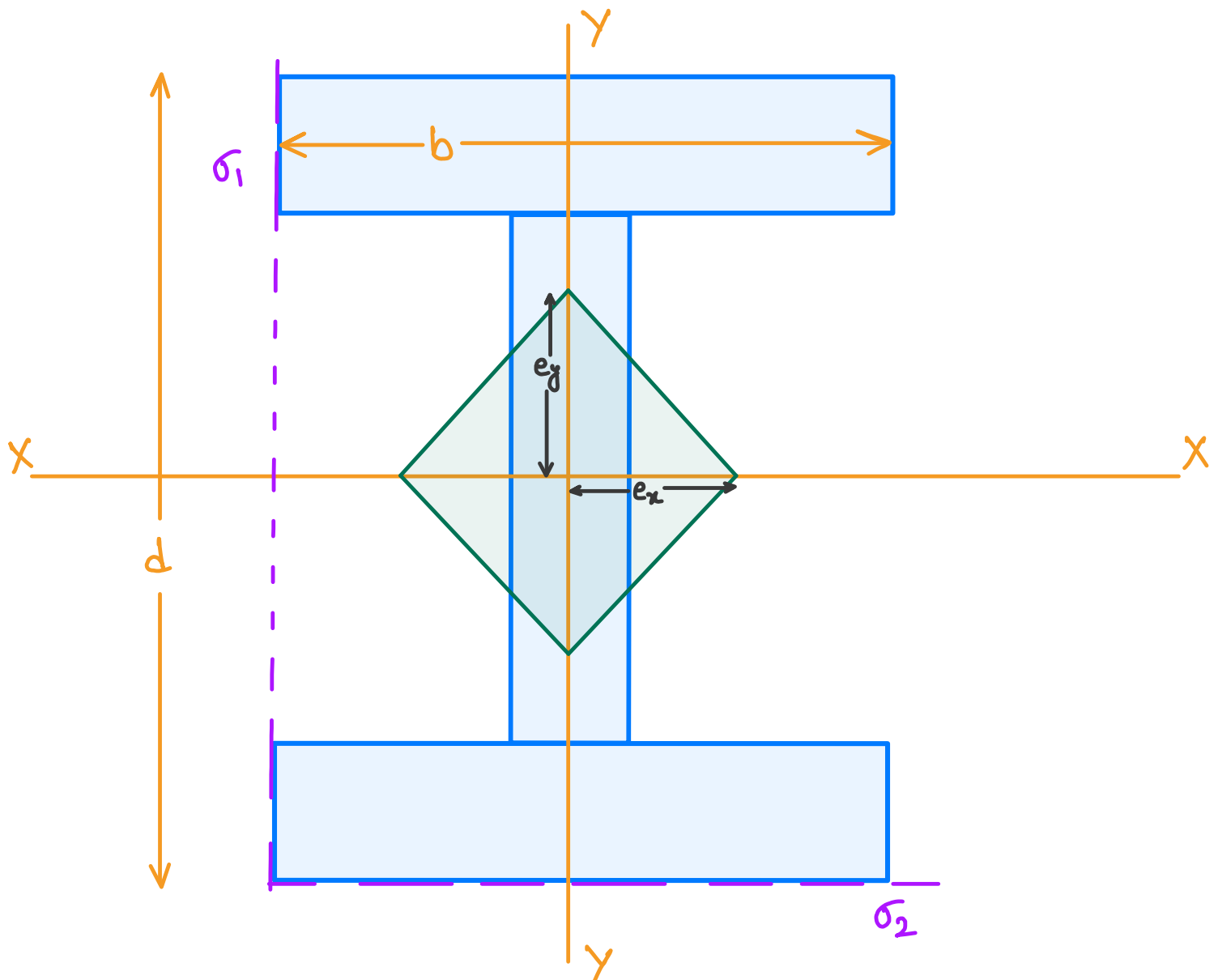
for example :- $\sigma_1 = 0 \Rightarrow -\frac{P}{A} + \frac{(Pe_x)b/2}{I_y} = 0$

$$-\frac{P}{A} + \frac{Pe_x b/2}{A r_y^2} = 0$$

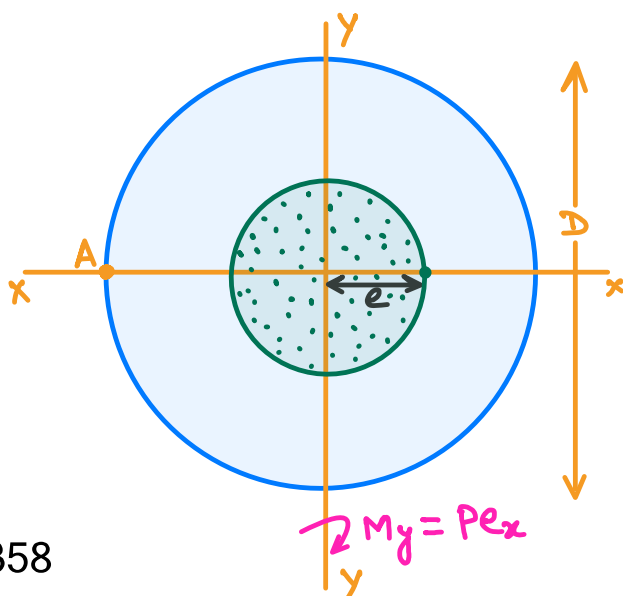
$$\Rightarrow e_x = \frac{r_y^2}{b/2}$$

Similarly,

$$\Rightarrow e_y = \frac{r_x^2}{d/2}$$



— for circular and Hollow circular section, Kern zone is circular.



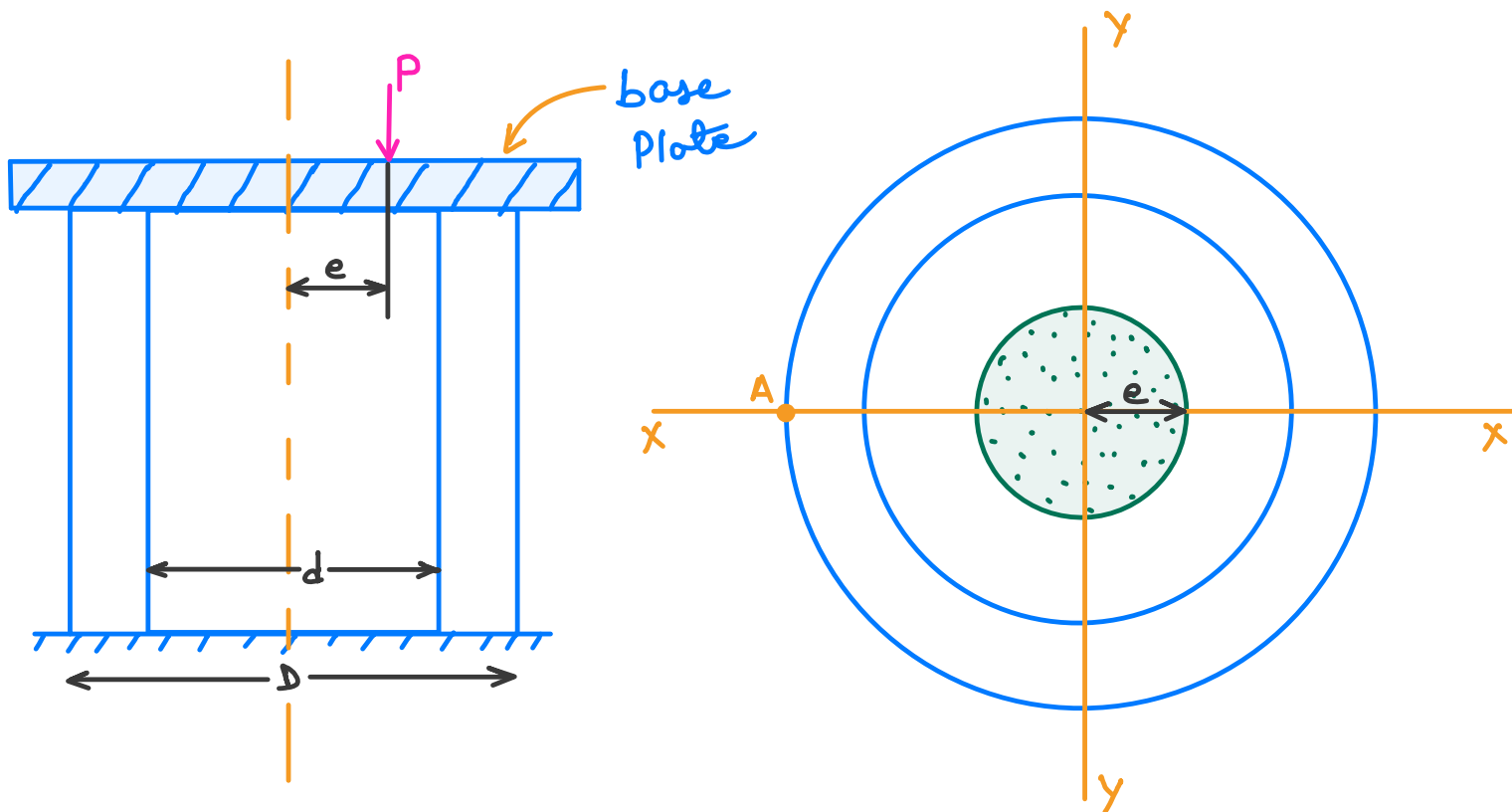
for e, $\sigma_A = 0$

$$\frac{-P}{A} + \frac{Pe \cdot D/2}{\frac{\pi D^4}{64}} = 0$$

$$e = \frac{D}{8}$$

$$2M_y = Pe_x$$

- Hence, Kern of circular section is circle of diameter $\frac{D}{4}$.



for e , $\sigma_A = 0$

$$\frac{-P}{A} + \frac{(Pe) D/2}{I_{yy}} = 0$$

$$\frac{-P}{\frac{\pi}{4}(D^2 - d^2)} + \frac{(Pe) D/2}{\frac{\pi}{64}[D^4 - d^4]} = 0$$

$$e = \frac{D^2 + d^2}{8D}$$

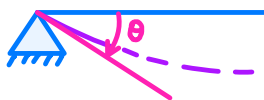
- Kern of Hollow circular section is circle of diameter $\frac{D^2 + d^2}{4D}$.

DEFLECTION OF BEAMS

- Deflection of the structure is observed due to
 - (a) load
 - (b) Temperature
 - (c) Settlement of support
 - (d) Fabrication defect due to creep.
 - (e) creep
- Deflection is caused by internal loadings such as Normal force, shear force, Bending moment, torsion.
- for beams and frames, deflection is primarily due to Bending moment.
- For Trusses, deflection is primarily due to Axial force.
- To visualise the deflected shape of structure due to loading, deflected diagram is drawn.

for eg:-

a>



$$\Delta = 0$$

$$\theta \neq 0$$

b>

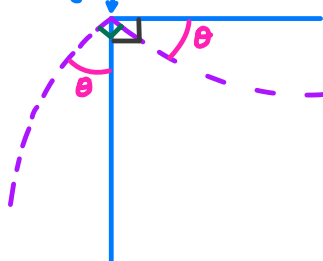


$$\theta = 0$$

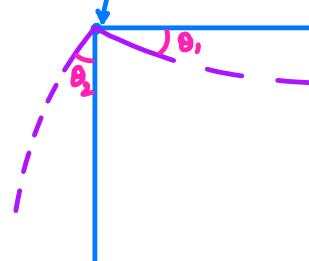
$$\Delta = 0$$

c>

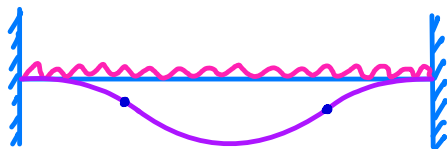
Rigid Joint



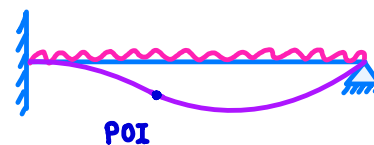
Pin Joint



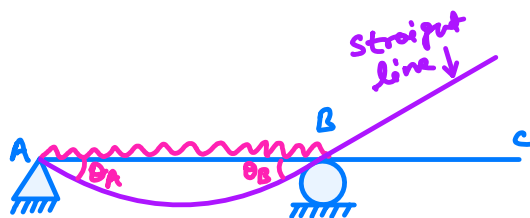
d. >



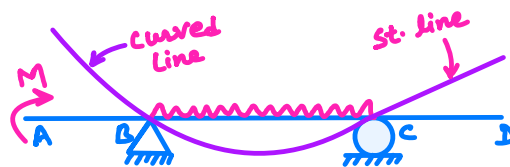
e. >



f. >



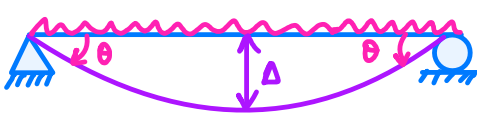
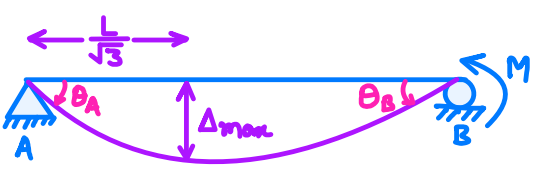
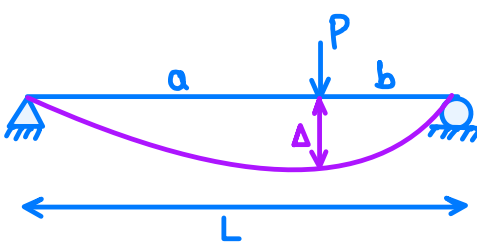
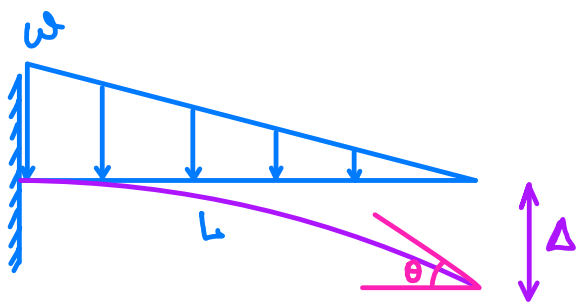
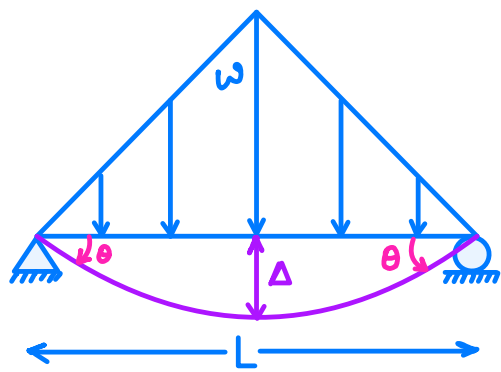
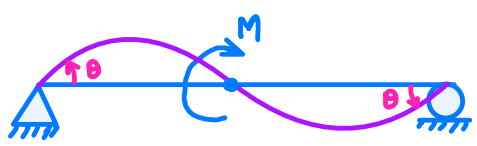
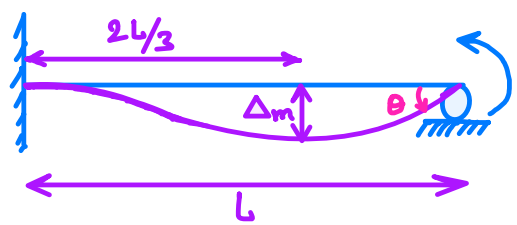
g. >

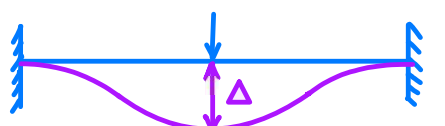
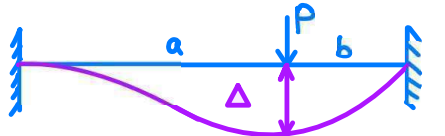
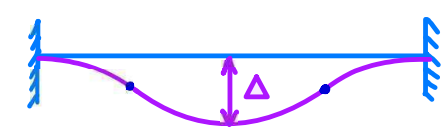


Slope and Deflection of Standard Cases :- GATE

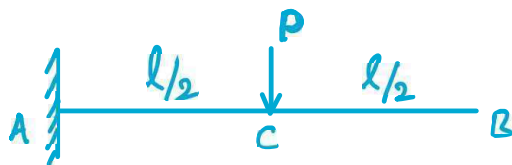
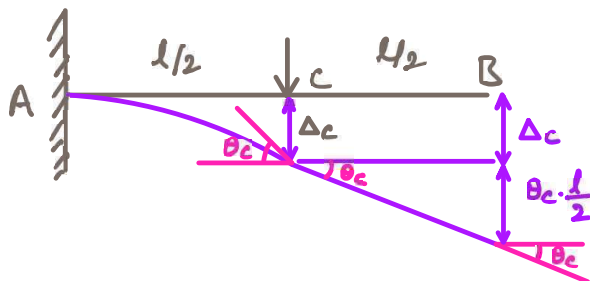
a. >		$\frac{PL^3}{3EI}$	$\frac{PL^2}{2EI}$
b. >		$\frac{ML^2}{2EI}$	$\frac{ML}{EI}$
c. >		$\frac{WL^4}{8EI}$	$\frac{WL^3}{6EI}$
d. >		$\frac{PL^3}{48EI}$	$\frac{PL^2}{16EI}$

361

e.		$\frac{5}{384} \frac{wL^4}{EI}$	$\frac{wL^3}{24EI}$
f.		$\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = \frac{ML}{3EI}$ $\theta_B = \frac{ML}{6EI}$
g.		$\frac{Pa^2b^2}{3EIL}$	—
h.		$\frac{wL^4}{30EI}$	$\frac{wL^3}{24EI}$
i.		$\frac{wL^4}{120EI}$	$\frac{5wL^3}{192EI}$
j.		0	$\frac{ML}{24EI}$
k.		$\frac{ML^2}{27EI}$	$\frac{ML}{4EI}$

L. >		$\frac{PL^3}{192 EI}$	—
m. >		$\frac{Pa^3 b^3}{3EI L^3}$	—
n. >		$\frac{wL^4}{384 EI}$	—

Q. >

find θ_B , Δ_B Solution.

Since. BM in Region CB = 0

$$\theta_B = \theta_c = \frac{P(\frac{l}{2})^2}{2EI} = \frac{Pl^2}{8EI}$$

$$\Delta_B = \Delta_c + \theta_c \frac{l}{2}$$

$$= \frac{P(\frac{l}{2})^3}{3EI} + \frac{Pl^2}{8EI} \cdot \frac{l}{2}$$

$$= \frac{5}{48} \cdot \frac{Pl^3}{EI}$$

Note:-

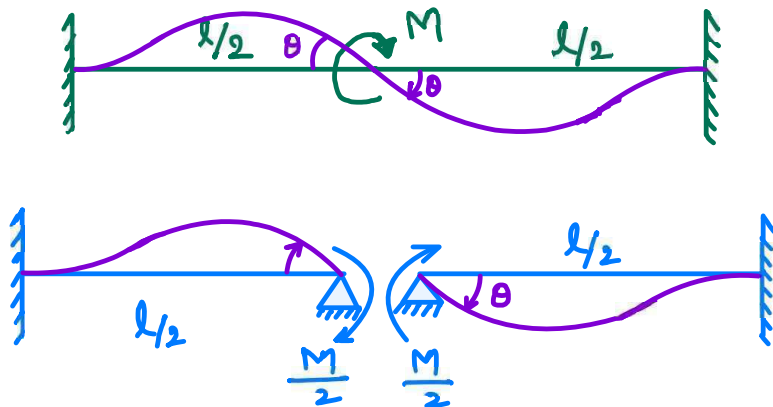
- If instead of point load at C, beam is loaded uniformly A to C find θ_B , Δ_B .

$$\theta_B = \theta_C = \frac{WL^3}{48EI}$$

$$\Delta_B = \Delta_C + \theta_C \cdot \frac{l}{2} = \frac{7}{384} \frac{Wl^2}{EI}$$

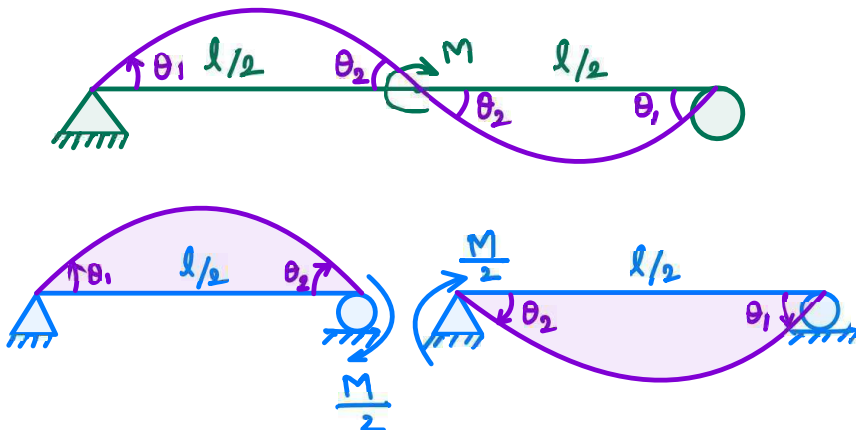
- In case of symmetrical condition wrt geometry, support, material for couple acting at the mid span, the mid span deflection is zero (However if the couple is not acting at mid span deflection at the location of couple application is not zero).

a.>



$$\theta = \frac{\left(\frac{M}{2}\right)\left(\frac{l}{2}\right)}{4EI} = \frac{ML}{16EI}$$

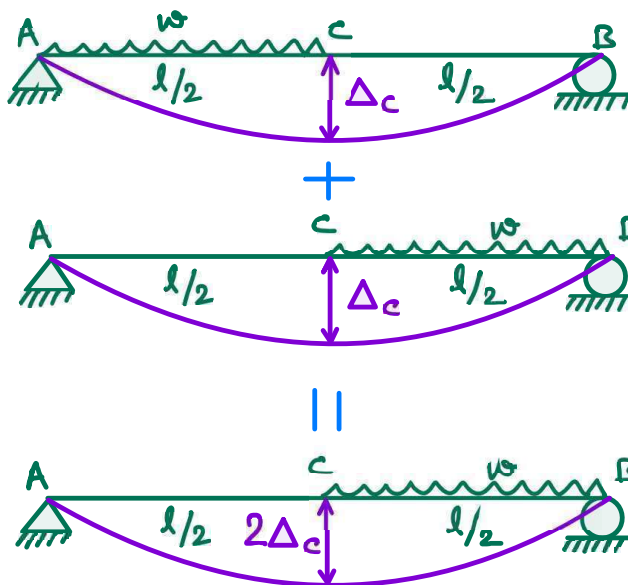
b.>



$$\theta_2 = \frac{\left(\frac{M}{2}\right)\left(\frac{l}{2}\right)}{3EI} = \frac{ML}{12EI}$$

$$\theta_1 = \frac{\left(\frac{M}{2}\right)\left(\frac{l}{2}\right)}{6EI} = \frac{Ml}{24EI}$$

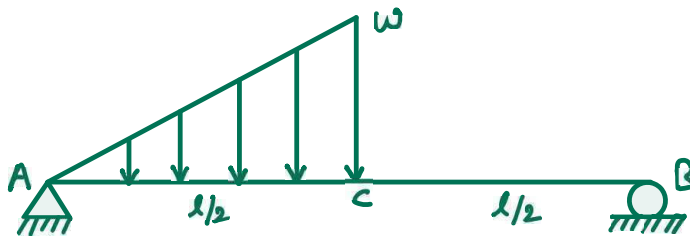
c. >

find δ_c .

$$2\Delta_c = \frac{5}{384} \frac{wl^4}{EI}$$

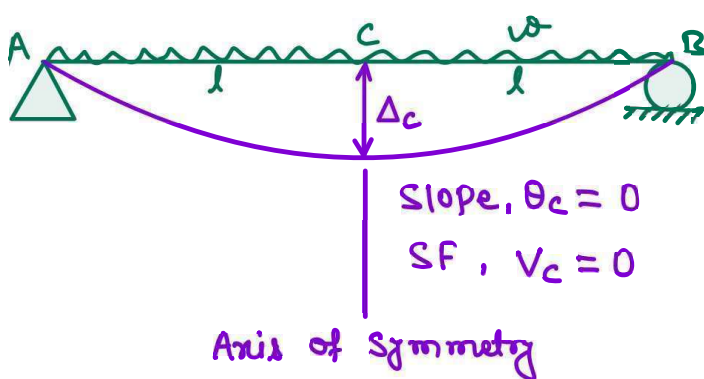
$$\Delta_c = \frac{5}{768} \frac{wl^4}{EI}$$

d. >

find Δ_c ?

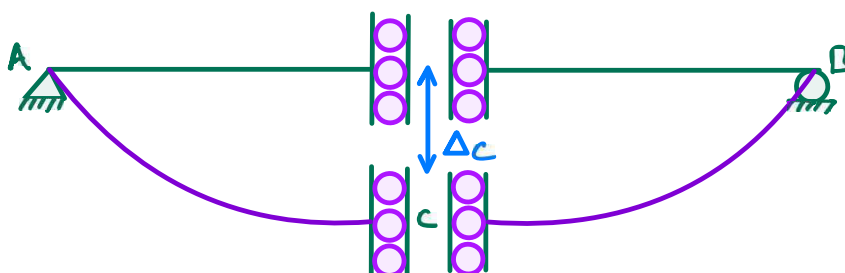
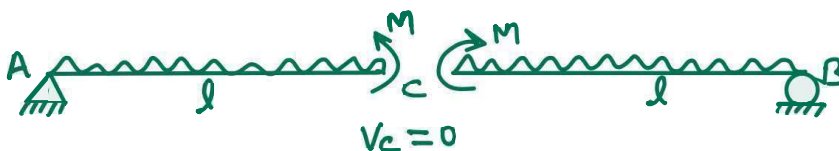
$$\delta_c = \frac{1}{2} \left(\frac{wl^4}{120 EI} \right)$$

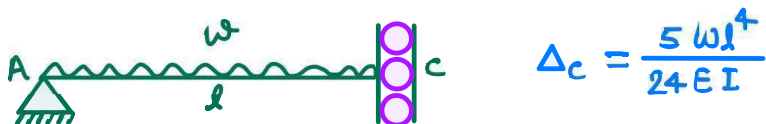
e. >



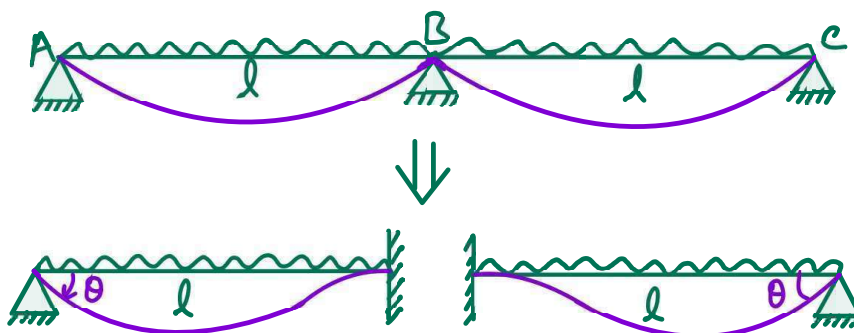
$$\Delta_c = \frac{5}{384} \frac{w(2l)^4}{EI}$$

$$\Delta_c = \frac{5}{24} \frac{wl^4}{EI}$$

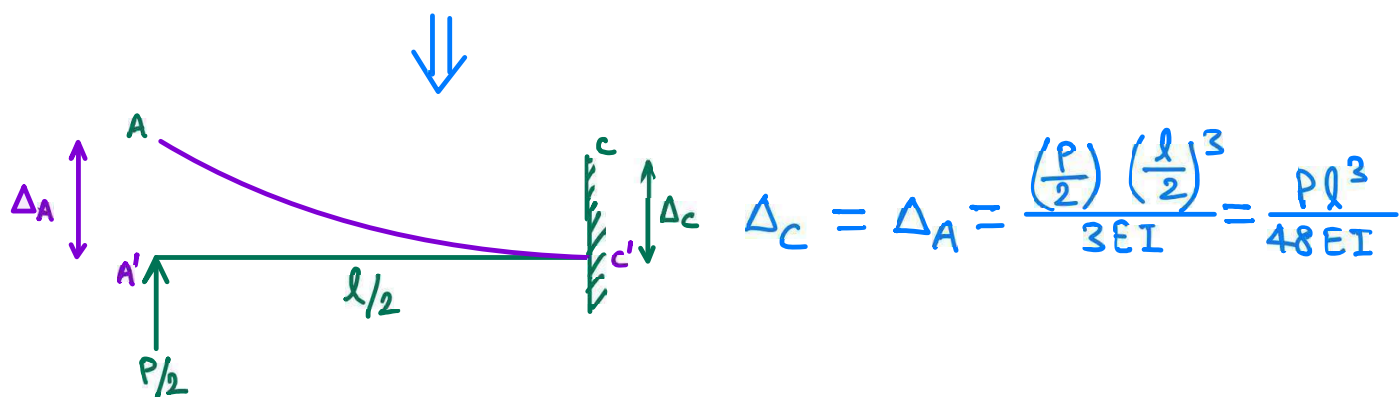
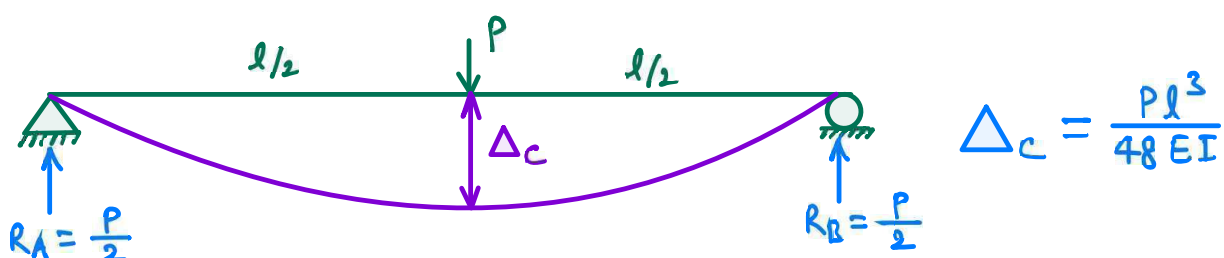




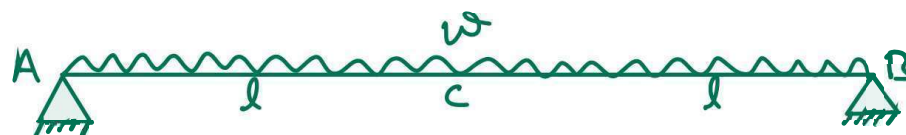
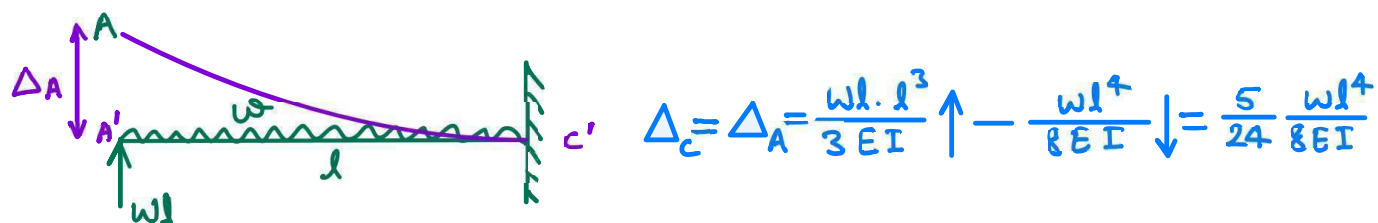
f.



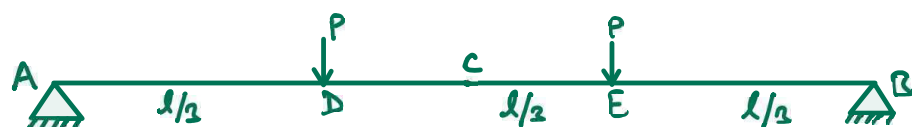
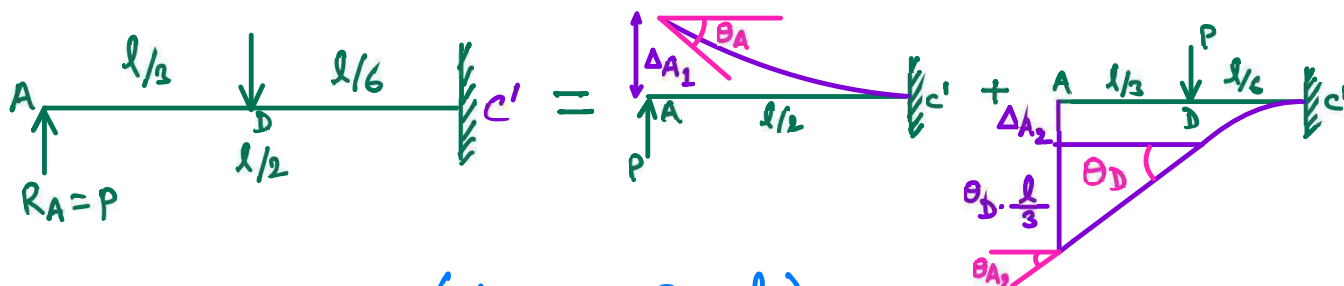
g.



h.

find $\Delta_c = ?$ 

I.

find Δ_c , θ_A .Solution

$$\Delta_c = \Delta_A = \Delta_{A_1} - (\Delta_{A_2} + \theta_D \cdot \frac{l}{3})$$

$$= \frac{P(\frac{l}{2})^3}{3EI} - \left\{ \frac{P(\frac{l}{6})^3}{3EI} + \frac{P(\frac{l}{6})^2}{2EI} \cdot \frac{l}{3} \right\}$$

$$\Delta_c = \frac{23}{648} \frac{Pl^3}{EI}$$

$$\theta_A = \theta_{A_1} + (-\theta_{A_2}) = \theta_{A_1} - \theta_D = \frac{P(\frac{l}{2})^2}{2EI} - \frac{P(\frac{l}{6})^2}{2EI}$$

$$\theta_A = \frac{Pl^2}{9EI}$$

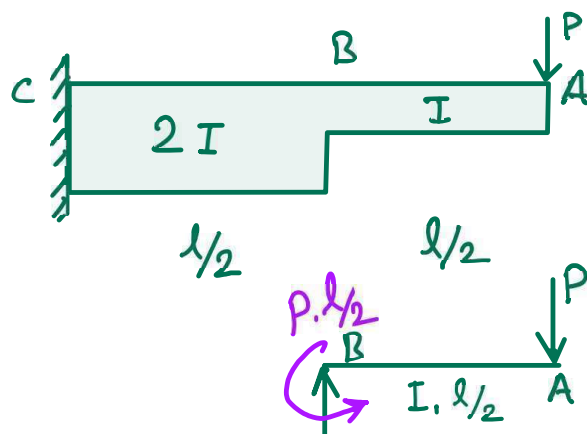
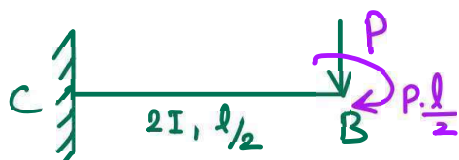
J.

find Δ_B , θ_B 

$$\Delta_B = \frac{Pl^3}{3EI} + \frac{Pa \cdot l^2}{2EI}$$

$$\theta_B = \frac{Pl^2}{2EI} + \frac{Pa \cdot l}{EI}$$

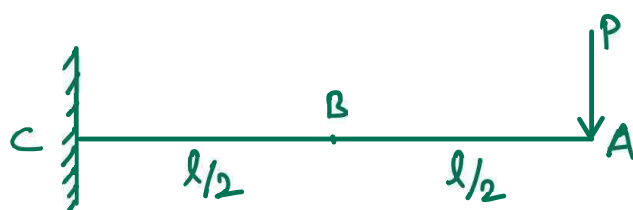
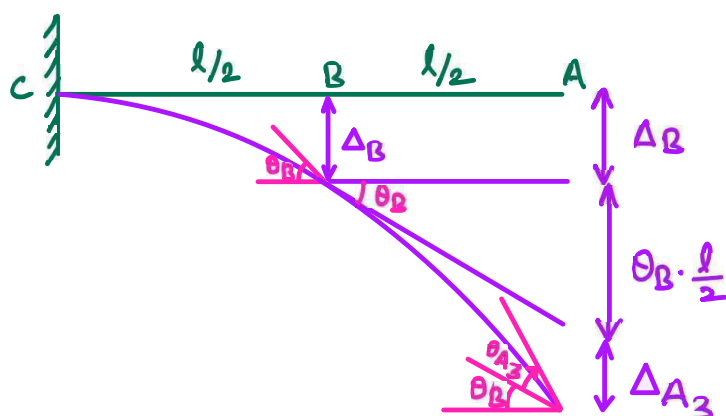
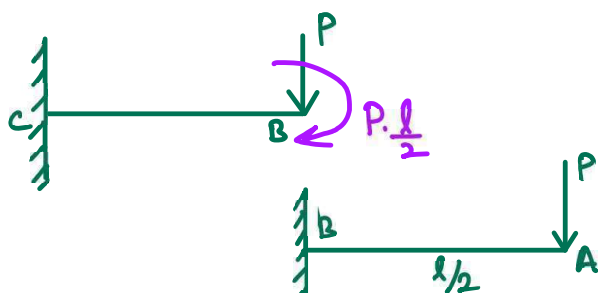
K. >

find Δ_B, θ_B 

$$\Delta_B = \frac{P\left(\frac{l}{2}\right)^3}{3E(2I)} + \frac{P\frac{l}{2} \cdot \left(\frac{l}{2}\right)^2}{2E(2I)} = \frac{5}{96} \frac{Pl^3}{EI}$$

$$\theta_B = \frac{P\left(\frac{l}{2}\right)^2}{2E(2I)} + \frac{P\frac{l}{2} \cdot \frac{l}{2}}{E(2I)} = \frac{3}{16} \frac{Pl^2}{EI}$$

L. >

find Δ_A, θ_A
(using previous concept)

$$\Delta_B = \frac{P(l/2)^3}{3EI} + \frac{P \cdot \frac{l}{2} (l/2)^2}{2EI} = \frac{5}{48} \frac{Pl^3}{EI}$$

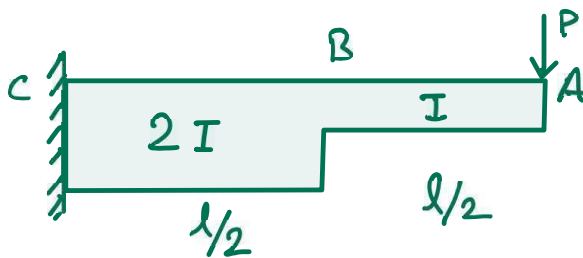
$$\theta_B = \frac{P(l/2)^2}{2EI} + \frac{P \cdot \frac{l}{2} (l/2)}{EI} = \frac{3}{8} \frac{Pl^2}{EI}$$

$$\Delta_{A_3} = \frac{P(l/2)^3}{3EI} = \frac{1}{24} \frac{Pl^3}{EI}$$

$$\begin{aligned} \Delta_A &= \Delta_B + \theta_B \cdot \frac{l}{2} + \Delta_{A_3} \\ &= \frac{5}{48} \frac{Pl^3}{EI} + \frac{3}{8} \frac{Pl^2}{EI} \cdot \frac{l}{2} + \frac{Pl^3}{24EI} = \frac{Pl^3}{3EI} \end{aligned}$$

$$\theta_A = \theta_B + \theta_{A_3} = \frac{P(\frac{l}{2})^2}{2EI} + \frac{P \cdot \frac{l}{2} (l/2)}{EI} + \frac{P(l/2)^2}{EI} = \frac{Pl^2}{2EI}$$

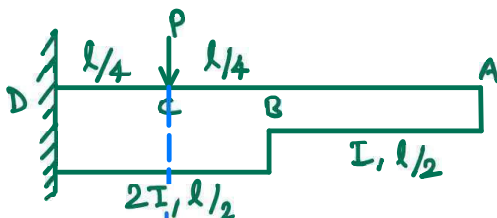
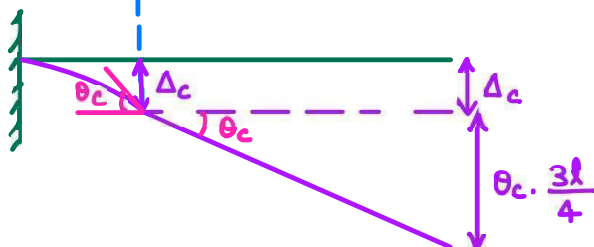
M. >

find θ_A , Δ_A .

$$\begin{aligned} \Delta_A &= \left[\frac{P(l/2)^3}{3E(2I)} + \frac{P \cdot \frac{l}{2} (\frac{l}{2})^2}{2E(2I)} \right] + \left[\frac{P(l/2)^2}{2E(2I)} + \frac{P \cdot \frac{l}{2} (l/2)}{E(2I)} \right] \frac{l}{2} + \frac{P(l/2)^3}{3E(I)} \\ &= \frac{3}{16} \frac{Pl^3}{EI} \end{aligned}$$

$$\theta_A = \frac{P(l/2)^2}{2E(2I)} + \frac{P \cdot \frac{l}{2} (l/2)}{E(2I)} + \frac{P(l/2)^2}{2EI} = \frac{5}{16} \frac{Pl^2}{EI}$$

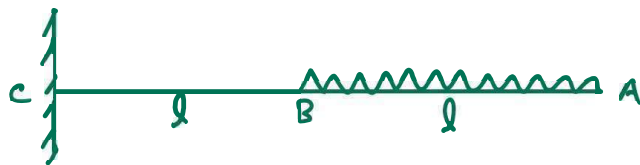
N. >

 θ_A , Δ_A = ?

$$\Delta_A = \frac{P(l/4)^3}{3E(2I)} + \frac{P(l/4)^2}{2E(2I)} \cdot \frac{3}{4}l = \frac{11}{768} \frac{Pl^3}{EI}$$

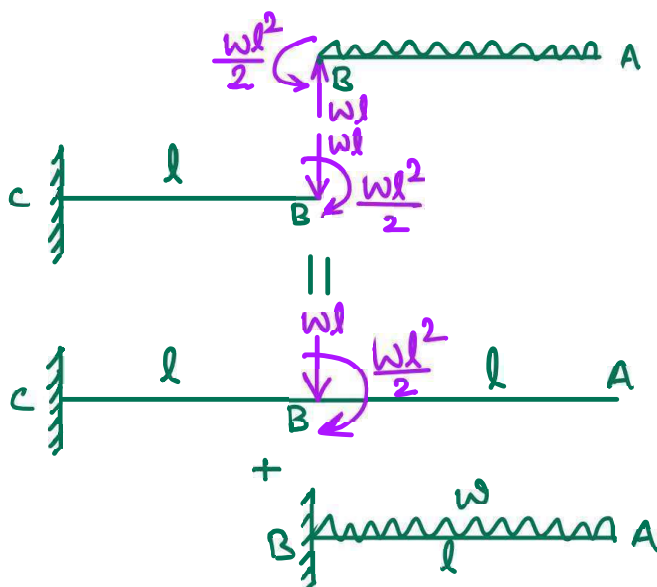
$$\theta_A = \theta_c = \frac{P(l/4)^2}{2E(2I)} = \frac{Pl^2}{64EI}$$

Q. >

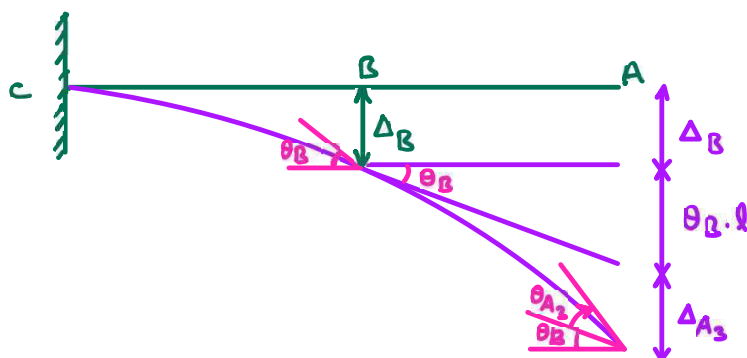


find Δ_A , θ_A ?

Soln



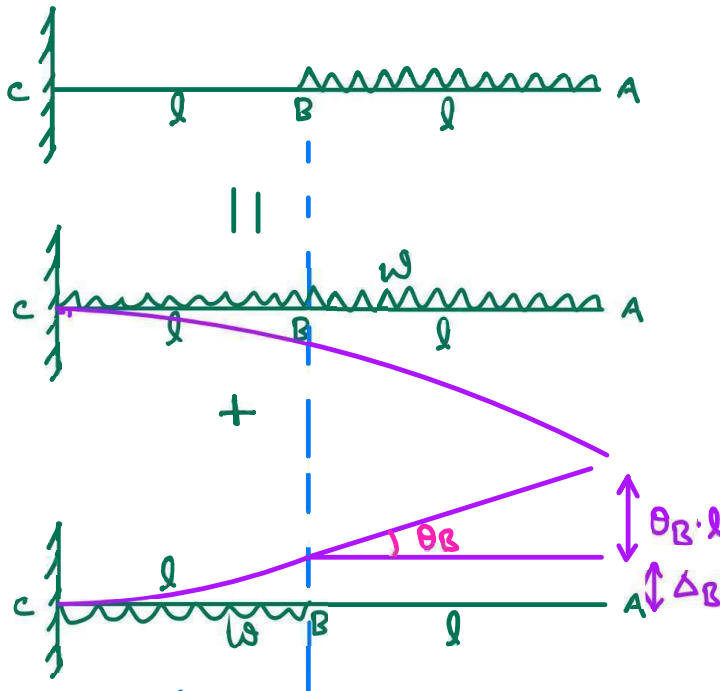
1st Approach.



$$\Delta_A = \left[\frac{wl \cdot l^3}{3EI} + \frac{\frac{wl^2}{2} \cdot l^2}{2EI} \right] + \left[\frac{wl \cdot l^2}{2EI} + \frac{\frac{wl^2}{2} \cdot l}{EI} \right] l + \frac{wl^4}{8EI} = \frac{41}{24} \frac{wl^4}{EI}$$

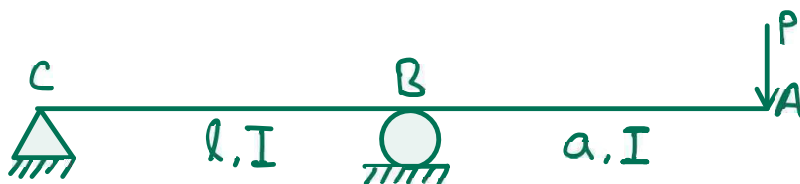
$$\theta_A = \frac{wl \cdot l^2}{2EI} + \frac{\frac{wl^2}{2} \cdot l}{EI} + \frac{wl^3}{6EI} = \frac{7}{6} \frac{wl^3}{EI}$$

2nd Approach. (Principle of superposition)

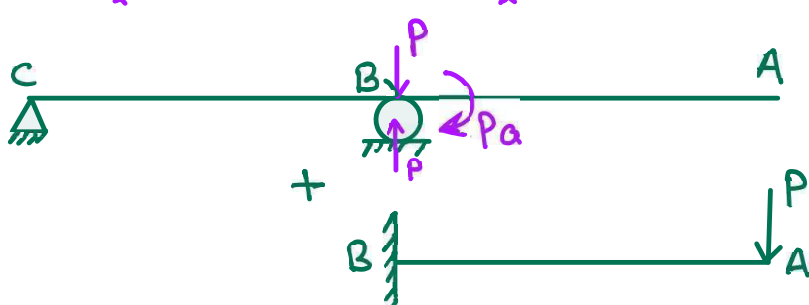
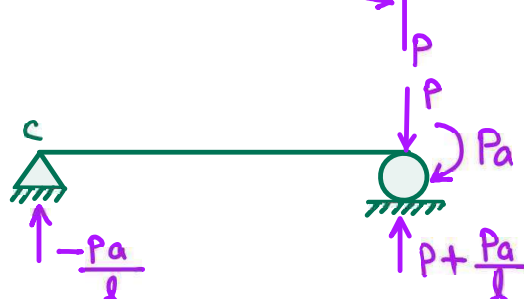
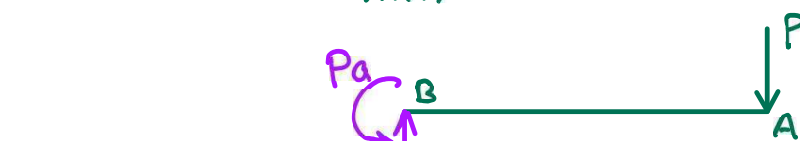


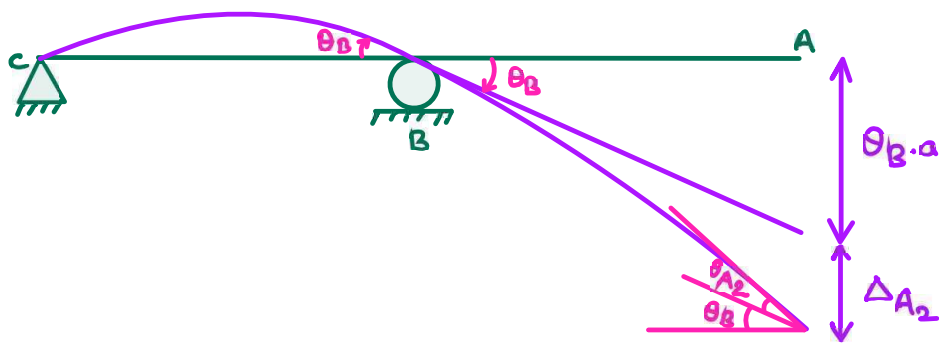
$$\Delta_A = \left| \frac{w(2l)^4}{8EI} - \left(\frac{wl^4}{8EI} - \frac{wl^3}{6EI} \cdot l \right) \right| = \frac{41}{24} \frac{wl^4}{EI}$$

$$\theta_A = \frac{w(2l)^3}{6EI} - \frac{wl^3}{6EI} = \frac{7}{6} \frac{wl^3}{EI}$$



find Δ_A, θ_A

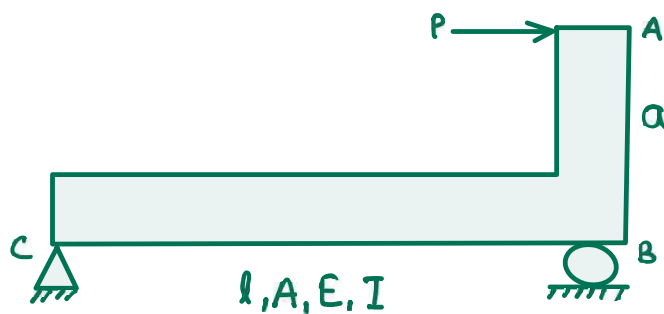




$$\Delta_A = \Delta_{A_1} + \Delta_{A_2} = \theta_B \cdot a + \Delta_{A_2} = \frac{Pa \cdot l \cdot a}{3EI} + \frac{Pa^3}{3EI} = \frac{Pa^2}{3EI} (l+a)$$

$$\theta_A = \theta_B + \theta_{A_2} = \frac{Pal}{3EI} + \frac{Pa^2}{2EI}$$

Q. >



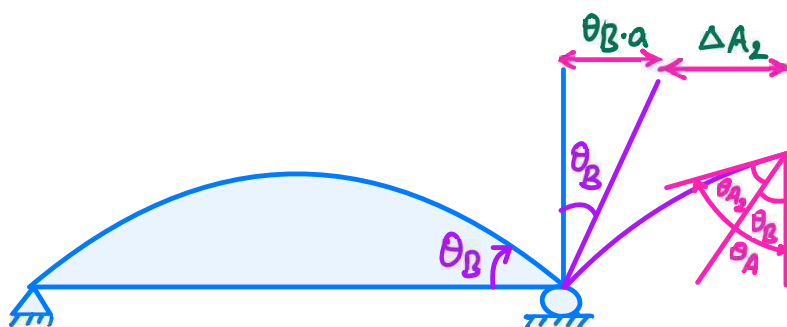
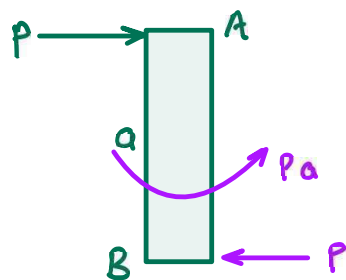
Assume $EI = \text{constant}$

find Δ_A, θ_A ?

i) if CB is Axially Rigid

ii) if CB is Axially Flexible.

Case-1



$$\Delta A = \Delta A_1 + \Delta A_2 = \theta_B \cdot a + \Delta A_2$$

$$= \frac{Pa \cdot l \cdot a}{3EI} + \frac{Pa^2}{3EI}$$

$$\theta_A = \theta_B + \theta_{A_2}$$

$$= \frac{Pal}{3EI} + \frac{Pa^2}{2EI}$$

Case - (ii) CB is axially flexible

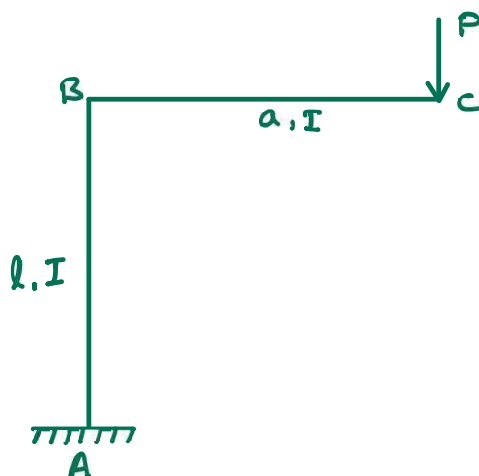
$$\Delta A = \Delta A_1 + \Delta A_2 + \Delta A_3$$

$$= \frac{Pa \cdot l \cdot a}{3EI} + \frac{Pa^3}{3EI} + \frac{PL}{AE}$$

$$\theta_A = \theta_B + \theta_{A_2}$$

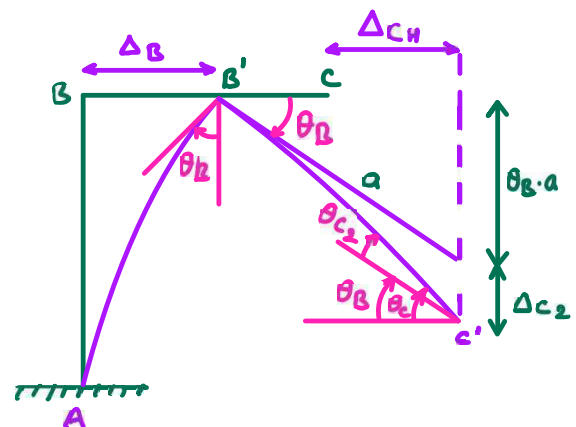
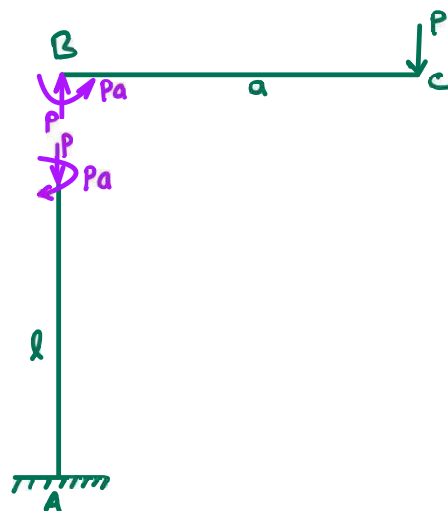
$$= \frac{Pal}{3EI} + \frac{Pa^2}{2EI}$$

R.7



Find Δ_{CH} , Δ_{CV} , θ_C .

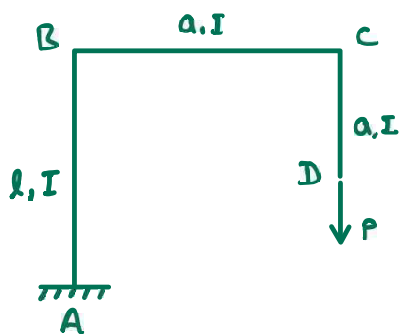
Solution



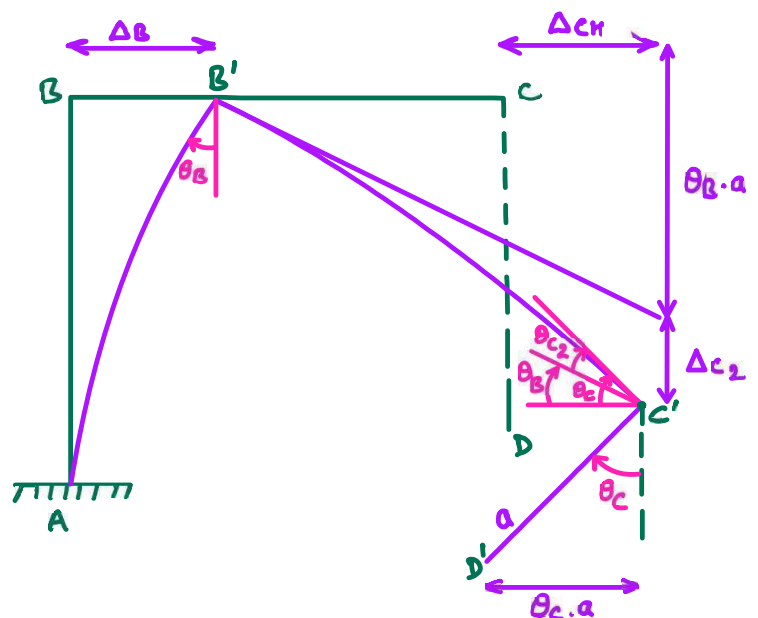
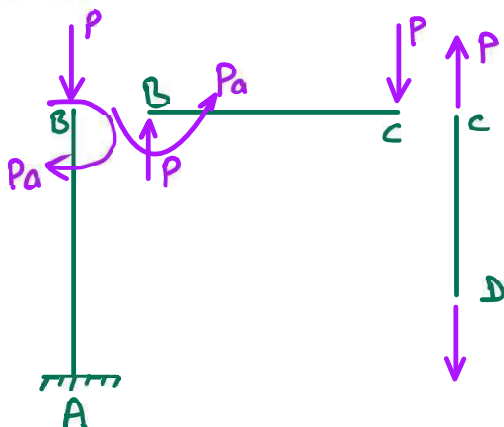
$$\Delta_{CH} = \Delta_B = \frac{Pa \cdot l^2}{2EI} \quad , \quad \theta_B = \frac{Pa \cdot l}{EI}$$

$$\Delta_{CV} = \Delta_{C1} + \Delta_{C2} = \theta_B \cdot a + \Delta_{C2} = \frac{Pa \cdot l}{EI} \cdot a + \frac{Pa^3}{3EI}$$

$$\theta_C = \theta_B + \theta_{C2} = \frac{Pa \cdot l}{EI} + \frac{Pa^2}{2EI}$$



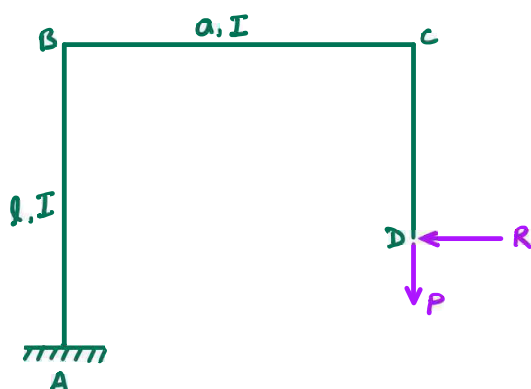
Solution



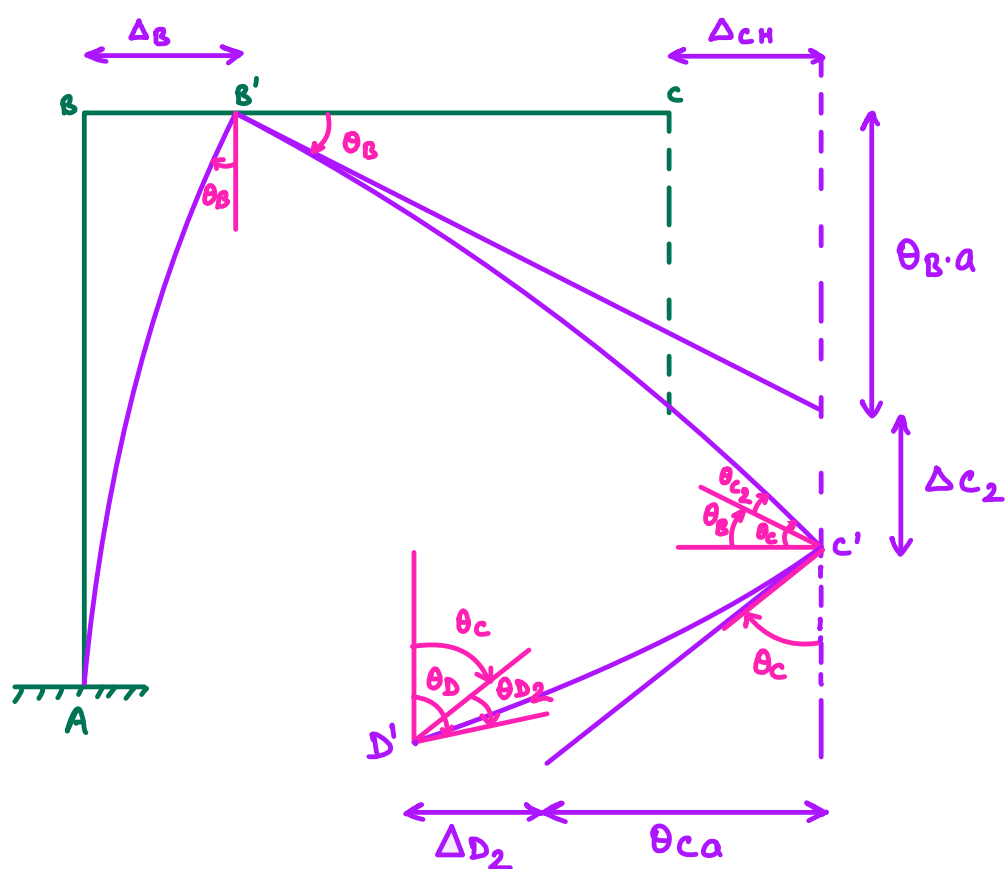
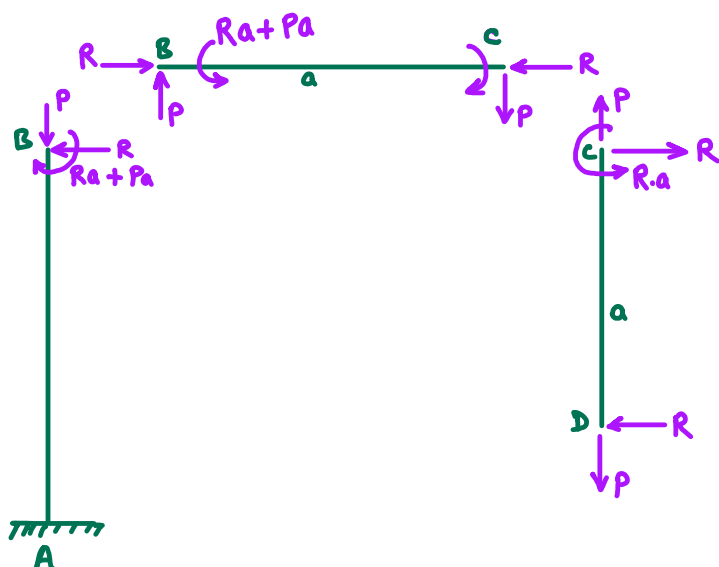
$$\Delta_{DH} = \Delta_{CH} - \theta_C a = \Delta_B - \theta_C a = \frac{Pa \cdot l^2}{2EI} - \left(\frac{Pa \cdot l}{EI} + \frac{Pa^2}{2EI} \right) a$$

$$\Delta_{DH} = \frac{Pa \cdot l^2}{2EI} - \frac{Pa^2 \cdot l}{EI} - \frac{Pa^3}{2EI}$$

$$\Delta_{DV} = \Delta_{CV} = \theta_B \cdot a + \Delta_{C2} = \frac{Pa \cdot l}{EI} \cdot a + \frac{Pa^3}{3EI}$$



find Δ_{DH} , Δ_{DV} , θ_D .



$$\Delta_{DH} = \Delta_{CH} - \theta_C a - \Delta_{D2} = \Delta_B - \theta_C a - \Delta_{D2}$$

$$\left[\frac{(R+P)a \cdot l^2}{2EI} - \frac{Rl^3}{3EI} \right] - \left[\left\{ \frac{(R+P)al}{EI} - \frac{Rl^2}{2EI} \right\} + \left\{ \frac{Ra}{EI} + \frac{Pa^2}{2EI} \right\} a - \frac{Ra^3}{3EI} \right]$$

Δ_B
 θ_B
 θ_{C2}
 Δ_{D2}

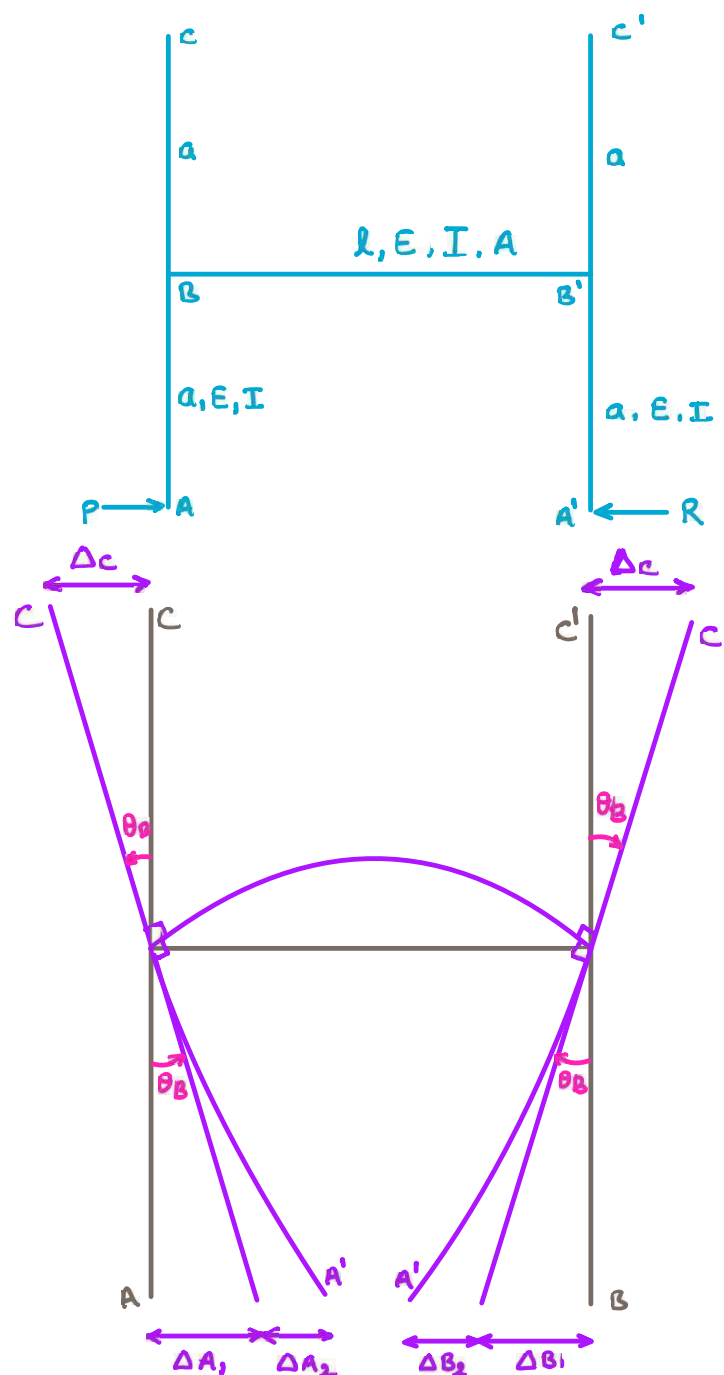
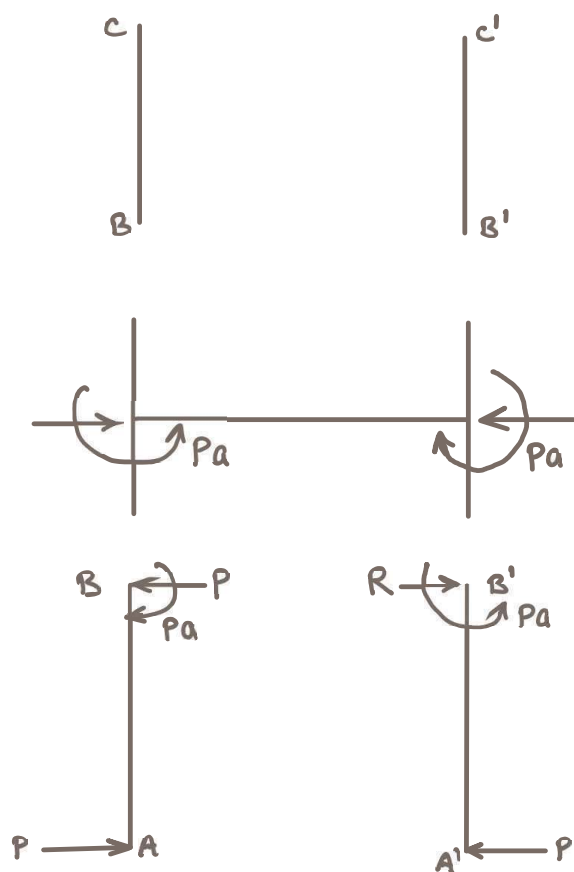
$$\theta_D = \theta_C + \theta_{D_2} = \theta_B + \theta_{C_2} + \theta_{D_2} = (\theta_B + \theta_{C_2}) + \frac{Ra^2}{2EI}$$

$$\Delta D_V = \Delta_C = \theta_B \cdot a + \Delta_{C_2} = \left\{ \frac{(R+P)al}{EI} - \frac{Rl^2}{2EI} \right\} a + \left\{ \frac{Ra \cdot a^2}{2EI} + \frac{Pa^3}{3EI} \right\}$$

Q. > Find the distance by which cc' moves apart and distance by which AA' moves closer if

- BB' is axially rigid.
- BB' is axially flexible.

Solution.



$$\begin{matrix} Pa \\ \curvearrowright \end{matrix} \begin{matrix} \theta_B \\ \curvearrowright \end{matrix} \begin{matrix} \theta_{B'} \\ \curvearrowright \end{matrix} \begin{matrix} Pa \\ \curvearrowright \end{matrix} = \begin{matrix} Pa \\ \curvearrowright \end{matrix} \begin{matrix} \theta_{B_1} \\ \curvearrowright \end{matrix} \begin{matrix} \theta_{B'_1} \\ \curvearrowright \end{matrix} \begin{matrix} Pa \\ \curvearrowright \end{matrix} + \begin{matrix} Pa \\ \curvearrowright \end{matrix} \begin{matrix} \theta_{B_2} \\ \curvearrowright \end{matrix} \begin{matrix} \theta_{B'_2} \\ \curvearrowright \end{matrix} \begin{matrix} Pa \\ \curvearrowright \end{matrix}$$

$$\theta_B = \theta'_B$$

$$\theta_B = \theta_{B_1} + \theta_{B_2} = \frac{Pa \cdot l}{3EI} + \frac{Pa \cdot l}{6EI} = \frac{Pa \cdot l}{2EI}$$

$$\Delta_c = \theta_B \cdot a = \frac{Pa \cdot l}{2EI} \cdot a$$

$$\Delta_A = \Delta_{A_1} + \Delta_{A_2} = \theta_B \cdot a + \Delta_{A_2} = \frac{Pa \cdot l}{2EI} \cdot a + \frac{Pa^3}{3EI}$$

a) when BB' is axially rigid.

$$\begin{aligned} \text{Distance moved by } CC' &= 2 \times \Delta_c \\ &= 2 \times \frac{Pa^2 l}{EI} \end{aligned}$$

$$\text{Distance moved by } AA' = 2 \Delta_A = \frac{Pa^2 l}{EI} + \frac{2}{3} \frac{Pa^3}{EI}$$

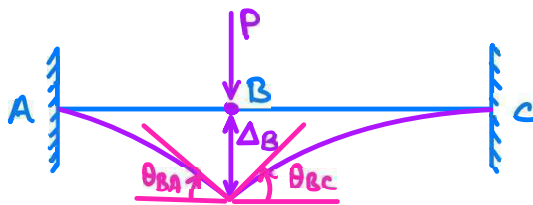
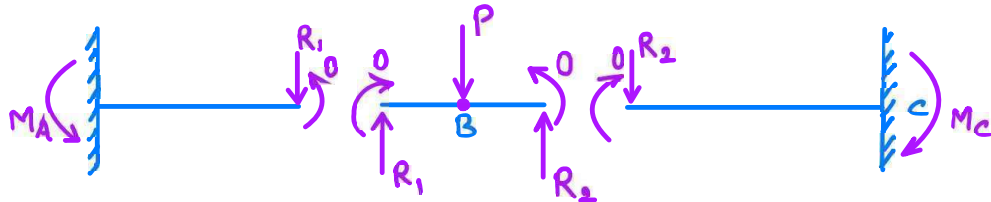
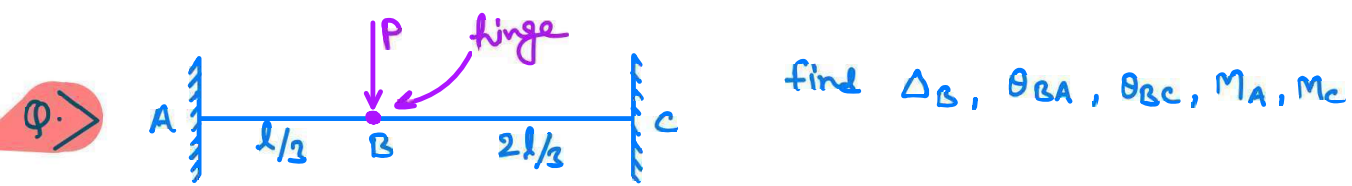
b) when BB' is axially flexible

$$\text{Distance moved by } CC' = 2 \Delta_c + \frac{PL}{AE} = \frac{Pa^2 l}{EI} + \frac{PL}{AE}$$

$$\text{Distance moved by } AA' = 2 \Delta_A + \frac{PL}{AE} = \frac{Pa^2 l}{EI} + \frac{2}{3} \frac{Pa^3}{EI} + \frac{PL}{AE}$$

Contraction of BB'

Statically Indeterminate cases:-



$$\Delta_1 = \Delta_2 = \Delta_B$$

$$\frac{R_1 (l/3)^3}{3EI} = \frac{R_2 (2l/3)^3}{3EI}$$

$$a) \Delta_1 = \Delta_2 = \Delta_B = \frac{\frac{8P}{9} (l/3)^3}{3EI} = \frac{8}{729} \frac{Pl^3}{EI}$$

$$R_1 = 8 R_2 \quad \text{--- (i)}$$

$$b) \theta_{BA} = \frac{R_1 (l/3)^2}{2EI} = \frac{8P (l/3)^2}{9 \cdot 2EI} = \frac{4}{81} \frac{Pl^2}{EI}$$

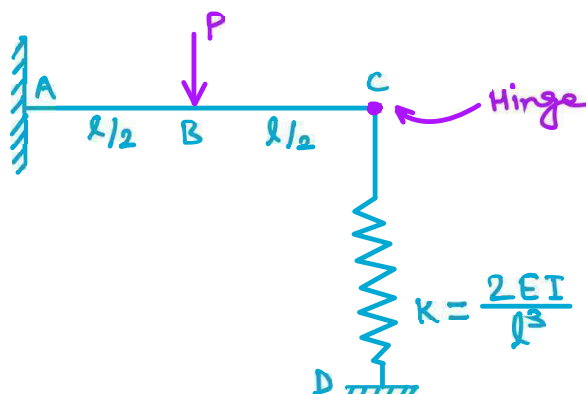
$$R_1 + R_2 = P \quad \text{--- (ii)}$$

$$c) \theta_{BC} = \frac{R_2 (2l/3)^2}{2EI} = \frac{P (2l/3)^2}{9 \cdot 2EI} = \frac{2}{81} \frac{Pl^2}{EI}$$

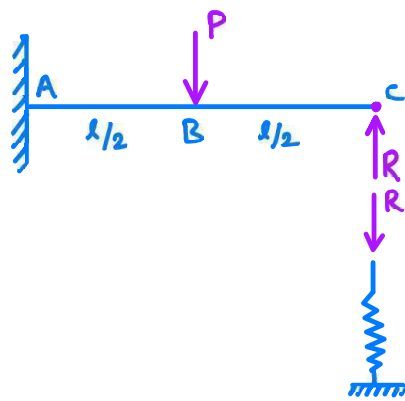
$$R_1 = \frac{8}{9} P, \quad R_2 = \frac{1}{9} P$$

$$d) M_A = R_1 \frac{l}{3} = \frac{8}{9} P \frac{l}{3} = \frac{8Pl}{27}$$

$$e) M_C = R_2 \frac{2l}{3} = \frac{P}{9} \cdot \frac{2l}{3} = \frac{2Pl}{27}$$



Compute the force in the spring.



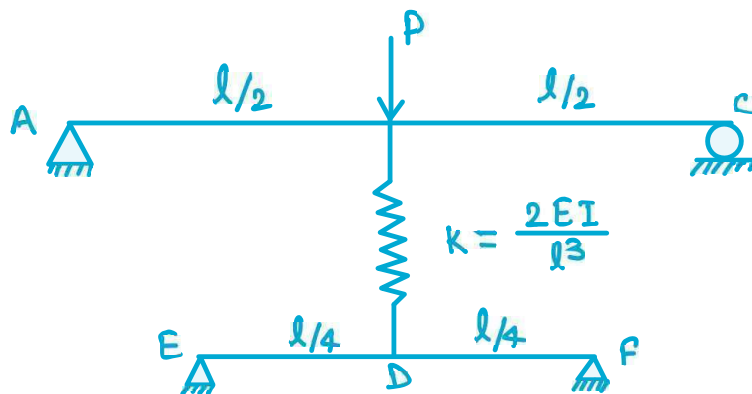
$$\Delta_c = \frac{P(l/2)^3}{3EI} + \frac{P(l/2)^2}{2EI} \cdot l/2 - \frac{Rl^3}{3EI} \quad \text{--- (i)}$$

$$\Delta_{\text{Spring}} = \frac{R}{k} = \frac{Rl^3}{2EI} \quad \text{--- (ii)}$$

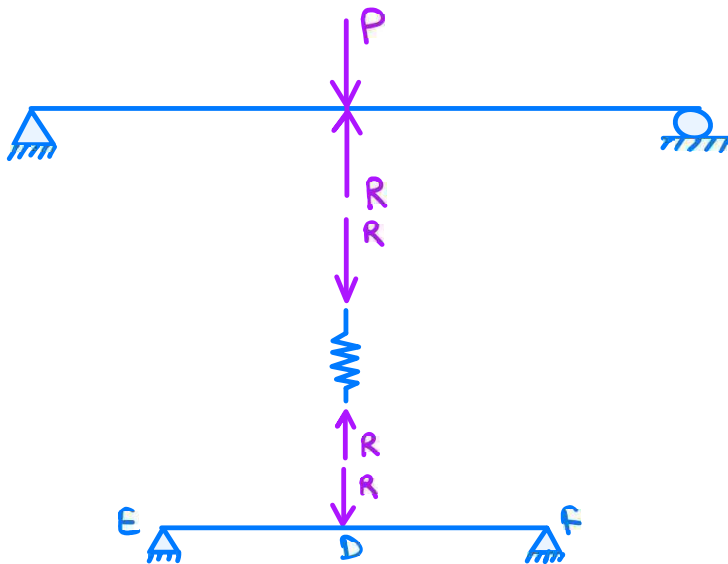
$$\Delta_c = \Delta_{\text{Spring}}$$

$$R = \frac{P}{8}$$

Φ>



Compute Δ_B & Stiffness of structure.



$$\Delta_B - \Delta_D = \Delta_{\text{Spring}}$$

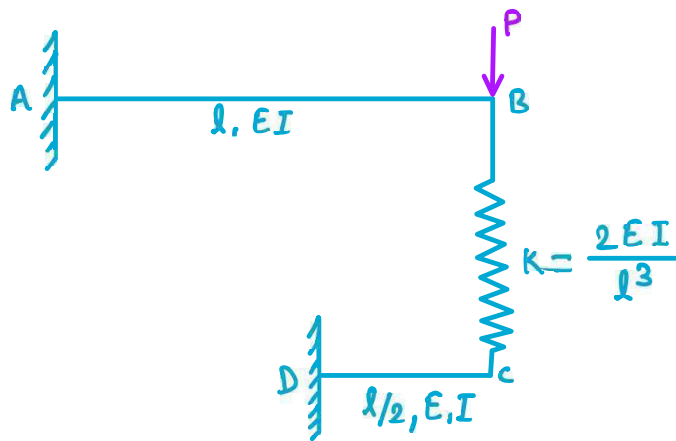
$$\frac{(P-R)l^2}{48EI} - \frac{R(l/2)^3}{48EI} = \frac{Rl^3}{2EI}$$

$$R = \frac{8}{201}P$$

$$\Delta_B = \frac{(P-R)l^3}{48EI} = \frac{193}{9648} \frac{Pl^3}{EI}$$

$$b) \text{ Stiffness of the structure} = \frac{P}{\Delta_B} = \frac{9648}{193} \frac{EI}{l^3}$$

Q. >

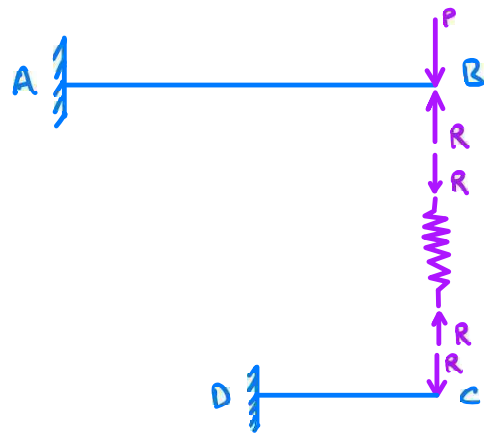
Compute Δ_B Solution:-

$$\Delta_B - \Delta_C = \Delta_{\text{spring}}$$

$$\frac{(P-R)l^3}{3EI} - \frac{R(l/2)^3}{3EI} = \frac{Rl^3}{2EI}$$

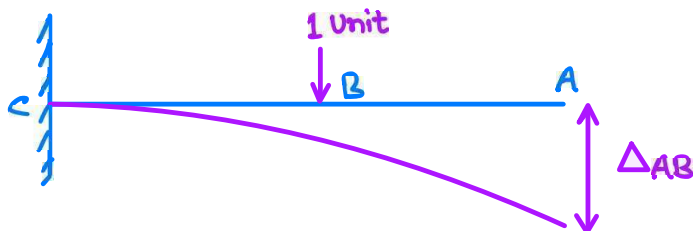
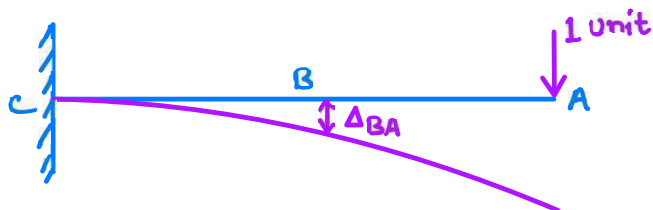
$$\Rightarrow R = \frac{8}{21}P$$

$$\Delta_B = \frac{(P-R)l^3}{3EI} = \frac{13}{63} \frac{Pl^3}{EI}$$



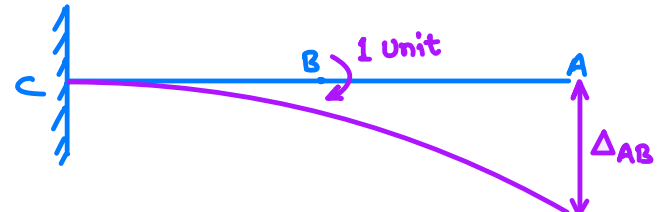
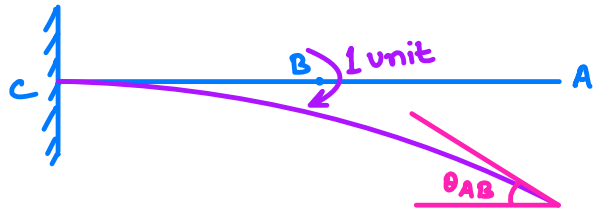
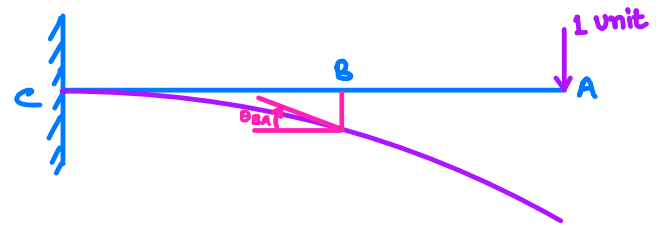
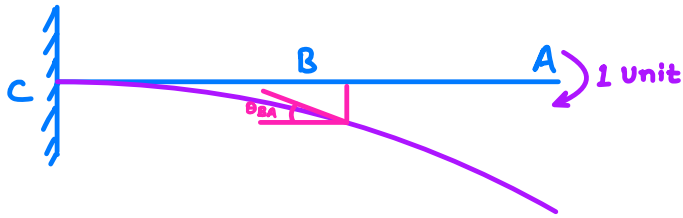
Maxwell Reciprocal Theorem.

— It is based upon "virtual work principle."

Ex:-

$$1 \cdot \Delta_{BA} = 1 \cdot \Delta_{AB}$$

$$\Delta_{BA} = \Delta_{AB}$$



$$1. \theta_{BA} = 1 \cdot \theta_{AB}$$

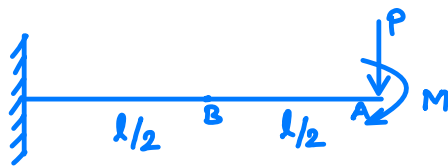
$$\theta_{BA} = \theta_{AB}.$$

$$1. \theta_{BA} = 1 \cdot \Delta_{AB}$$

$$\theta_{BA} = \Delta_{AB}.$$

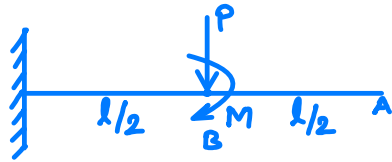
- In any beam, frame or truss, the deflection or slope at any point "A" due to load/moment at any point B is equal to deflection or slope at point B due to load/moment at point "A".

for example



$$\Delta_{BA} = \frac{P(l/2)^3}{3EI} + \frac{P \cdot l/2 \cdot (l/2)^2}{2EI} + \frac{M(l/2)^2}{2EI} = \frac{5}{48} \frac{Pl^3}{EI} + \frac{Ml^2}{8EI}$$

$$\theta_{BA} = \frac{P(l/2)^2}{2EI} + \frac{P \cdot l/2 \cdot l/2}{EI} + \frac{M \cdot l/2}{EI} = \frac{3}{8} \frac{Pl^2}{EI} + \frac{Ml}{2EI}$$



$$\Delta_{AB} = \left(\frac{P(l/2)^3}{3EI} + \frac{P(l/2)^2 \cdot l/2}{2EI} \right) + \left(\frac{M(l/2)^2}{2EI} + \frac{M(l/2) \cdot l/2}{EI} \right)$$

$$= \frac{5}{48} \frac{Pl^3}{EI} + \frac{Ml^2}{8EI}$$

$$\theta_{AB} = \frac{P(l/2)^2}{2EI} + \frac{M(l/2)}{EI} = \frac{Pl^2}{8EI} + \frac{Ml}{2EI}$$

Equate Magnitude of P & M to get result.

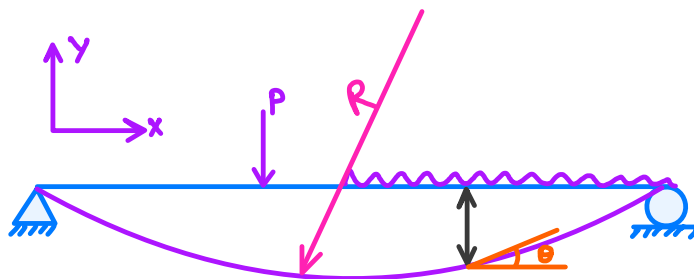
⇒ Deflection can also be computed by following methods :-

- a> Double integration Technique
- b> Macaulay's Method
- c> Area Moment method. (obj.)
- d> Conjugate Beam method. (obj.)
- e> Energy method

These methods give Slope and deflection due to BM only.

It can give Slope & deflection due to all types of internal forces.

a> Double Integration Method



Here, $\frac{1}{R} = \text{curvature}$

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}$$

$y = \text{deflection at any point}$

$\theta = \frac{dy}{dx} = \text{slope at any point.}$

Since $\frac{dy}{dx}$ is very small, $\frac{dy}{dx}$ & $\left(\frac{dy}{dx}\right)^2$ can be neglected.

$$\frac{1}{R} = \frac{d^2y}{dx^2} \text{ --- (A)}$$

from Bending formula,

$$\frac{M}{I} = \frac{E}{R} \Rightarrow \frac{1}{R} = \frac{M}{EI} \text{ --- (B)}$$

from (A) & (B)

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \text{ --- (i)}$$

By double integrating this equation, deflection at given point can be computed (thereby this method is termed as double integration method).

$$\frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{1}{EI} \frac{dM}{dx}$$

$$\frac{d^3y}{dx^3} = \frac{V}{EI} \text{ --- (ii)}$$

$$\frac{d^4y}{dx^4} = \frac{dV}{dx} \cdot \frac{1}{EI}$$

$$\frac{d^4y}{dx^4} = \frac{w}{EI} \text{ --- (iii)}$$

Note:-

- If the beam is having some discontinuity in the elastic deflected curve like internal hinges, this method is not applicable.
- If flexural rigidity (EI) or Moment changes from section to section then also this method is not recommended.
- It is recommended for complex loading system for which calculation of BM is difficult.

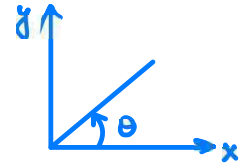
Sign convention

(a) $\left(\begin{array}{c} \uparrow \\ | \\ \downarrow \end{array}\right)$ (+) BM , $\left(\begin{array}{c} \downarrow \\ | \\ \uparrow \end{array}\right)$ (-)ve BM , $\uparrow \downarrow$ (+) SF , $\downarrow \uparrow$ (-) SF , $\uparrow$ (+) loading
 $\downarrow$ (-) loading.

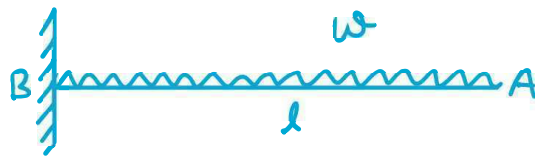
(b) If y is positive, deflection is upward & if y is (-)ve deflection is downward.

(c) If $\theta = \frac{dy}{dx}$ (+)ve. (Slope is anticlockwise)

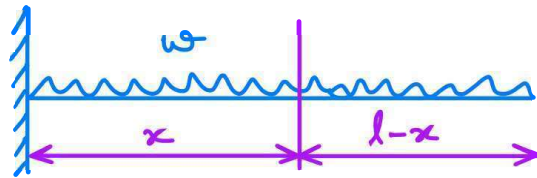
$\theta = \frac{dy}{dx}$ (-)ve (Slope is clockwise)



Q: Compute Δ_A, θ_A using double integration method.



Solution.



$$M_x = \frac{-w(l-x)^2}{2}$$

$$EI \frac{d^2y}{dx^2} = M_x = \frac{-w(l-x)^2}{2}$$

$$EI \frac{dy}{dx} = + \frac{w(l-x)^3}{6} + c_1$$

$$EI y = - \frac{w(l-x)^4}{24} + c_1 x + c_2$$

Apply known/Boundary conditions

$$\text{at } x=0, \frac{dy}{dx} = 0 \Rightarrow EI(0) = \frac{wl^3}{6} + c_1$$

$$c_1 = -\frac{wl^3}{6}$$

$$\text{at } x=0, y=0 \Rightarrow EI(0) = -\frac{wl^4}{24} + c_2$$

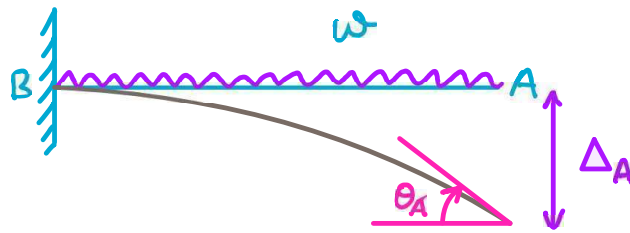
$$c_2 = \frac{wl^4}{24}$$

$$\frac{dy}{dx} = \frac{1}{EI} \left[\frac{w(l-x)^3}{6} - \frac{wl^3}{6} \right]$$

At point A, $x=l$, $\theta_A = \frac{dy}{dx} = -\frac{wl^3}{6EI}$

$$y = \frac{1}{EI} \left[-\frac{w(l-x)^4}{24} - \frac{wl^3}{6}x + \frac{wl^4}{24} \right]$$

At point A, $y=l$, $\Delta_A = \frac{1}{EI} \left[-\frac{wl^3}{6}l + \frac{wl^4}{24} \right] = \frac{-wl^4}{8EI}$

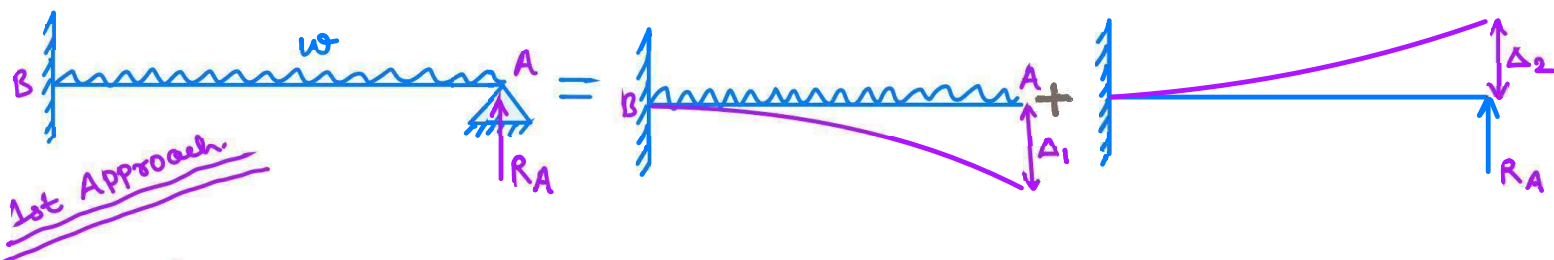


Q. >



Complete the reaction at propped end.

Solution.

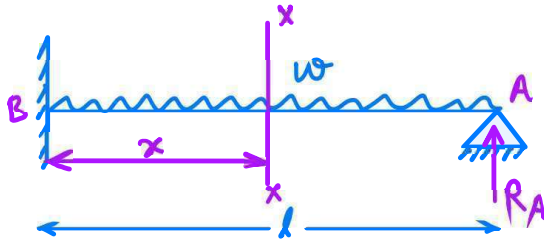


$$\Delta_A = 0 \text{ (using compatibility equation)}$$

$$\Delta_A = \Delta_1 - \Delta_2 = \frac{wl^4}{8EI} - \frac{R_A l^3}{3EI} = 0$$

$$R_A = \frac{3}{8} wl$$

2nd Approach (Double integration)



$$EI \frac{d^2 y}{dx^2} = M_x = R_A(l-x) - \frac{w(l-x)^2}{2}$$

$$EI \frac{dy}{dx} = -R_A \frac{(l-x)^2}{2} - \frac{w(l-x)^3}{3 \cdot 2} (-1) + C_1$$

$$EI y = +R_A \frac{(l-x)^3}{3 \cdot 2} - \frac{w(l-x)^4}{3 \cdot 2 \cdot 4} + C_1 x + C_2$$

$$\text{At } x=0, \frac{dy}{dx}=0 \Rightarrow EI(0) = -R_A \frac{l^2}{2} + \frac{wl^3}{6} + C_1$$

$$C_1 = R_A \frac{l^2}{2} - \frac{wl^3}{6} \quad \text{--- (I)}$$

$$\text{At } x=0, y=0 \Rightarrow EI(0) = \frac{R_A l^3}{6} - \frac{wl^4}{24} + 0 + C_2$$

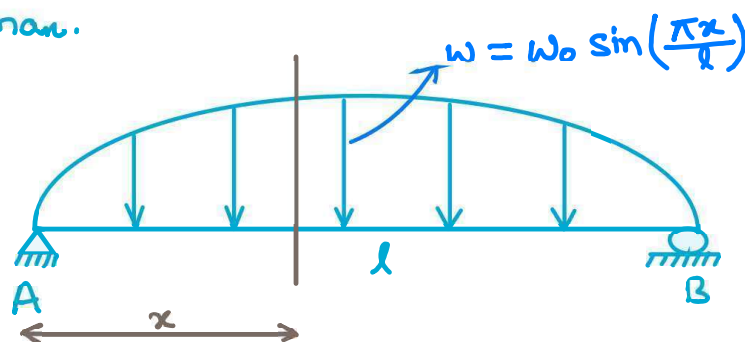
$$C_2 = \frac{wl^4}{24} - \frac{R_A l^3}{6} \quad \text{--- (II)}$$

$$\text{At } x=l, y=0 \Rightarrow EI(0) = 0 - 0 + C_1 l + C_2$$

$$C_2 = -C_1 l \quad \text{--- (III)}$$

$$\frac{wl^4}{24} - \frac{R_A l^3}{6} = -\left(R_A \frac{l^2}{2} - \frac{wl^3}{6}\right)l \Rightarrow \underline{\underline{R_A = \frac{3}{8}wl}}$$

Q. Compute $S_{\max}$.



Solution

Here Δ_{max} would be at $x = \frac{l}{2}$ (due to symmetric loading)

$$EI \frac{d^2 y}{dx^2} = M_x \quad (\text{Difficult to compute})$$

$$\text{Hence, } EI \frac{d^4 y}{dx^4} = w_x = -w_0 \sin\left(\frac{\pi x}{l}\right)$$

$$EI \frac{d^3 y}{dx^3} = w_0 \frac{l}{\pi} \cos\left(\frac{\pi x}{l}\right) + c_1$$

$$EI \frac{d^2 y}{dx^2} = w_0 \frac{l^2}{\pi^2} \sin\left(\frac{\pi x}{l}\right) + c_1 x + c_2 = M_x$$

$$\text{at } x=0, M_x=0 \Rightarrow 0 = 0 + c_1(0) + c_2 \Rightarrow c_2=0$$

$$\text{at } x=l, M_x=0 \Rightarrow 0 = 0 + c_1(l) + 0 \Rightarrow c_1=0$$

$$EI \frac{dy}{dx} = w_0 \frac{l^3}{\pi^3} \cos\left(\frac{\pi x}{l}\right) + c_3$$

$$EI y = -w_0 \frac{l^4}{\pi^4} \sin\left(\frac{\pi x}{l}\right) + c_3 x + c_4$$

$$\text{At } x=0, y=0 \Rightarrow EI(0) = 0 + c_3(0) + c_4 \Rightarrow c_4=0$$

$$\text{At } x=l, y=0 \Rightarrow EI(0) = 0 + c_3(l) + 0 \Rightarrow c_3=0$$

$$y = \frac{1}{EI} \left[-w_0 \frac{l^4}{\pi^4} \sin \frac{\pi x}{l} \right]$$

$$\text{for } \Delta_{max}, \quad x = \frac{l}{2},$$

$$\Delta_{max} = -\frac{w_0 l^4}{EI \pi^4} \sin\left(\frac{\pi}{l} \cdot \frac{l}{2}\right)$$

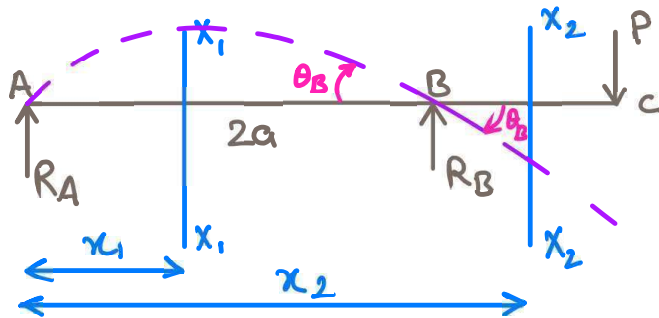
$$\Delta_{max} = \frac{-w_0 l^4}{EI \pi^4} \quad ***$$

Q.7

Compute Δ_c .



Soln



$$\sum f_x = 0$$

$$R_A + R_B = P \quad \text{--- (i)}$$

$$M_A = 0$$

$$R_B \times 2a = P \times 3a.$$

$$R_B = 3/2 P, \quad R_A = -P/2.$$

$$EI \frac{d^2 y_1}{dx_1^2} = R_A x_1 = -\frac{P}{2} x_1 \quad 0 \leq x_1 \leq 2a.$$

$$EI \frac{dy_1}{dx_1} = -\frac{P}{2} \frac{x_1^2}{2} + C_1$$

$$EI y_1 = -\frac{P \cdot x_1^3}{12} + C_1 x_1 + C_2.$$

$$x_1 = 0, \quad y_1 = 0 \Rightarrow 0 = 0 + 0 + C_2 \Rightarrow C_2 = 0$$

$$x_1 = 2a, \quad y_1 = 0 \Rightarrow 0 = -\frac{P(2a)^3}{12} + C_1(2a) + 0$$

$$C_1 = \frac{Pa^2}{3}.$$

$$EI y_1 = -\frac{Px_1^3}{12} + \frac{Pa^2}{3} x_1 \quad \text{--- (i)}$$

$$EI \frac{d^2 y_2}{dx_2^2} = -P(3a - x_2)$$

$$EI \frac{dy_2}{dx_2} = -\frac{P(3a-x_2)^2}{2}(-1) + C_3$$

$$EI y_2 = -\frac{P(3a-x_2)^3}{6} + C_3 x_2 + C_4$$

$$\text{At } x_2 = 2a, y_2 = 0 \Rightarrow 0 = -\frac{P(3a-2a)^3}{6} + C_3(2a) + C_4$$

$$2a C_3 + C_4 = \frac{Pa^3}{6} \quad \text{————— (A)}$$

$$\text{at } x_1 = x_2 = 2a, \frac{dy_1}{dx_1} = \frac{dy_2}{dx_2}$$

$$\frac{1}{EI} \left[-\frac{P(2a)^2}{4} + \frac{Pa^2}{3} \right] = \frac{1}{EI} = \left[-\frac{Pa^2}{2} + C_3 \right]$$

$$C_3 = -\frac{7}{6} Pa^2 \quad \text{————— (B)}$$

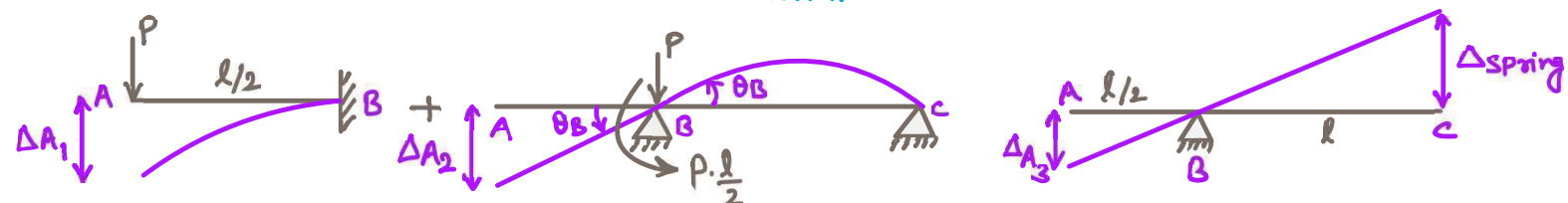
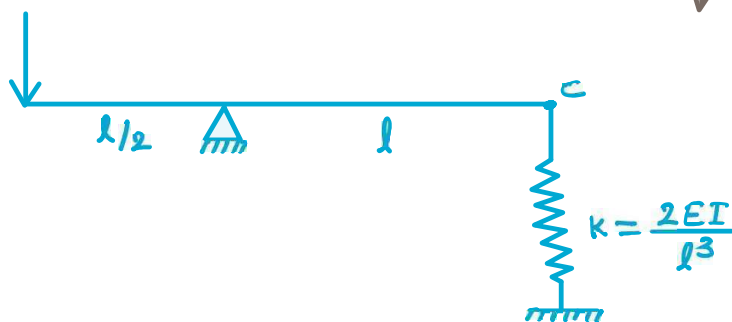
from (A) & (B)

$$C_4 = \frac{5}{2} Pa^3$$

$$EI y_2 = -\frac{P(3a-x_2)^3}{6} - \frac{7}{6} Pa^2 x_2 + \frac{5}{2} Pa^3$$

$$\text{from } \Delta_c, x_2 = 3a \Rightarrow \Delta_c = \frac{1}{EI} \left[-\frac{P(3a-3a)^3}{6} - \frac{7}{6} Pa^2(3a) + \frac{5}{2} Pa^3 \right]$$

$$\Delta_c = -\frac{Pa^3}{EI} \downarrow$$

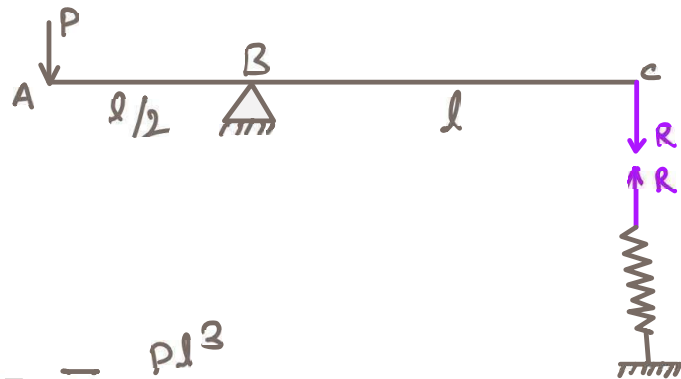


$$\Delta_A = \Delta A_1 + \Delta A_2 + \Delta A_3.$$

$$\Delta A_1 = \frac{P(l/2)^3}{3EI} = \frac{Pl^3}{24EI}$$

$$\Delta A_2 = \frac{(P \cdot l/2) \cdot l \cdot l/2}{3EI} = \frac{Pl^3}{12EI}$$

$$\frac{\Delta A_3}{l/2} = \frac{\Delta_{SP}}{l} \Rightarrow \Delta A_3 = \frac{\Delta_{SP} \cdot l}{2}$$



$$M_B = 0$$

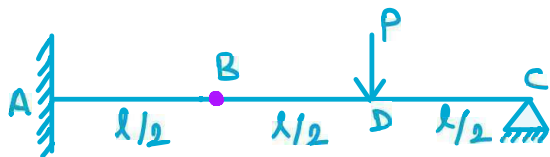
$$\Rightarrow P \cdot \frac{l}{2} = R \cdot l$$

$$R = P/2$$

$$\Delta_{SP} = R/k = P/2 \cdot \frac{l^3}{2EI} = \frac{Pl^3}{4EI}$$

$$\Delta A = \frac{Pl^3}{24EI} + \frac{Pl^3}{12EI} + \frac{Pl^3}{4EI} \cdot \frac{1}{2} = \frac{Pl^3}{4EI}$$

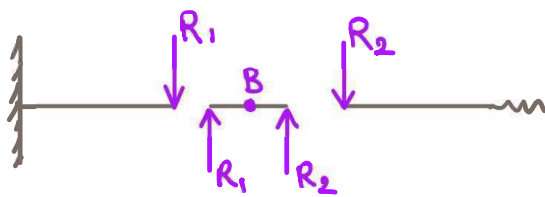
Q.7



Compute Δ_B & Δ_D

Solution

$$M_B = 0 \Rightarrow P \cdot \frac{l}{2} = R_C l \Rightarrow R_C = \frac{P}{2}$$

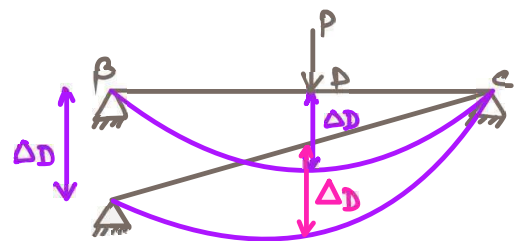


$$R_1 + R_2 = 0$$

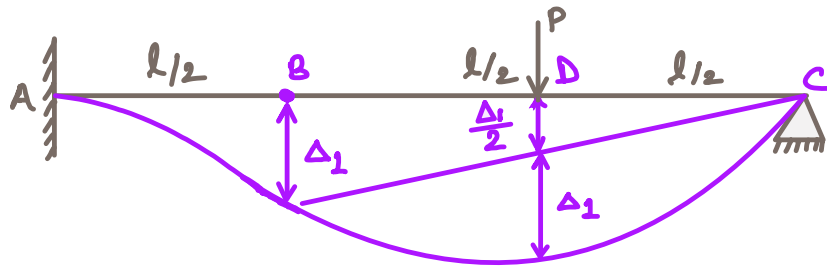
$$\frac{R_1 (l/2)^3}{3EI} = \Delta_B = \frac{Pl^3}{48EI}$$

$$= R_1 = \frac{P}{2}$$

$$R_2 = -\frac{P}{2}$$



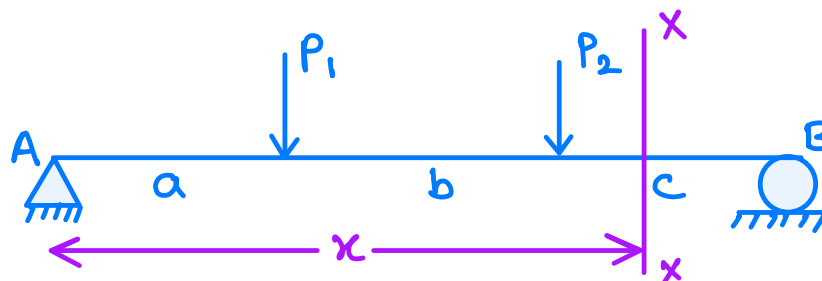
$$\Delta_B = \Delta_D = \frac{Pl^3}{48EI}$$



$$\Delta_D = \Delta_1 + \frac{\Delta_1}{2} = \frac{3\Delta_1}{2} = \frac{3}{2} \frac{Pl^3}{48EI} = \frac{1}{32} \frac{Pl^3}{EI}$$

B> Macaulay's Method.

- When Bending moment equation changes along the length of the beam due to various load, a convenient method of writing the Bending moment for entire beam is used and termed as "Macaulay's method".



Here B.M equation is written for last segment but this is valid for entire span of the beam under the following condition i.e. if (expression < 0) it is taken zero (0).

$$EI \frac{d^2y}{dx^2} = M_x = R_A x - P_1(x-a) - P_2\{x-(a+b)\}$$

$$EI \frac{dy}{dx} = R_A \frac{x^2}{2} + C_1 - \frac{P_1(x-a)^2}{2} - P_2 \frac{\{x-(a+b)\}^2}{2}$$

$$EI y = RA \frac{x^3}{6} + C_1 x + C_2 - \frac{P_1(x-a)^3}{6} - \frac{P_2\{x-(a+b)\}^3}{6}$$

At $x=0$, $y=0$, $\Rightarrow 0 = 0 + C_1(0) + C_2 \Rightarrow C_2 = 0$

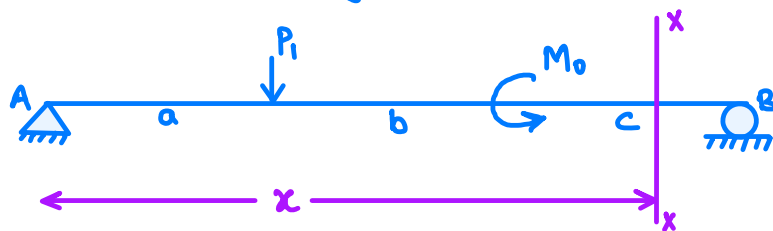
At $x=l$, $y=0 \Rightarrow 0 = RA \cdot \frac{l^3}{6} + C_1 l + 0 - \frac{P_1(l-a)^3}{6} - \frac{P_2(l-(a+b))^3}{6}$

$$C_1 =$$

Note:-

- Constant terms C_1 & C_2 should be included in the 1st term.
- Following modifications are to be carried out in the given cases while using Macaulay's method.

a> In case of Moment given :-

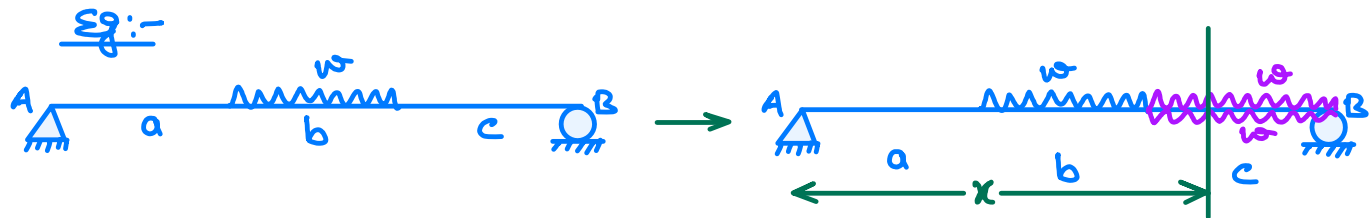


$$EI \frac{d^2 y}{dx^2} = M_x = RA \cdot x - P_1(x-a) - M_0\{x-(a+b)\}^0$$

$$EI \frac{dy}{dx} = RA \cdot \frac{x^2}{2} + C_1 - \frac{P_1(x-a)^2}{2} - \frac{M_0\{x-(a+b)\}^1}{1}$$

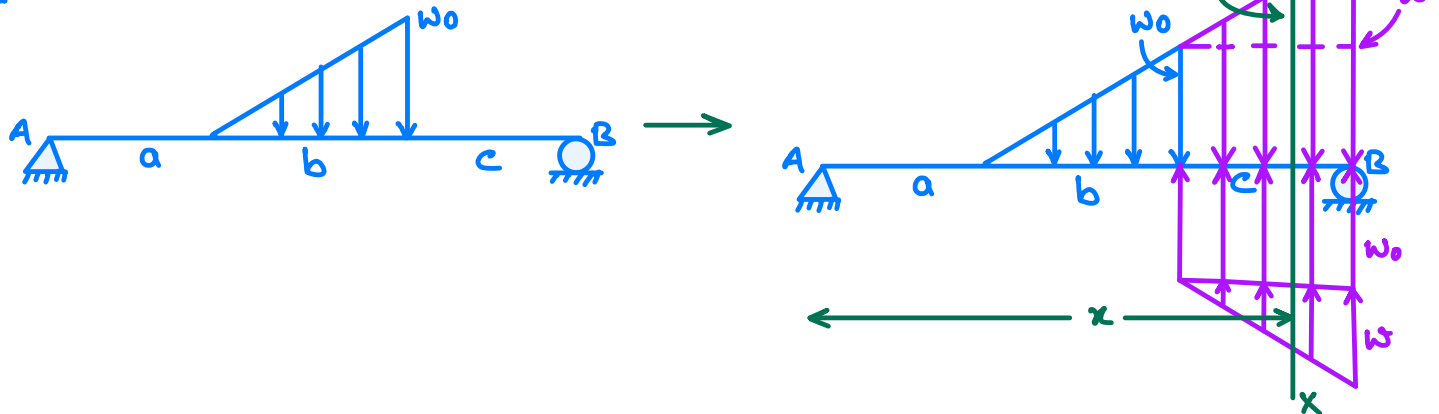
$$EI y = RA \cdot \frac{x^3}{6} + C_1 x + C_2 - \frac{P_1(x-a)^3}{3 \cdot 2} - \frac{M_0\{x-(a+b)\}^2}{2}$$

b> In case of Part UDL, UVL i.e. when UDL or UVL is not continuing upto the last segment, we need to modify the load so that the distributed loading becomes continuous upto the last segment.



$$EI \frac{d^2y}{dx^2} = M_x = R_A \cdot x - \frac{w(x-a)^2}{2} + \frac{w\{x-(a+b)\}^2}{2}$$

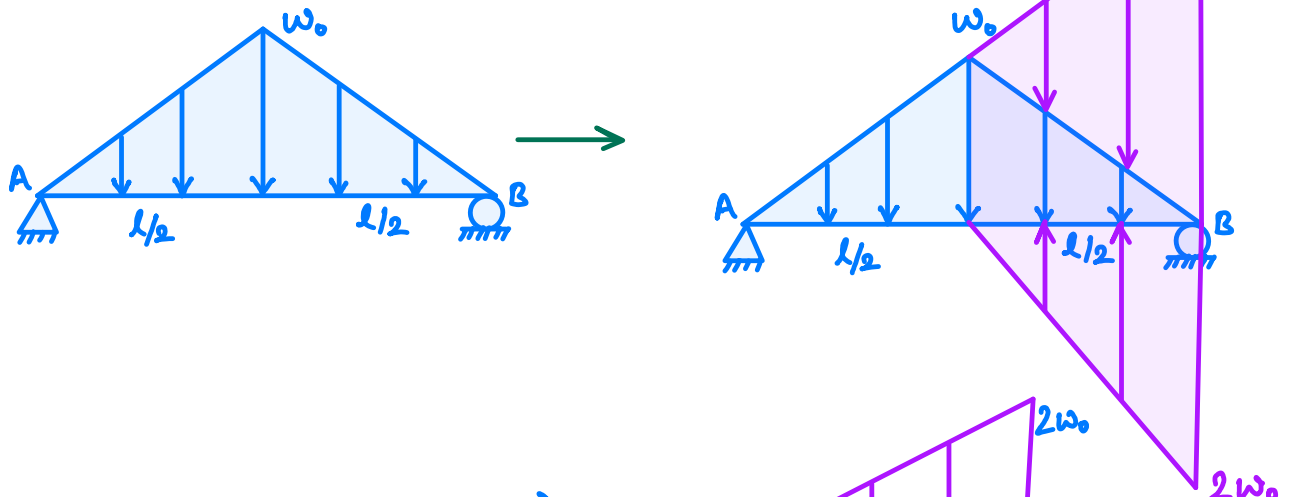
Ex:-



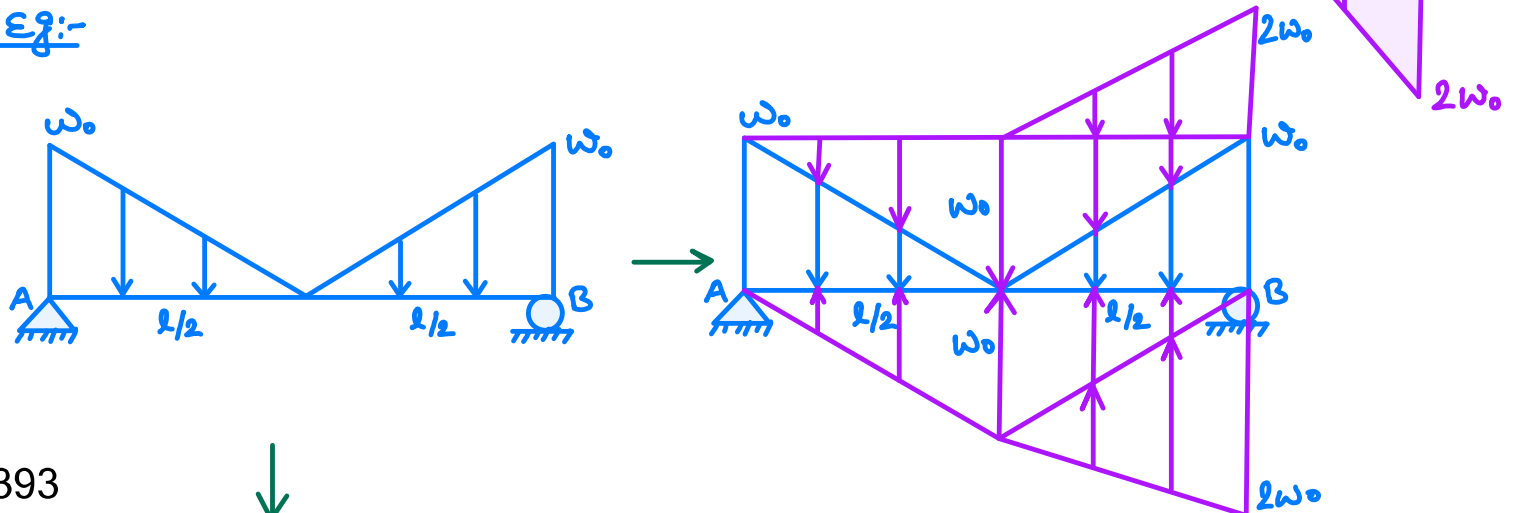
$$EI \frac{d^2y}{dx^2} = M_x = R_A \cdot x - \frac{1}{2}(x-a)(w_0 + w') \frac{(x-a)}{3} + \frac{w_0\{x-(a+b)\}^2}{2} + \frac{1}{2}\{x-\frac{a+b}{3}\}^2 w'$$

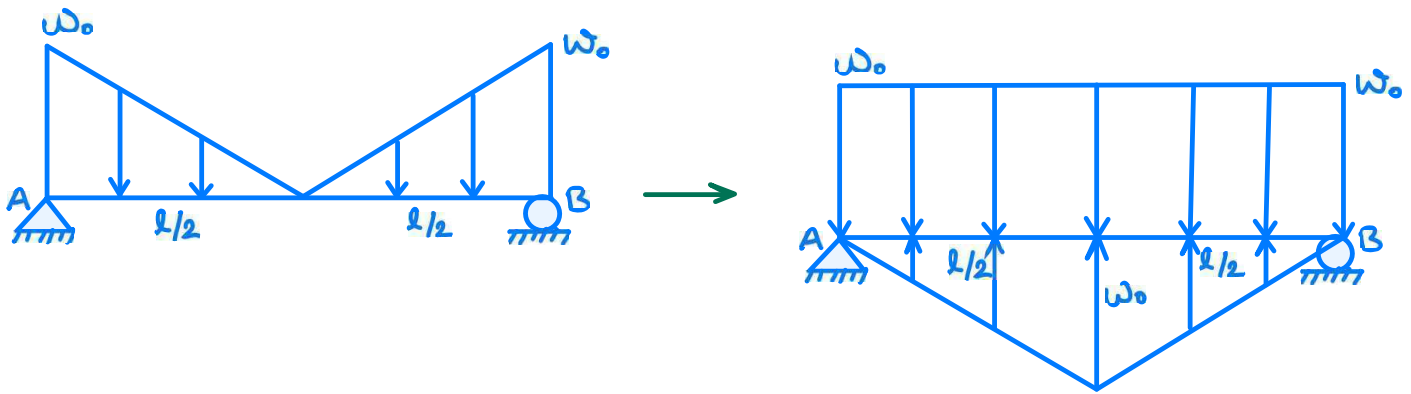
$$\frac{w'}{x-(a+b)} = \frac{w_0}{b} \Rightarrow w' = \frac{w_0\{x-(a+b)\}}{b}$$

Ex:-

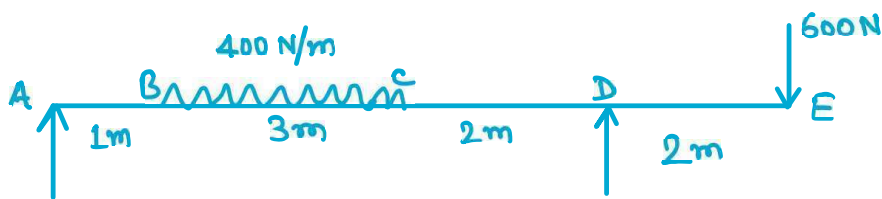
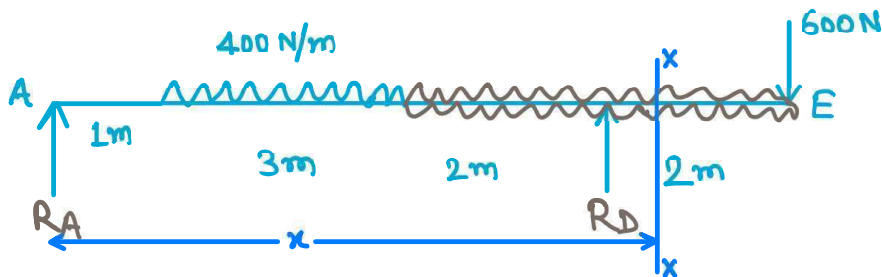


Ex:-



OR

Q> Compute the deflection at 5m from Support A.

Solution.

$$R_A + R_D = 400 \times 3 + 600 = 1800 \text{ N}$$

$$R_D \times 6 - 600 \times 8 - 400 \times 3 \times \left(\frac{3}{2} + 1\right) = 0$$

$$R_D = 1300 \text{ N}$$

$$R_A = 500 \text{ N.}$$

$$EI \frac{d^2 y}{dx^2} = M_x = 500x - \frac{400(x-1)^2}{2} + 400 \frac{(x-4)^2}{2} + 1300(x-6)$$

$$EI \frac{dy}{dx} = 500 \frac{x^2}{2} + C_1 - \frac{400(x-1)^3}{6} + \frac{400(x-4)^3}{6} + 1300 \frac{(x-6)^2}{2}$$

$$EI y = 500 \frac{x^3}{6} + C_1 x + C_2 - \frac{400(x-1)^4}{24} + \frac{400(x-4)^4}{24} + \frac{1300(x-6)^3}{6}$$

$$x=0, y=0 \Rightarrow 0 = 0 + C_1(0) + C_2 \Rightarrow C_2 = 0$$

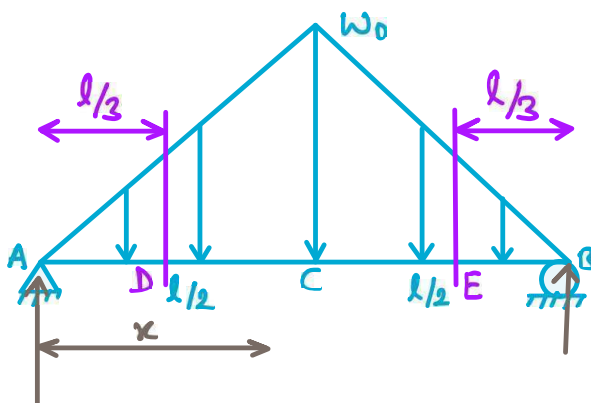
$$x=6, y=0 \Rightarrow 0 = 500 \cdot \frac{6^3}{6} + C_1(6) + 0 - 400 \frac{(6-1)^4}{24} + \frac{400(6-4)^4}{24}$$

$$C_1 = -1308.33$$

$$EI y = 500 \frac{x^3}{6} - 1308.33 x - \frac{400(x-1)^4}{24} + \frac{400(x-1)^4}{24} + 400 \frac{(x-4)^4}{24} + \frac{1300(x-6)^3}{6}$$

$$\text{at } x=5\text{m}, y_5 = 500 \cdot \frac{5^3}{6} - 1308.33 \times 5 - \frac{400(5-1)^4}{24} + \frac{400(5-4)^4}{24}$$

$$y_5 = \frac{-375}{EI} \downarrow$$



Complete $\Delta_D, \theta_E, \Delta_{max}$.

Solution $R_A = R_B = \frac{1}{2} \times l \times w_0 \times \frac{1}{2} = \frac{w_0 l}{4}$

$$EI \frac{d^2 y}{dx^2} = M_x = \frac{w_0 l}{4} x - \frac{1}{2} x \cdot \frac{w_0}{(l/2)} \cdot x \cdot \frac{x}{3} \quad (0 \leq x \leq \frac{l}{2})$$

$$EI \frac{dy}{dx} = \frac{w_0 l x^2}{8} - \frac{w_0 x^3}{12 l} + C_1$$

$$EI y = \frac{w_0 l x^3}{24} - \frac{w_0 x^4}{60 l} + C_1 x + C_2$$

$$x=0, y=0 \Rightarrow 0 = 0 + 0 + C_2 \Rightarrow C_2 = 0$$

$$x = l/2, \frac{dy}{dx} = 0 \Rightarrow 0 = \frac{w_0 l (l/2)^2}{8} - \frac{40 (l/2)^5}{12 l} + C_1$$

$$C_1 = \frac{-5}{192} w l^3$$

$$EI \frac{dy}{dx} = \frac{w_0 l x^2}{8} - \frac{w_0 x^4}{12 l} - \frac{5}{192} w l^3$$

$$\text{at pt D, } x = \frac{l}{3} \Rightarrow \theta_D = \frac{1}{EI} \left[\frac{w_0 l}{8} \left(\frac{l}{3} \right)^3 - \frac{w_0 (l/3)^4}{12 l} - \frac{5}{192} w l^3 \right]$$

$$\theta_D = \frac{-205/15552}{EI}$$

$$\theta_E = -\theta_D = \frac{205}{15552 EI}$$

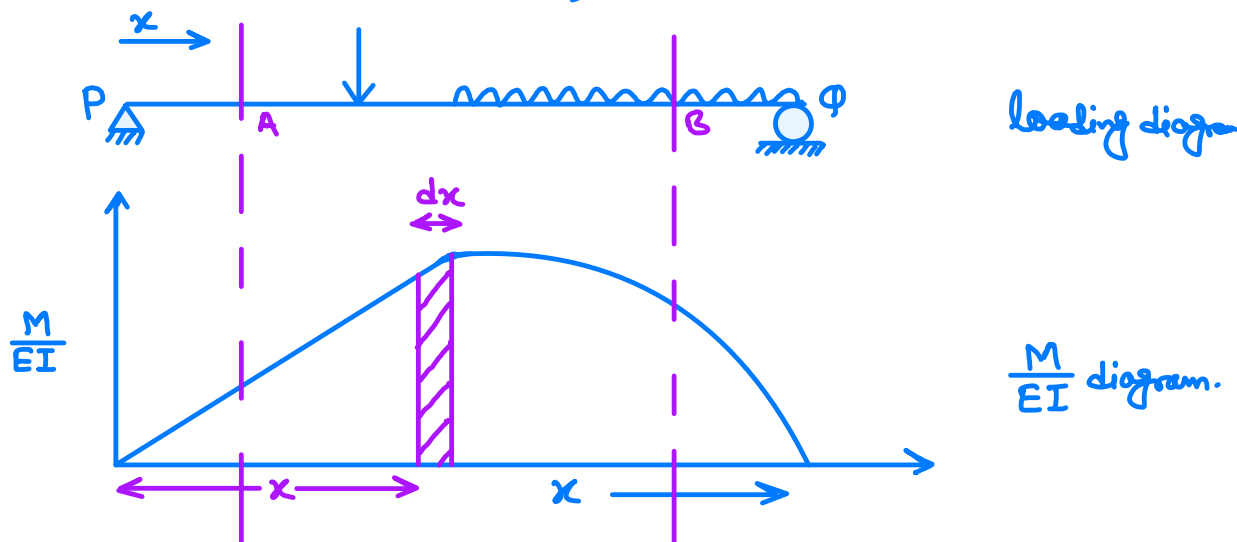
$$\text{At pt D, } x = \frac{l}{3}, \Delta_D = \frac{w_0 l \left(\frac{l}{3}\right)^3}{24} - \frac{w_0 \left(\frac{l}{3}\right)^5}{60 l} + \frac{-5}{192} w l^3 \cdot \frac{l}{3} + 0$$

$$\Delta_D = \frac{-7.2}{1000 EI}$$

$$\text{Also at } x = \frac{l}{2}, \Delta = \Delta_{\max} = \frac{-1/120}{EI}$$

c. > Moment Area method (Mohr's method)

- This method determines Slope and deflection due to Bending Only.
- It is useful particularly for beams which is subjected to series of concentrated loads or having segments with different moment of inertia.
- It is not applicable for case where internal hinge is present because continuity of slope is not maintained by elastic curve (deformed shape).



$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$\text{as } \frac{dy}{dx} = \theta$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M}{EI}$$

$$\frac{d\theta}{dx} = \frac{M}{EI}$$

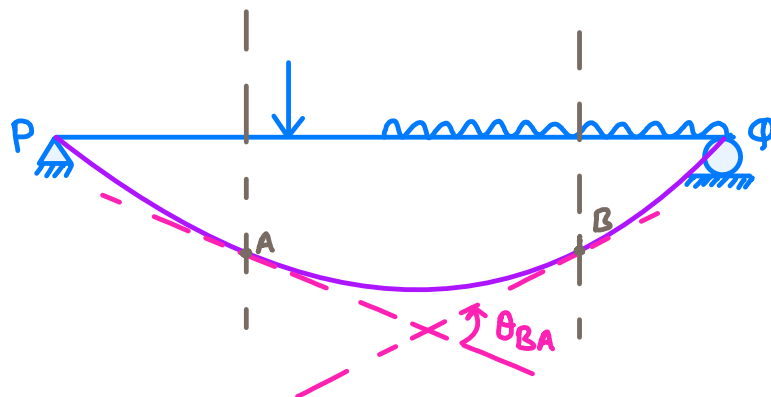
$$d\theta = \frac{M}{EI} dx$$

$$\int_A^B d\theta = \int_A^B \frac{M}{EI} dx$$

$$\theta_B - \theta_A = \text{Area under } \frac{M}{EI} \text{ diagram between A and B.}$$

$$\theta_{BA} = \theta_B - \theta_A = \text{Slope of "B" wrt to tangent drawn at "A"}$$

Theorem - 1 → The change of slope (θ_{BA}) between two points (A & B) on the elastic curve equals to area of $\frac{M}{EI}$ diagram between these two points (A & B).

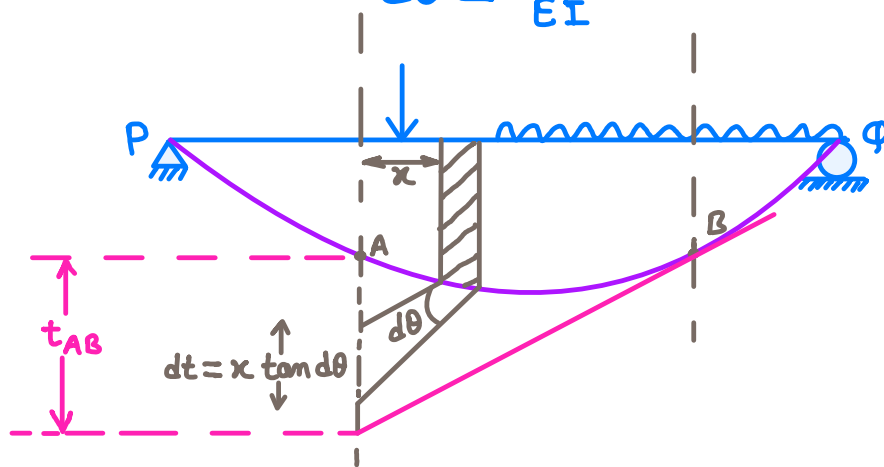


Sign Convention for Slope :-

- Angle is measured Anti-clockwise from tangent at "A" (i.e point of smaller "x") to tangent at "B" (i.e point of larger "x") if area $\frac{M}{EI}$ diagram is (+ve) or else (if $A = \frac{M}{EI}$ is (-ve)) angle is measured clockwise.

Now, $\frac{d\theta}{dx} = \frac{M}{EI}$

$$d\theta = \frac{M}{EI} dx$$



Since, $d\theta$ is very small, $\tan d\theta \approx d\theta$

Hence, $dt = x d\theta$

$$dt = x \frac{M}{EI} dx$$

$$t_{AB} = \int_A^B dt = \int_A^B x \frac{M}{EI} dx$$

$$t_{AB} = \bar{x} \int_A^B \frac{M}{EI} dx.$$

$$\left[\int x dA = \bar{x} A = \bar{x} \int dA \right]$$

Theorem-2 Deflection of any point A on the elastic curve wrt tangent drawn at another point "B" (t_{AB}) is equal to moment of area under $\frac{M}{EI}$ diagram between point "A" & "B" about point "A".

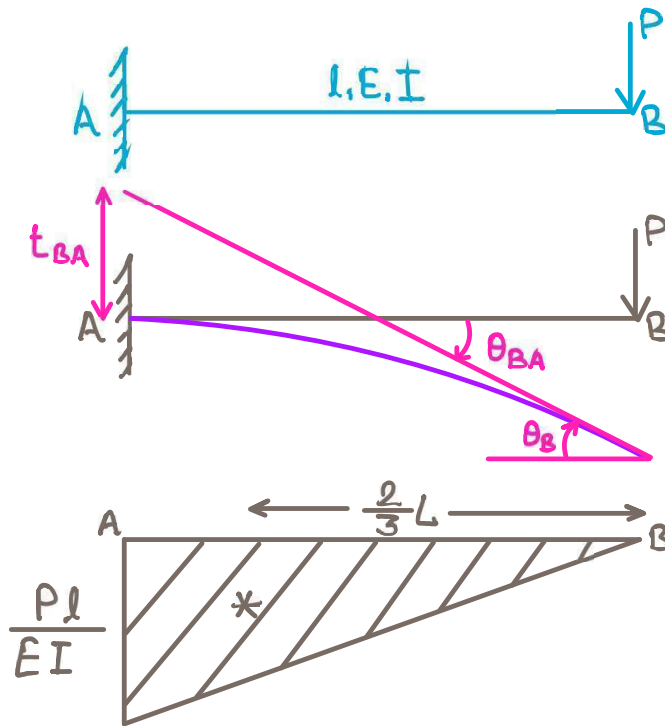
Sign convention for deflection

- point A on elastic curve is above tangent extended from B upto A, if moment of $\frac{M}{EI}$ diagram between A and B about point A is (+ve) or else (if moment of $\frac{M}{EI}$ diagram is (-ve)), point A on elastic curve would be below the tangent extended from "B".

Note :-

- To find out slope and deflection at any point using Area-Moment method we need to find a point where slope is either known or zero.

Q. >

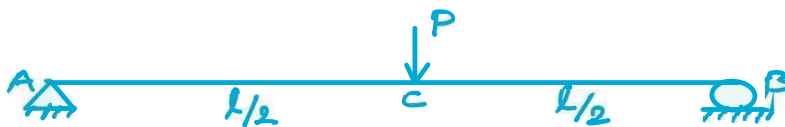
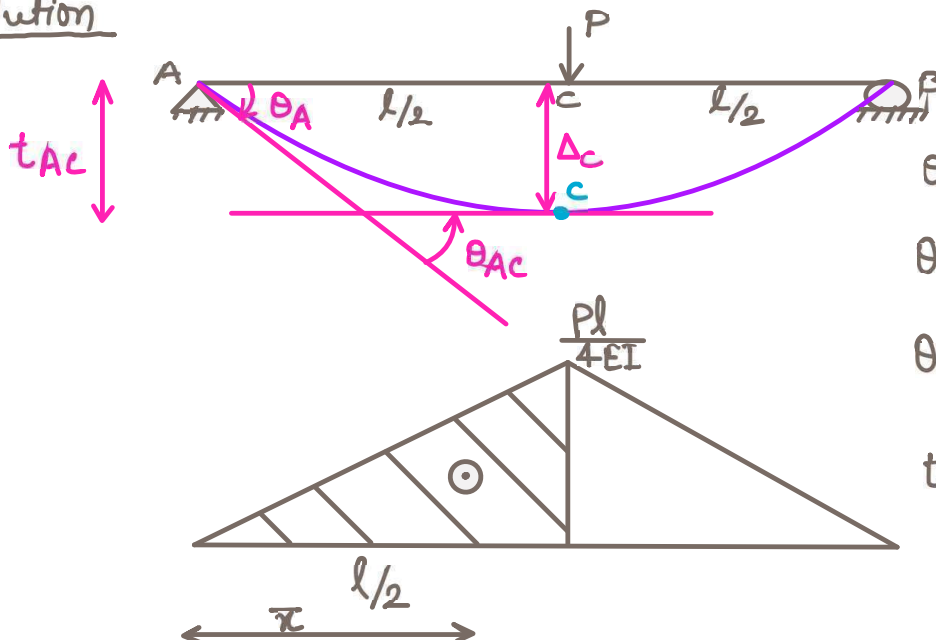
Solutioncompute θ_B, Δ_B .

$$\theta_{BA} = \theta_B - \theta_A = -\frac{1}{2} \cdot l \cdot \frac{Pl}{EI} = -\frac{Pl^2}{2EI}$$

$$\theta_B = -\frac{Pl^2}{2EI} \quad \text{OR} \quad \frac{Pl^2}{2EI}$$

$$t_{BA} = \left(-\frac{Pl^2}{2EI}\right) \frac{2}{3}l = -\frac{Pl^3}{3EI} = \frac{Pl^3}{3EI} \downarrow$$

Q. >

compute Δ_C, θ_A .Solution

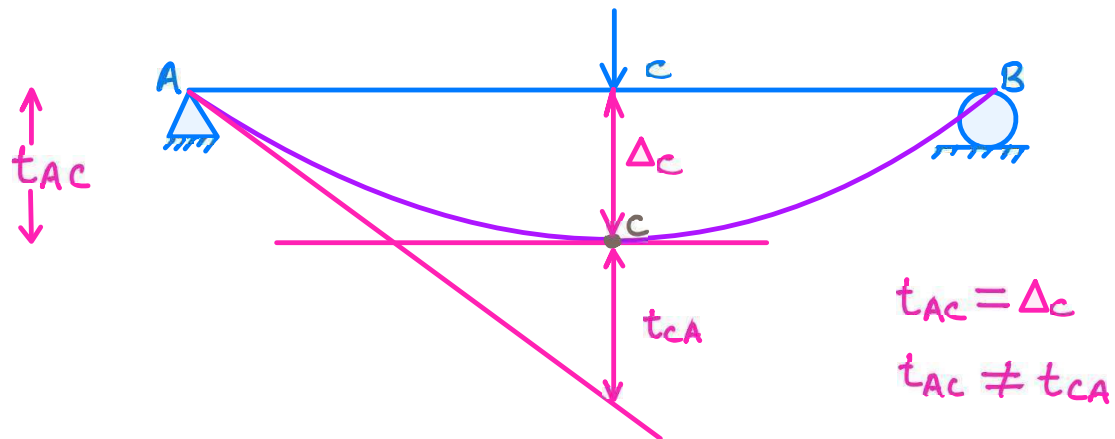
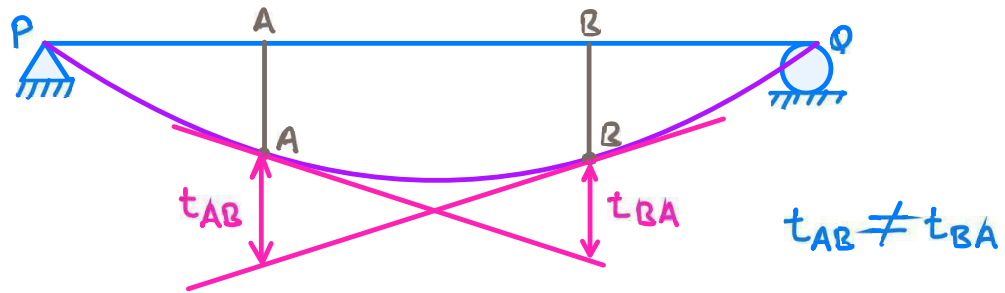
$$\theta_{AC} = \theta_A - \theta_C = \frac{1}{2} \cdot \frac{l}{2} \cdot \frac{Pl}{4EI}$$

$$\theta_{AC} = \frac{Pl^2}{16EI}$$

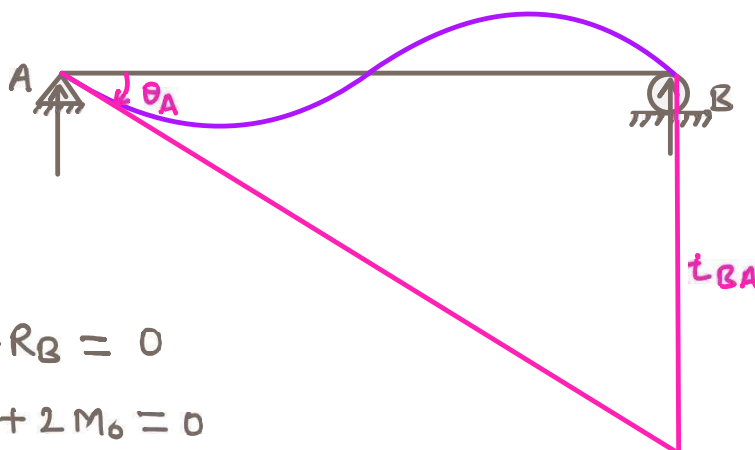
$$\theta_A = -\frac{Pl^2}{16EI} \quad \text{OR compute } \theta_{CA}$$

$$t_{AC} = \Delta_C = \frac{Pl^2}{16EI} \times \frac{2}{3} \times \frac{l}{2}$$

$$\Delta_C = \frac{Pl^3}{48EI} \downarrow$$

NOTE :-

Q. >

Compute, θ_A .Solution

$$\tan \theta_A = \frac{t_{BA}}{L}$$

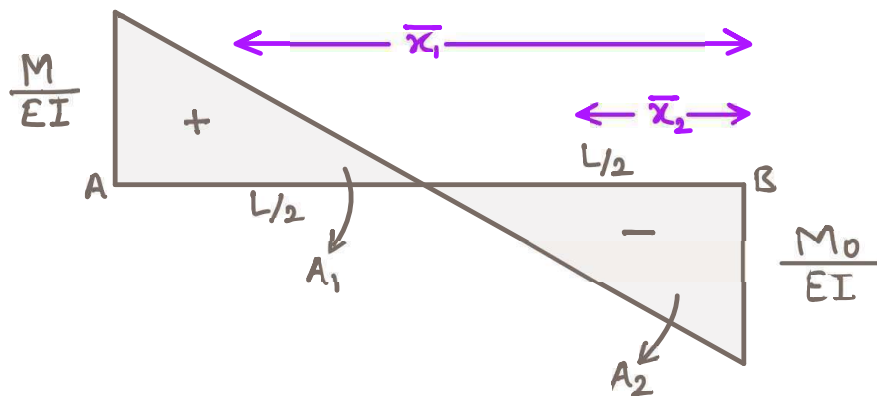
If θ is small $\theta \approx \tan \theta$

$$\theta_A = \frac{t_{BA}}{L}$$

$$R_A + R_B = 0$$

$$R_A \cdot L + 2M_0 = 0$$

$$R_A = \frac{-2M_0}{L}, \quad R_B = \frac{2M_0}{L}$$



$$A_1 = \frac{1}{2} \cdot \frac{L}{2} \cdot \frac{M_0}{EI} = \frac{M_0 L}{4EI}$$

$$\bar{x}_1 = \frac{L}{2} + \frac{2}{3} \cdot \frac{L}{2} = \frac{5}{6}L$$

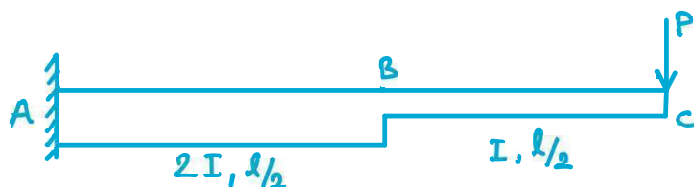
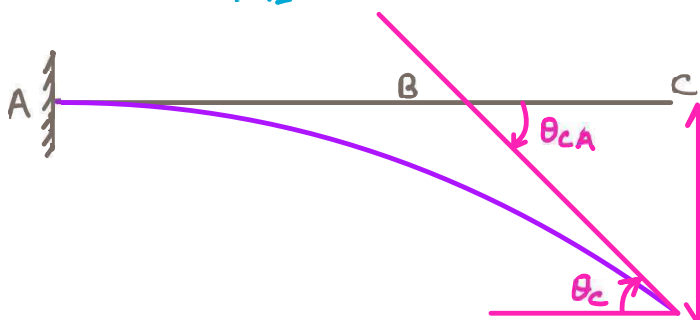
$$A_2 = -\frac{M_0 L}{4EI}$$

$$\bar{x}_2 = \frac{1}{3} \cdot \frac{L}{2} = \frac{L}{6}$$

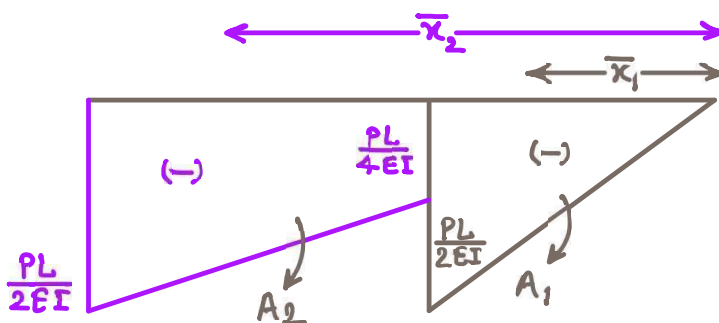
$$t_{BA} = A_1 \bar{x}_1 + A_2 \bar{x}_2 = \frac{M_0 L}{4EI} \cdot \frac{5}{6}L + \left(-\frac{M_0 L}{4EI}\right) \cdot \frac{L}{6} = \frac{M_0 L^2}{6EI}$$

$$\theta_A = \frac{M_0 L^2}{6EI} \cdot \frac{1}{L} = \frac{M_0 L}{6EI}$$

Q.7

Compute Δ_c, θ_c 

$$t_{CA} = \Delta_c \quad \text{and} \quad \theta_{CA} = \theta_c$$



$$\theta_{CA} = \theta_c - \theta_A = A_1 + A_2$$

$$t_{CA} = \Delta_c = A_1 \bar{x}_1 + A_2 \bar{x}_2$$

$$A_1 = \frac{1}{2} \cdot \frac{L}{2} \times \frac{PL}{2EI} = \frac{PL^2}{8EI}$$

$$\bar{x}_1 = \frac{2}{3} \cdot \frac{L}{2} = \frac{L}{3}$$

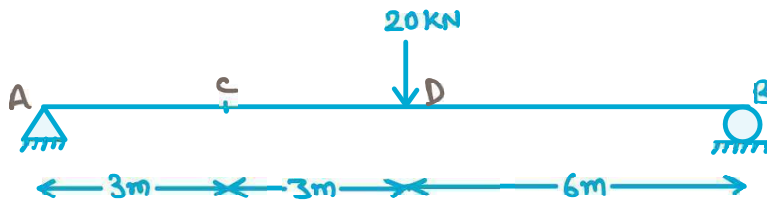
$$A_2 = \frac{1}{2} \left(\frac{PL}{4EI} + \frac{PL}{2EI} \right) \frac{L}{2} = \frac{3}{16} \frac{PL^2}{EI}$$

$$\bar{x}_2 = \frac{L}{2} + \frac{\left(\frac{1}{4} + 2 \cdot \frac{1}{2}\right) \cdot \frac{L}{2} \cdot \frac{1}{3}}{\frac{1}{4} + \frac{1}{2}} = \frac{7}{9}L$$

$$\theta_c = \frac{PL^2}{8EI} + \frac{3}{16} \frac{PL^2}{EI} = -\frac{5}{16} \frac{PL^2}{EI} \quad (\text{c.w})$$

$$40 t_{CA} = \frac{PL^2}{8EI} \times \frac{L}{3} + \frac{3}{16} \frac{PL^2}{EI} \times \frac{7}{9}L = \Delta_c = -\frac{3}{16} \frac{PL^2}{EI} \quad \downarrow$$

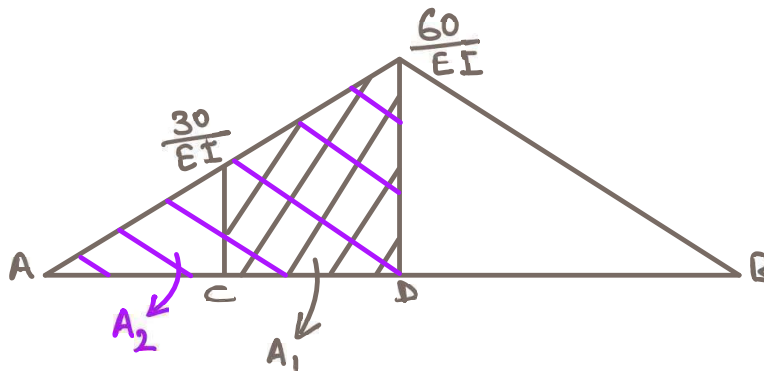
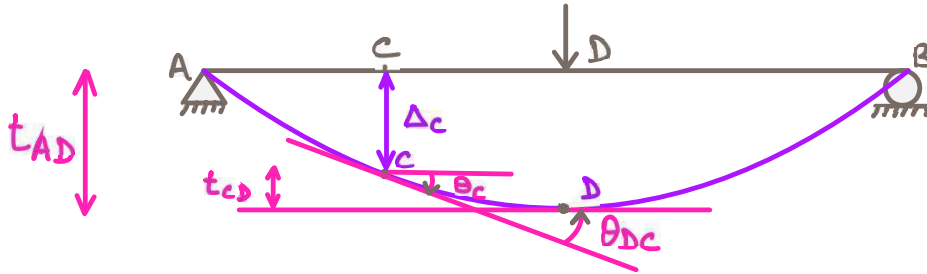
Q.7



$$E = 2 \times 10^5 \text{ N/mm}^2$$

$$I = 6 \times 10^6 \text{ mm}^4$$

find θ_c & $\Delta_c = ?$



$$\theta_{Dc} = \theta_D - \theta_c = A_1 = \frac{1}{2} \times \left(\frac{30}{EI} + \frac{60}{EI} \right) \times 3 = 0.1125 \text{ rad.}$$

$$t_{AD} = A_2 \bar{x}_2 = \frac{1}{2} \times 6 \times \frac{60}{EI} \times \frac{2}{3} \times 6 = \frac{720}{EI}$$

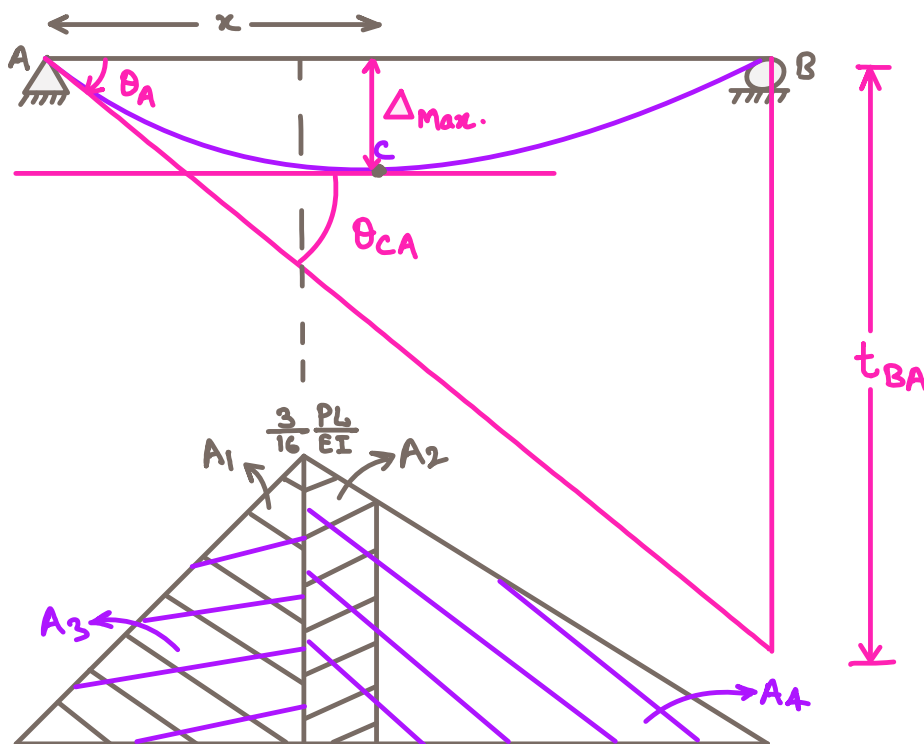
$$t_{CD} = A_1 \bar{x}_1 = \frac{1}{2} \times \left(\frac{30}{EI} + \frac{60}{EI} \right) \times 3 \times \frac{3}{3} \left(\frac{2 \times 60 + 30}{30 + 60} \right) = \frac{225}{EI}$$

$$\Delta_c = t_{AD} - t_{CD} = \frac{720}{EI} - \frac{225}{EI} = \frac{495}{EI} = \frac{495}{2 \times 10^5 \times 6 \times 10^6}$$

$$= 0.4125 \text{ mm.}$$

Q.8

Compute Δ_{max} .



Let C be the point at distance "x" from A at which slope is zero or deflection is maximum.

$$t_{BA} = A_3 \bar{x}_3 + A_4 \bar{x}_4 = \frac{1}{2} \cdot \frac{L}{4} \times \frac{3}{16} \frac{PL}{EI} \times \left(\frac{3L}{4} + \frac{L}{4} \times \frac{1}{3} \right) + \frac{1}{2} \times \frac{3L}{4} \times \frac{3}{16} \frac{PL}{EI} \left(\frac{3}{4} L \cdot \frac{2}{3} \right)$$

$$t_{BA} = \frac{7}{128} \frac{PL^3}{EI}$$

$$\theta_A = \frac{t_{BA}}{L} = \frac{7}{128} \frac{PL^2}{EI} \quad \text{--- (i)}$$

$$\theta_{CA} = A_1 + A_2 = \frac{1}{2} \times \frac{L}{4} \times \frac{3}{16} \frac{PL}{EI} + A_2 \quad \text{--- (ii)}$$

from (i) & (ii)

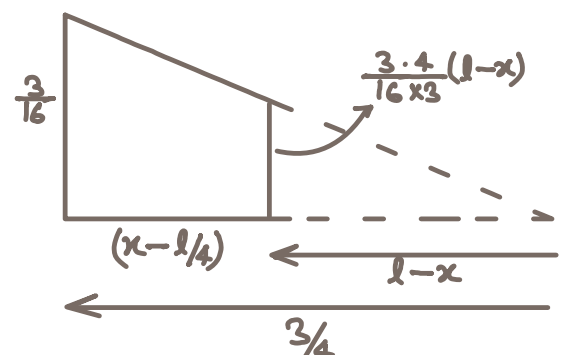
$$\frac{7}{128} \frac{PL^2}{EI} = \frac{3PL^2}{16 \times 8 EI} + A_2$$

$$A_2 = \frac{4}{128} \frac{PL}{EI} \quad \text{--- (iii)}$$

$$A_2 = \frac{1}{2} \left(\frac{3}{16} + \frac{(l-x)}{4} \right) \frac{PL}{EI} \left(x - \frac{L}{4} \right) \quad \text{--- (iv)}$$

from (iii) & (iv)

$$\underline{x = 0.441 \text{ m}}$$

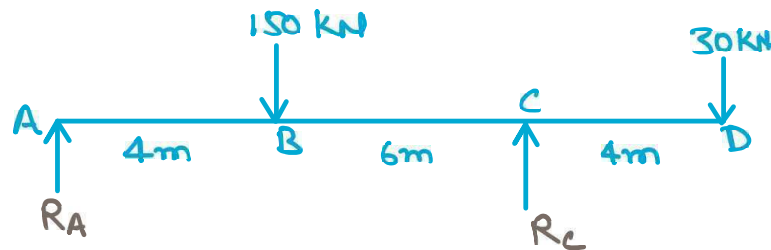


Now $\Delta_{max} = t_{Ac} = t_{Bc}$

$$\Delta_{max} = \frac{1}{2}(l-x) \frac{(l-x)}{4} \cdot \frac{2}{3}(l-x) \Big|_{x=0.441l}$$

$$\Delta_{max} = 0.0146 \frac{PL^3}{EI}$$

Q. > Compute Δ_D



Solution

$$R_A + R_C = 150 + 30 = 180 \text{ kN.}$$

$$R_C \times 10 = 150 \times 4 + 30 \times 14$$

$$R_C = 102 \text{ kN}$$

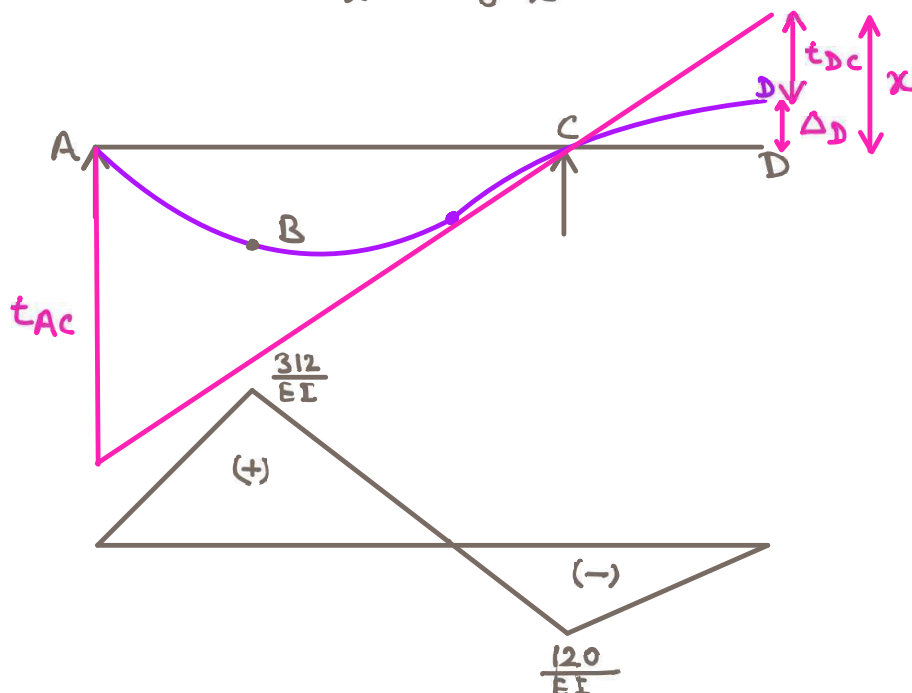
$$R_A = 78 \text{ kN}$$

$$M_B = R_A \times 4 = 78 \times 4 = 312 \text{ kN-m}$$

$$M_C = -30 \times 4 = -120 \text{ kN-m}$$

Let at distance x from B,

$$BM = 0 \Rightarrow \frac{312}{x} = \frac{120}{6-x} \Rightarrow x = 4.33 \text{ m.}$$



$$\begin{aligned}
 t_{AC} &= A_1 \bar{x}_1 + A_2 \bar{x}_2 + A_3 \bar{x}_3 \\
 &= \left(\frac{1}{2} \times 4 \times \frac{312}{EI} \times \frac{2}{3} \times 4 \right) + \left\{ \frac{1}{2} \times 4 \cdot 33 \times \frac{312}{EI} \times 4 + \left(4 \cdot 33 \times \frac{1}{3} \right) \right\} \\
 &\quad + \left\{ -\frac{1}{2} \times (6 - 4 \cdot 33) \times \frac{120}{EI} \left(4 + 4 \cdot 33 + \frac{1 \cdot 67}{3} \times 2 \right) \right\}
 \end{aligned}$$

$$t_{AC} = \frac{4394.64}{EI} \approx \frac{4395}{EI}$$

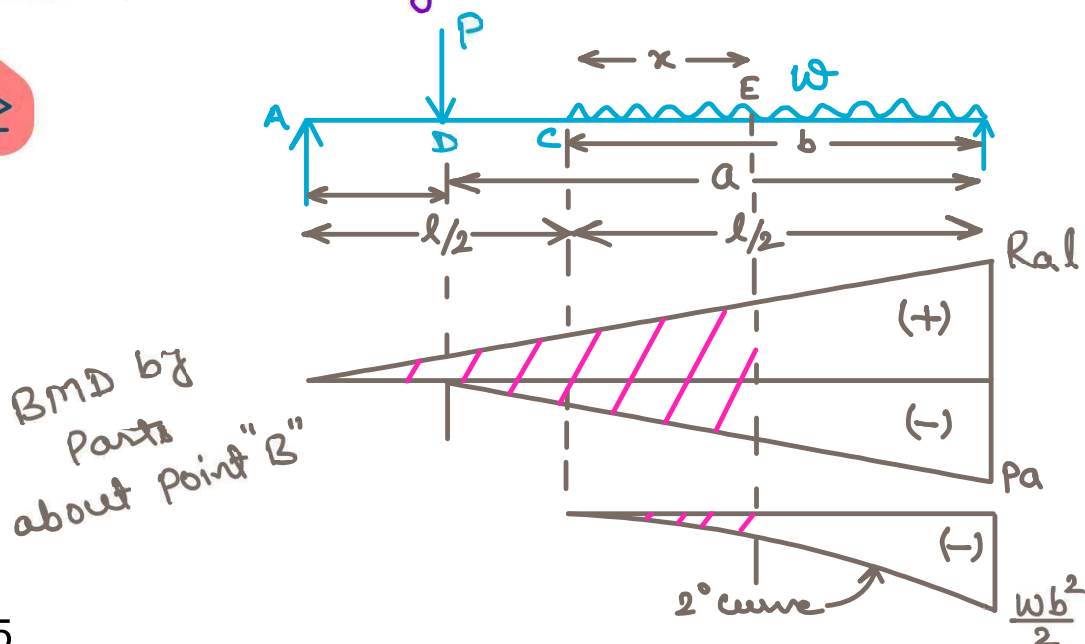
$$\frac{x}{4} = \frac{t_{AC}}{10} \Rightarrow x = \frac{4395}{EI} \times \frac{4}{10} = \frac{1758}{EI}$$

$$t_{DC} = -\frac{1}{2} \times 4 \times \frac{120}{EI} \times \frac{2}{3} \times 4 = \frac{-640}{EI}$$

$$\Delta_D = x - t_{DC} = \frac{1758}{EI} - \frac{640}{EI} = \frac{1118}{EI}$$

Note :-

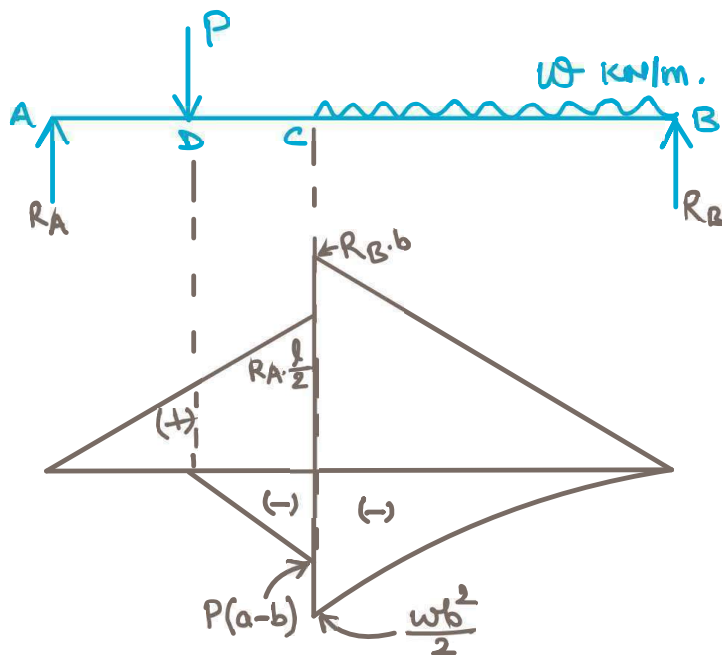
- when number of loading on a beam are large it is difficult to calculate the area under resultant $\frac{M}{EI}$ diagram and its C.G.
- In such case, instead of considering the total loading on the section, $\frac{M}{EI}$ diagram is drawn for each loading separately and is termed as "BMD By PARTS".



Note :-

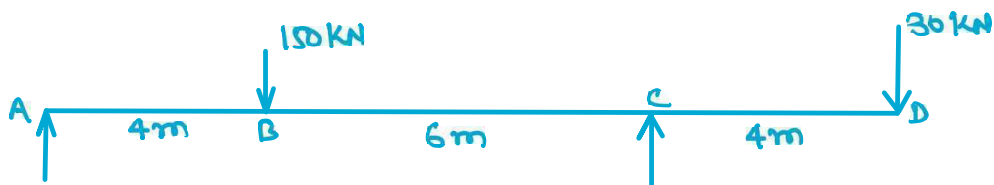
- To Draw BMD by Parts, Any section on beam is chosen & it is considered as fixed & BMD upto that section is drawn.

Eg:-



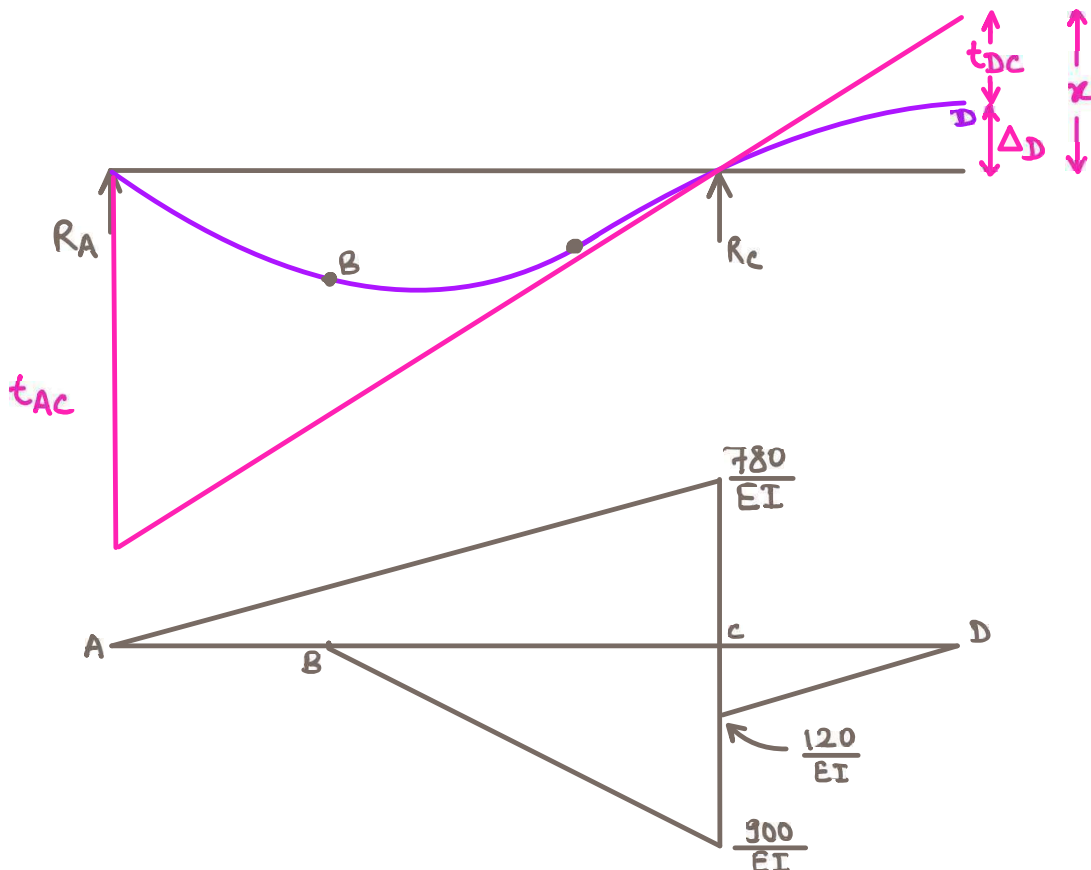
BMD by parts
about point "C"

Q>



Compute Δ_D

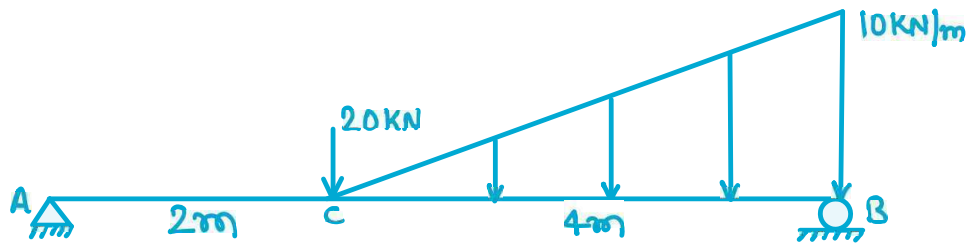
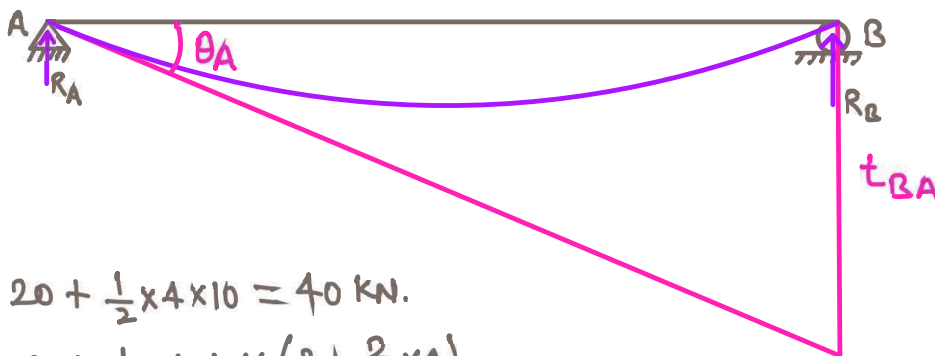
Solution



$$\Delta_D = x - t_{DC}$$

$$t_{Ac} = \frac{1}{2} \times 10 \times \frac{780}{EI} \times \frac{2}{3} \times 10 - \frac{1}{2} \times 6 \times \frac{900}{EI} \times \left(4 + \frac{2}{3} \times 6\right) = \frac{4400}{EI}$$

Q.7

Compute θ_A .Solution

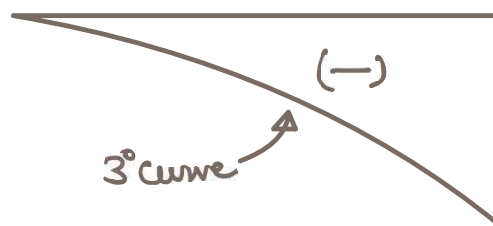
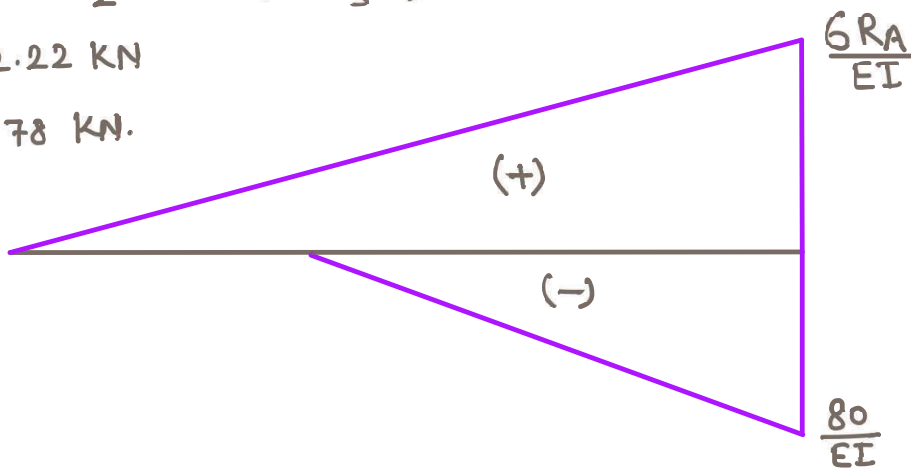
$$\theta_A = \frac{t_{BA}}{L}$$

$$R_A + R_B = 20 + \frac{1}{2} \times 4 \times 10 = 40 \text{ kN.}$$

$$R_B \times 6 = 20 \times 2 + \frac{1}{2} \times 4 \times 10 \times \left(2 + \frac{2}{3} \times 4\right)$$

$$R_B = 22.22 \text{ kN}$$

$$R_A = 17.78 \text{ kN.}$$



$$\frac{10 \times 4^2}{6EI} = \frac{26.67}{EI}$$

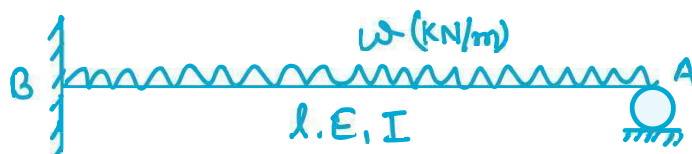
Distance of CG from vertex
 $\downarrow$
 $\frac{bh}{n+2}$

$$t_{BA} = \left(\frac{1}{2} \times 6 \times 6 \times \frac{17.78}{EI} \times \frac{6}{3} \right) - \left(\frac{1}{2} \times 4 \times \frac{80}{EI} \times \frac{4}{3} \right) - \left(4 \times \frac{26.67}{EI} \times \left(\frac{1}{3+1} \right) \times \frac{4}{3+2} \right)$$

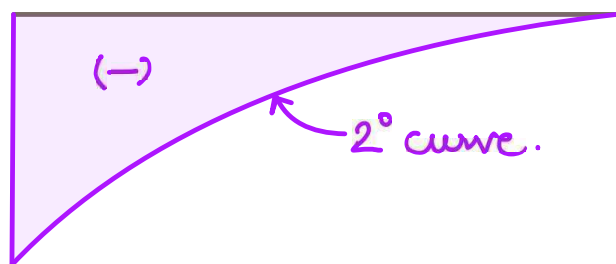
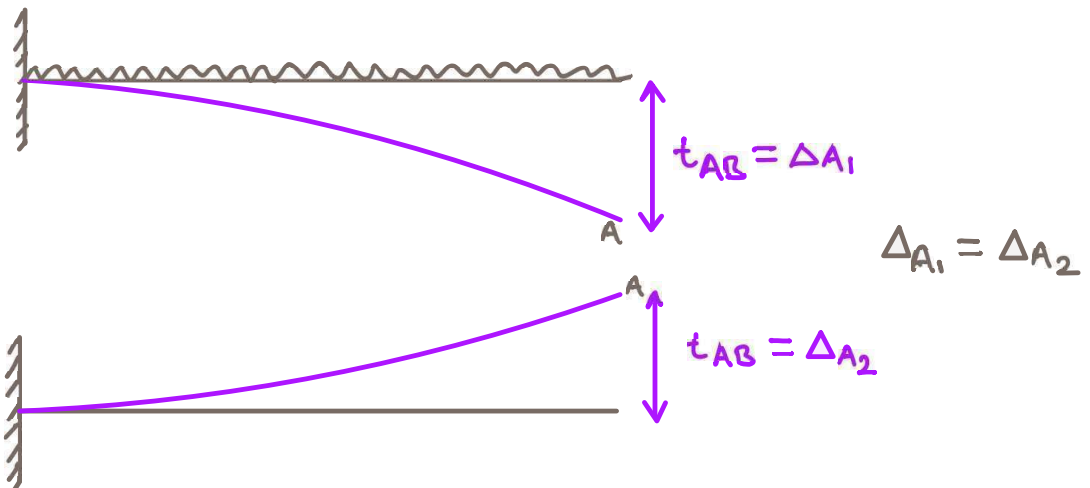
$$t_{BA} = \frac{405.41}{EI}$$

$$\theta_B = \frac{t_{BA}}{L} = \frac{405.41}{EI \times 6} = \frac{67.57}{EI}$$

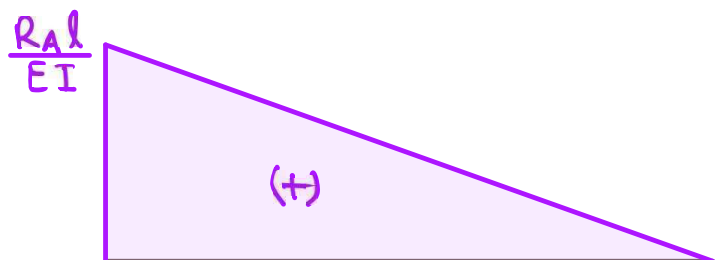
Q. > compute the reaction at the propped end.



Solution :-



$$\frac{wl^2}{2EI}$$



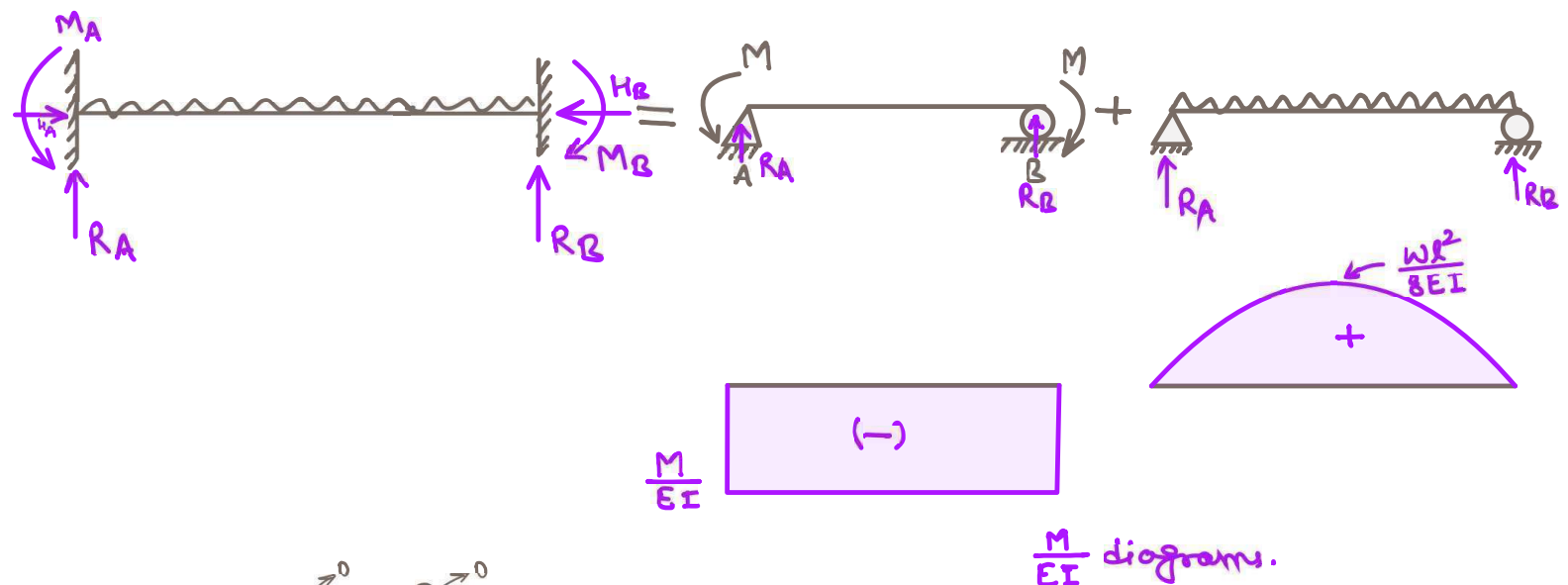
$$t_{AB} = \Delta_{A_1} = -l \cdot \frac{wl^2}{2EI} \cdot \frac{1}{(2+1)} \cdot \frac{3l}{4} = -\frac{wl^4}{8EI}$$

$$t_{AB} = \Delta_{A_2} = \frac{1}{2} \cdot l \cdot \frac{R_A l}{EI} \cdot \frac{2}{3} l = \frac{R_A l^3}{3EI}$$

$$\frac{wl^4}{8EI} = \frac{R_A l^3}{3EI}$$

$$R_A = \frac{3}{8} wl$$

Q>

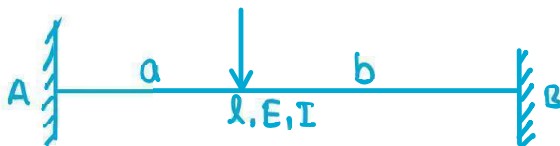
Solution

$$\theta_{BA} = \theta_B^0 - \theta_A^0$$

$$= -\frac{M}{EI} \cdot l + l \cdot \frac{wl^2}{8EI} \cdot \frac{2}{3} = 0$$

$$M = \frac{wl^2}{12}$$

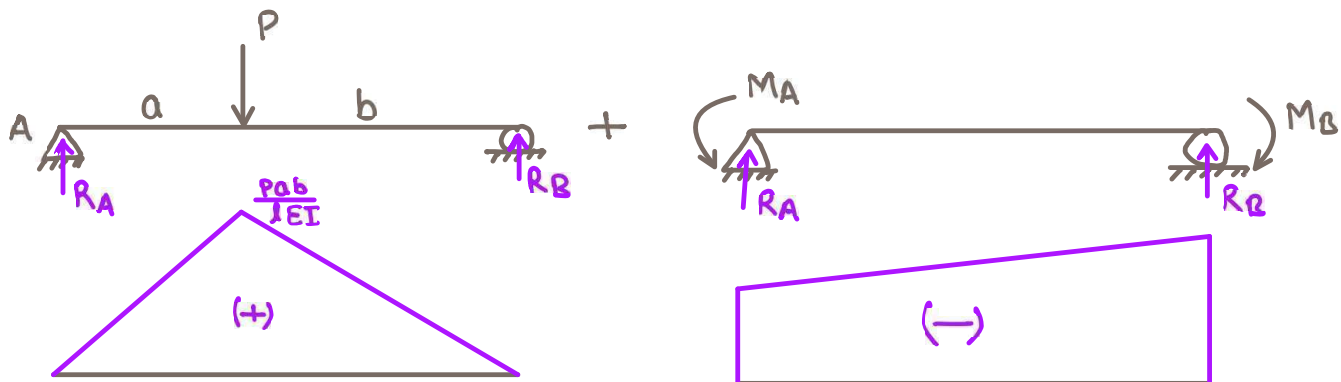
Q>



Solution :-

$$\theta_{BA} = 0$$

$$t_{BA} = 0$$



$$\theta_{BA} = \frac{1}{2} \cdot \frac{Pab \cdot l}{EI} - \frac{1}{2} \cdot \frac{(M_A + M_B)l}{EI} = 0$$

$$Pab - (M_A + M_B)l = 0 \quad \text{--- (i)}$$

$$t_{BA} = \frac{1}{2} \cdot a \cdot \frac{Pab}{EI} \left(\frac{a}{3} + b \right) + \frac{1}{2} \cdot b \cdot \frac{Pab}{EI} \left(\frac{2}{3}b \right) - \frac{1}{2} \cdot \frac{(M_A + M_B)l}{EI} \cdot l \left(\frac{2M_A + M_B}{M_A + M_B} \right) \cdot \frac{l}{3}$$

$$\frac{Pab}{l} (2b + a) (a + b) - (2M_A + M_B) l^2 = 0$$

$$Pab(2b + a) - (2M_A + M_B) l^2 = 0$$

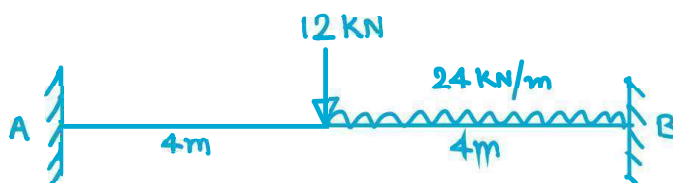
from (i) & (ii)

$$M_A = \frac{Pab^2}{l^2}$$

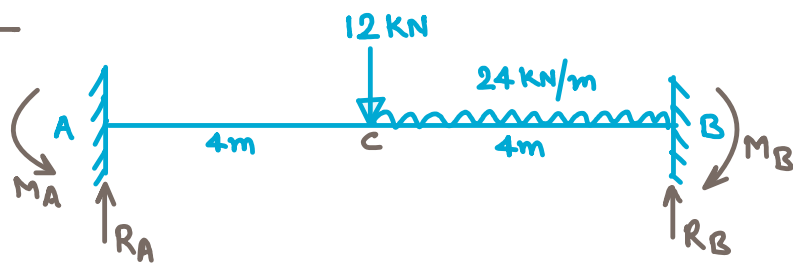
$$M_B = \frac{Pa^2b}{l^2}$$

Q. >

compute M_A & M_B

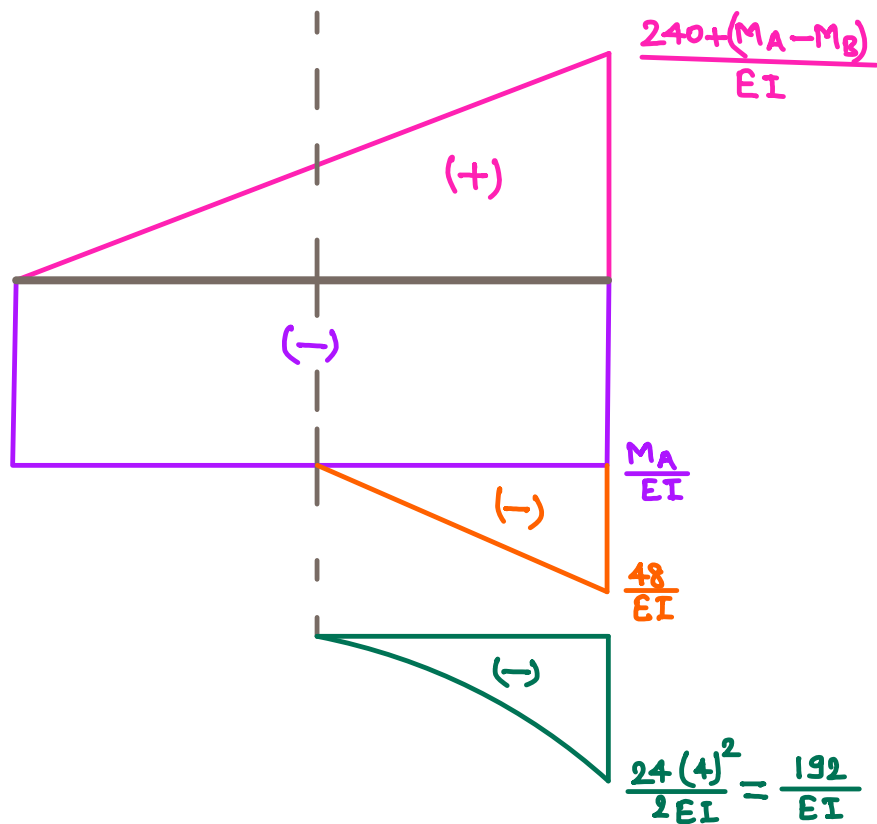
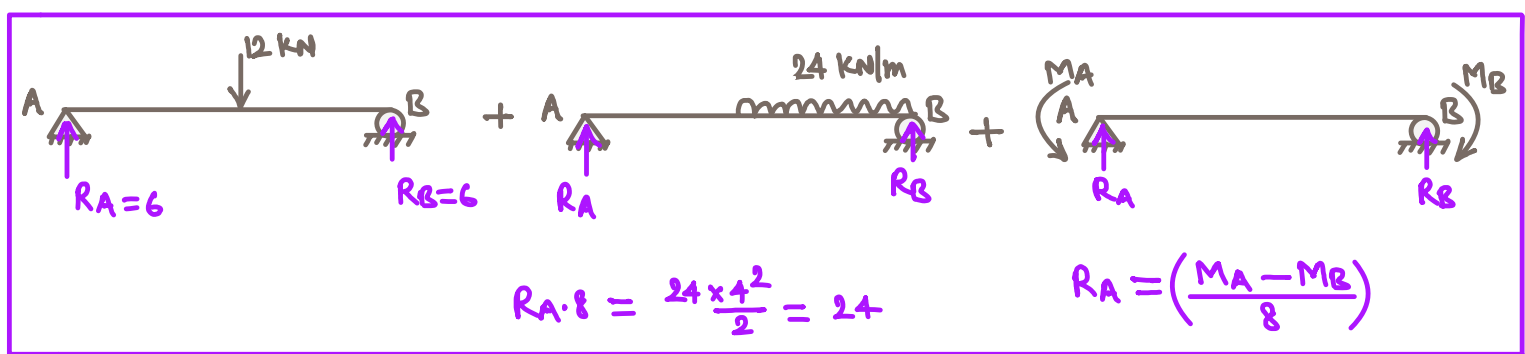


Solution:-



$$R_A \times 8 + M_B - M_A - 12 \times 4 - 24 \times \frac{4^2}{2} = 0$$

$$R_A = 30 + \left(\frac{M_A - M_B}{8} \right)$$



$$\theta_{BA} = 0$$

$$t_{BA} = 0$$

$$\theta_{BA} = \frac{1}{2} \left[\frac{240 + (M_A - M_B)}{EI} \right] \cdot 8 - \frac{M_A}{EI} \cdot 8 - \frac{1}{2} \frac{48}{EI} \cdot 4 - \frac{1}{3} \cdot 4 \cdot \frac{192}{EI} = 0$$

$$M_A + M_B = 152 \quad \text{--- (i)}$$

$$t_{BA} = \frac{1}{2} \left[240 + \frac{(M_A - M_B)}{EI} \right] \cdot 8 \cdot \frac{8}{3} - \frac{M_A \cdot 8}{EI} \cdot 4 - \frac{1}{2} \cdot \frac{48}{EI} \cdot 4 \cdot \frac{4}{3} - \frac{1}{3} \cdot 4 \cdot \frac{192}{EI} \cdot \frac{4}{3} = 0$$

$$2M_A + M_B = 204 \quad \text{--- (i)}$$

from (i) & (ii)

$$M_A = 52 \text{ KN-m}$$

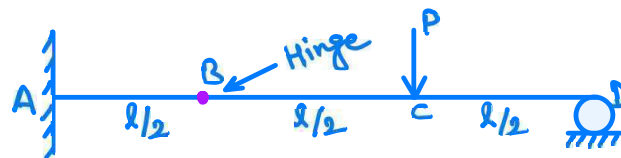
$$M_B = 100 \text{ KN-m.}$$

Note:-

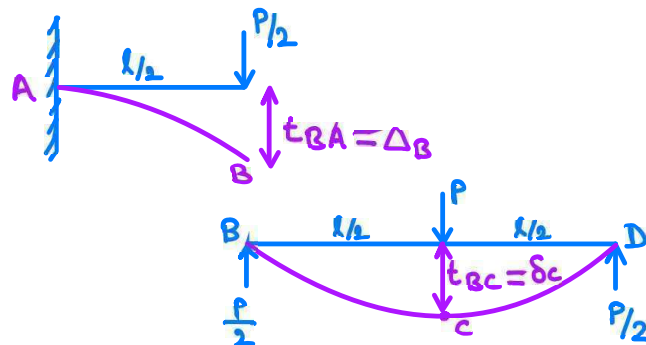
Area-Moment method can not be applied easily in case of discontinuity in the deflected shape/elastic curve.

for eg:-

● Compute Δ_B, Δ_C

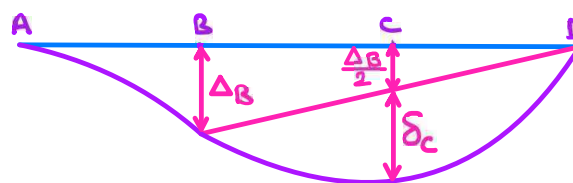


Solution.



$$t_{BA} = \Delta_B = \frac{1}{2} \cdot \frac{l}{2} \cdot \frac{P}{2} \cdot \frac{l}{2} \left(\frac{2}{3} \cdot \frac{l}{2} \right) \cdot \frac{1}{EI} = \frac{Pl^3}{48EI}$$

$$t_{BC} = \delta_C = \frac{1}{2} \cdot \frac{l}{2} \cdot \frac{Pl}{4} \cdot \frac{2l}{6} \cdot \frac{1}{EI} = \frac{Pl^3}{48EI}$$



$$\Delta_C = \frac{\Delta_B}{2} + \delta_C = \frac{Pl^3}{32EI}$$

D> Conjugate Beam method

- All methods of Slope and deflection calculation discussed till now i.e. Double integration, Macaulay's method, Moment-Area method calculate Slope and deflection using Elastic curve.
- However conjugate beam method is based on the principle of Statics, Hence in case where there is discontinuity in the deflected shape this method is most suitable.
- Beam which is loaded with " $\frac{M}{EI}$ " diagram is called conjugate beam.
- According to this method :-
 - (A) Shear force at any point in conjugate beam is equal to Slope at that point in real beam.
 - (B) Bending moment at any point in the conjugate beam is equal to deflection at that point in real beam.

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M}{EI} \Rightarrow \frac{d\theta}{dx} = \frac{M}{EI}$$

$$\int d\theta = \theta = \int \frac{M}{EI} dx \quad \left\{ \text{If } \frac{M}{EI} = w \right\}$$

$$\theta = \int w dx = V$$

Slope in real beam

Shear force in conjugate beam.

Now,

$$\theta = \frac{dy}{dx} = \int \frac{M}{EI} dx$$

$$\int dy = y = \iint \frac{M}{EI} dx dx = \iint w dx \cdot dx = \int V \cdot dx$$

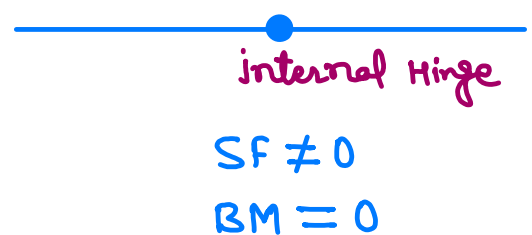
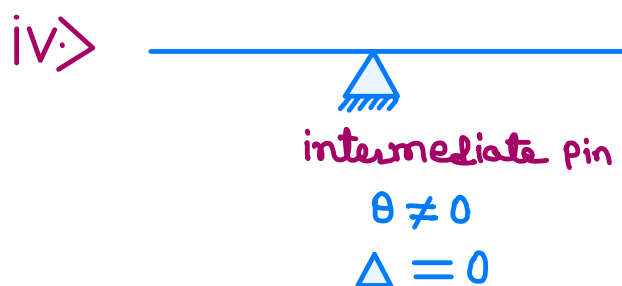
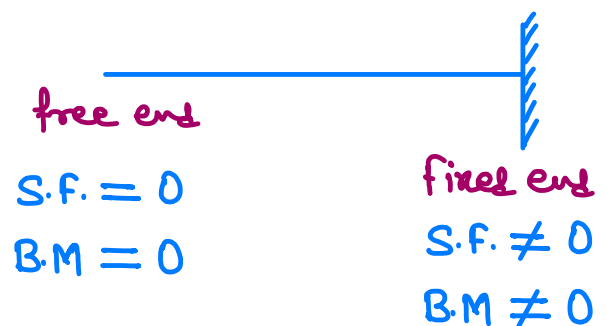
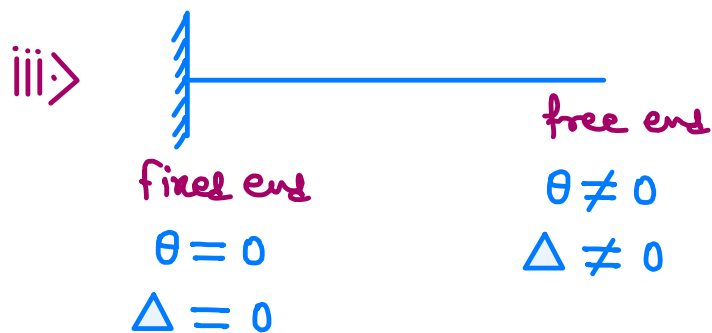
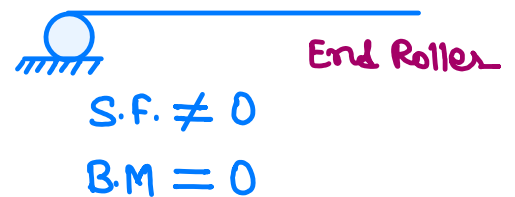
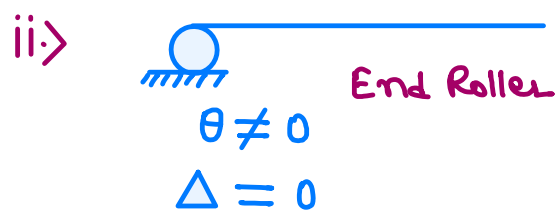
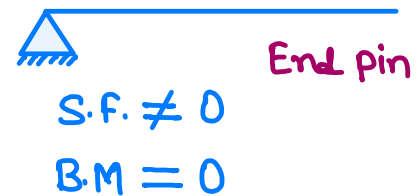
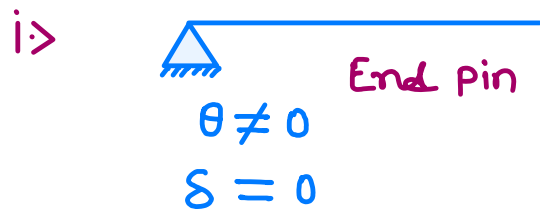
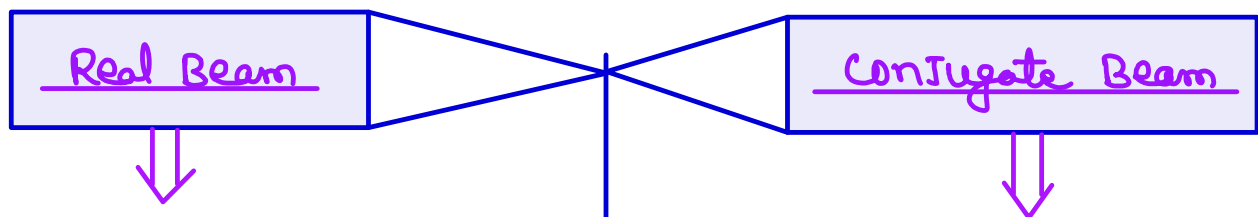
$$y = M$$

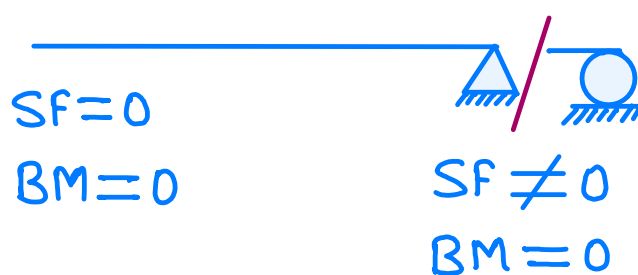
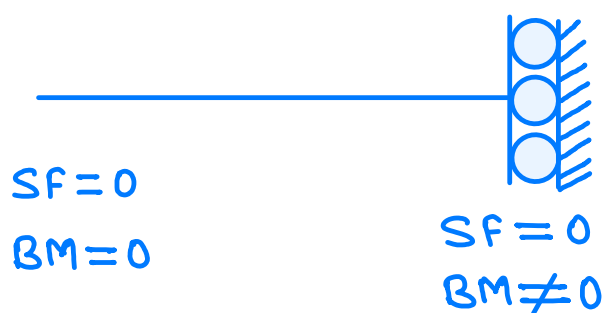
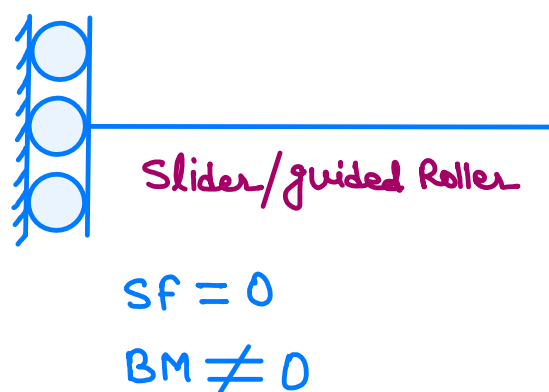
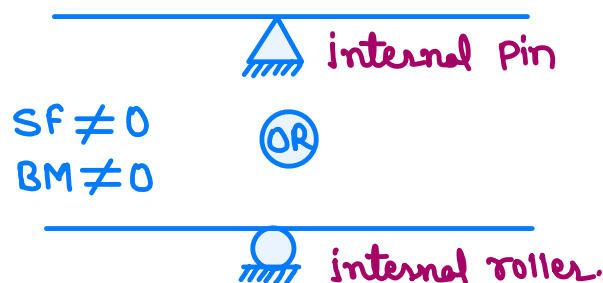
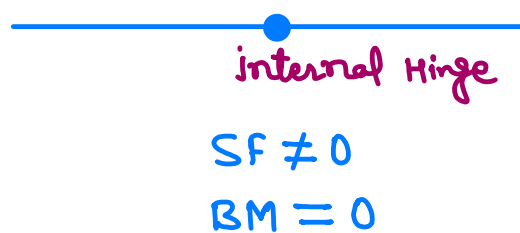
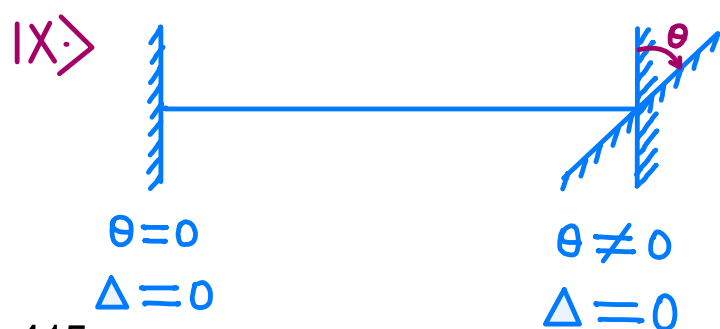
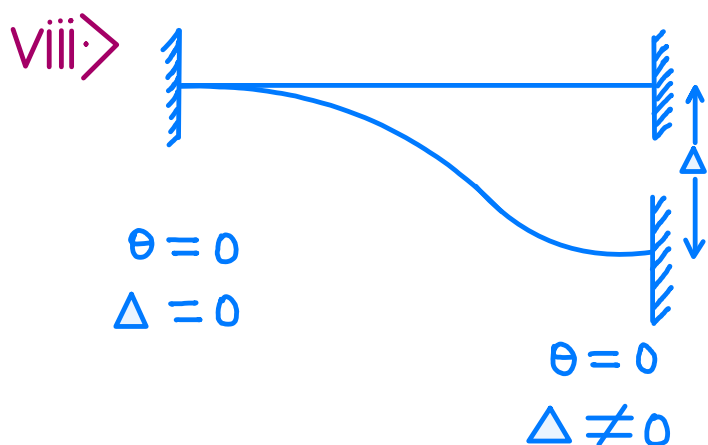
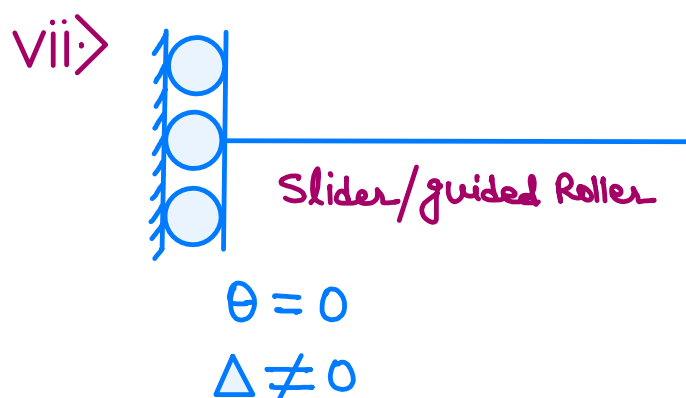
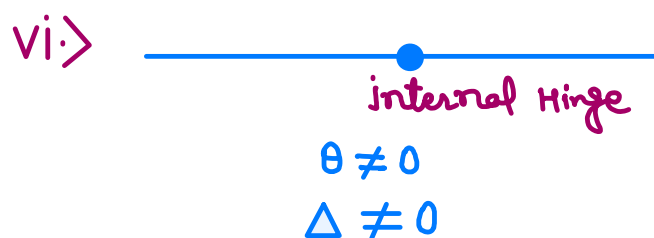
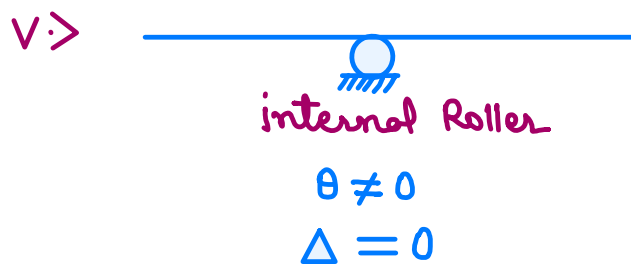
Deflection in real beam

Bending moment in conjugate beam.

★ Conjugate beam supports :-

- While drawing the conjugate beam it is necessary that the shear & moment develops at the support of a conjugate beam corresponding to slope and deflection of real beam at the location of supports in conjugate beam.



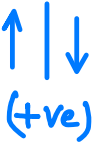


NOTE :-

- i> Determinate beam have determinate and stable conjugate beam.
- ii> Indeterminate beam have unstable conjugate beam.

Sign convention :-

i> BM:-  

ii> SF :-  

iii> If $\frac{M}{EI}$ diagram is (+ve), loading on conjugate beam is upward and if $\frac{M}{EI}$ diagram is (-ve), loading on conjugate beam is downward.

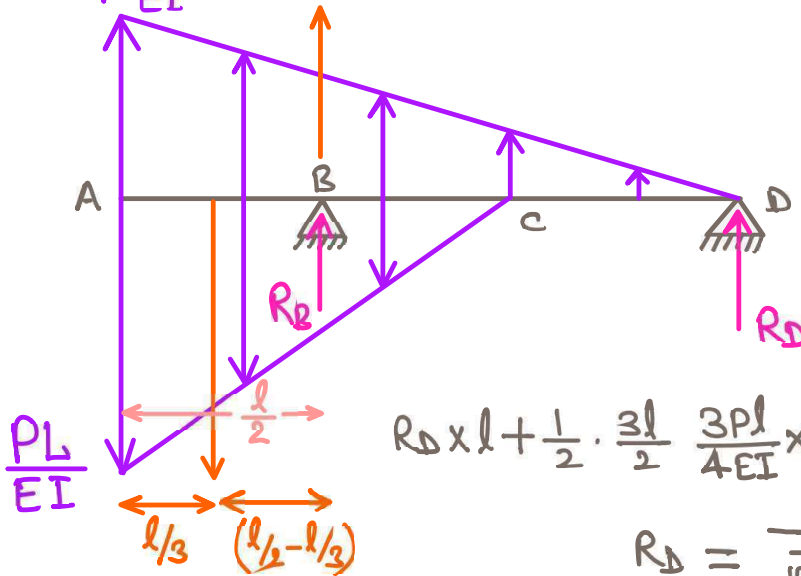
iv> If Shear force in conjugate beam is (+ve), Slope in real beam will be anticlockwise & if in conjugate beam is (-ve), Slope in real beam will be clockwise.

v> If Bending moment (B.M) in conjugate beam is positive (+ve), deflection in real beam will be upward & if B.M. in conjugate beam is (-ve), deflection in real beam will be downward.

vi> Downward deflection is taken as negative (-ve).


$$\frac{3}{4} \frac{Pl}{EI}$$
$$M_B = 0$$

$$R_D = \frac{P}{2}$$



$$R_D \times l + \frac{1}{2} \cdot \frac{3l}{2} \frac{3Pl}{4EI} \times 0 + \frac{1}{2} \times l \times \frac{Pl}{EI} \left(\frac{l}{2} - \frac{l}{3} \right) = 0$$

$$R_D = \frac{-Pl^2}{12EI}$$

$$R_B + R_D + \frac{1}{2} \cdot \frac{3l}{2} \cdot \frac{3Pl}{4EI} - \frac{1}{2} l \cdot \frac{Pl}{EI} = 0$$

$$R_B + \left(\frac{-Pl^2}{12EI}\right) + \frac{9Pl^2}{16EI} - \frac{Pl^2}{2EI} = 0$$

$$R_B = \frac{Pl^2}{48EI}$$

$$M_B = \frac{-Pl^2}{12EI} \cdot l + \frac{1}{2} \cdot l \left(\frac{3}{4} \cdot \frac{2}{3} \times 1 \right) \frac{Pl}{EI} \cdot \frac{l}{3} - \frac{1}{2} \cdot \frac{l}{2} \cdot \left(\frac{1}{2} \cdot 1 \right) \frac{Pl}{EI} \cdot \frac{l}{2} \cdot \frac{1}{3}$$

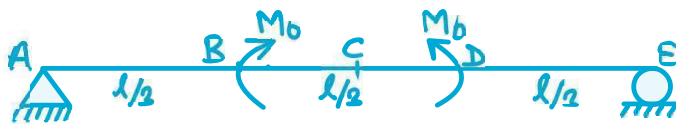
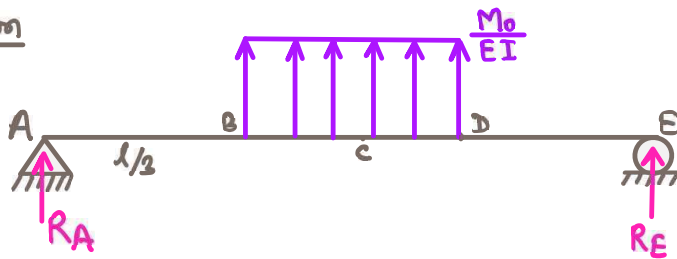
$$M_B = \frac{-Pl^3}{48EI}$$

$$\Rightarrow \Delta_B = \frac{Pl^3}{48EI} \downarrow$$

$$M_c = \frac{-Pl^2}{12EI} \cdot \frac{l}{2} + \left(\frac{3}{4}, \frac{2}{3}, \frac{1}{2}\right) \frac{Pl}{EI} \cdot \left(\frac{l}{2}, \frac{1}{3}\right) = \frac{-Pl^3}{32EI}$$

$$\Rightarrow \Delta_c = \frac{Pl^3}{32EI} \downarrow$$

Q. >

compute Δ_c , θ_A .Solution

for real beam

$$R_A \times l + M_0 - M_0 = 0$$

$$R_A = 0, \quad R_E = 0$$

for conjugate beam,

$$R_A + R_E + \frac{M_0}{EI} \cdot \frac{l}{2} = 0 \Rightarrow R_A + R_E = -\frac{M_0 l}{3EI} \quad \text{--- (i)}$$

$$R_A = R_E \quad \text{--- (ii)}$$

from (i) & (ii)

$$R_A = R_E = -\frac{M_0 l}{6EI}$$

$$(S.F.)_A = R_A = -\frac{M_0 l}{6EI}$$

$$\Rightarrow \theta_A = \frac{M_0 l}{6EI} \quad \downarrow$$

$$(BM)_C = -\frac{M_0 l}{6EI} \cdot \frac{l}{2} + \frac{M_0}{EI} \cdot \frac{l}{2} \cdot \frac{l}{4} = -\frac{5}{72} \frac{M_0}{EI}$$

$$\Rightarrow \Delta_C = \frac{5 M_0}{72 EI} \quad \downarrow$$

Q. >

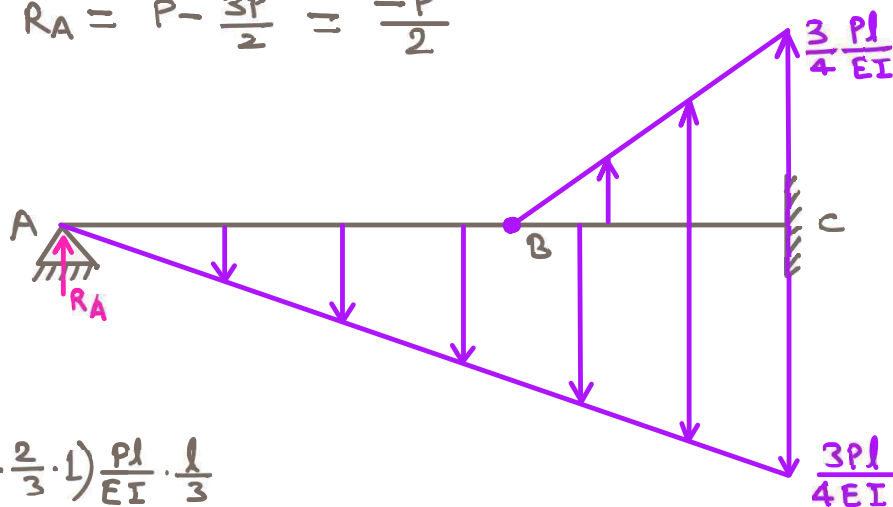
compute Δ_c EI is constant.

Solution

for Real beam: $R_B \times l = P \cdot \frac{3}{2} l \Rightarrow R_B = \frac{3P}{2}$

$$R_A + R_B = P$$

$$\Rightarrow R_A = P - \frac{3P}{2} = -\frac{P}{2}$$



for conjugate beam:

$$R_A \times l = \frac{1}{2} \cdot l \cdot \left(\frac{3}{4} \cdot \frac{2}{3} \cdot l \right) \frac{Pl}{EI} \cdot \frac{l}{3}$$

$$R_A = \frac{Pl^2}{12EI}$$

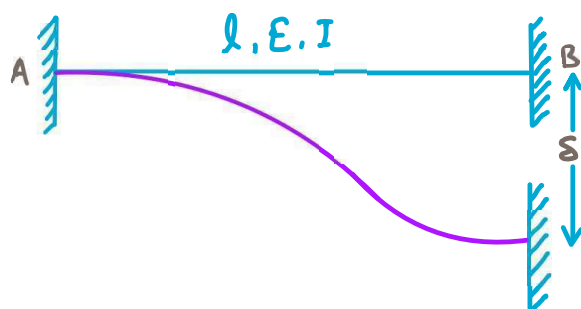
$$M_c = \frac{Pl^2}{12EI} \cdot \frac{3}{2} l + \frac{1}{2} \cdot \frac{l}{2} \cdot \frac{3}{4} \frac{Pl}{EI} \cdot \frac{1}{3} \frac{l}{2} - \frac{1}{2} \cdot \frac{3}{2} \cdot l \cdot \frac{3}{4} \frac{Pl}{EI} \cdot \frac{1}{3} \cdot \frac{3l}{2}$$

$$M_c = -\frac{Pl^2}{8EI}$$

$$\Delta_c = \frac{Pl^3}{8EI} \downarrow$$

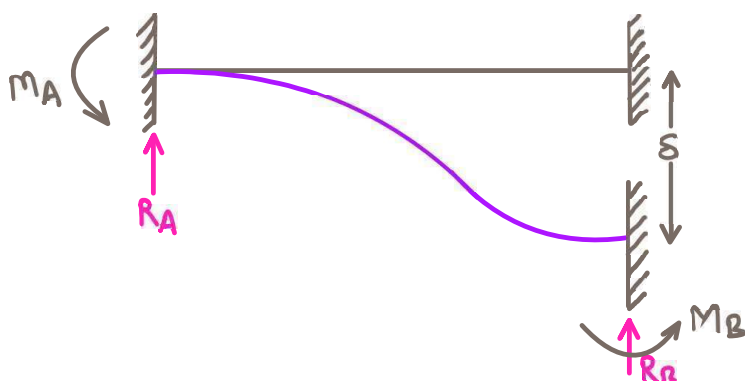
Std. cases:-

Q>



Compute End reactions & moments.

Solution

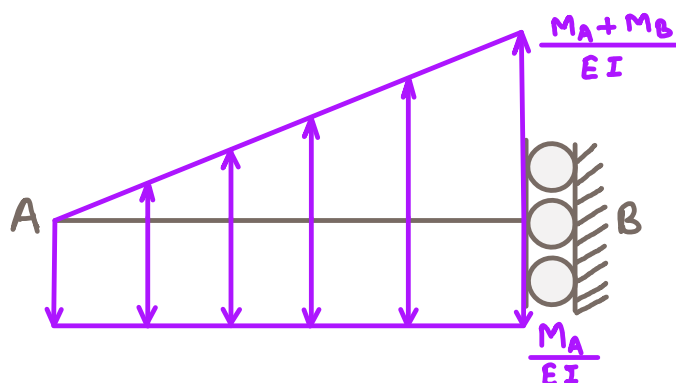


for Real beam:

$$R_A \times l = M_A + M_B$$

$$R_A = \frac{M_A + M_B}{l}$$

$$R_B = -\left(\frac{M_A + M_B}{l} \right)$$



$$BM_B = -\delta \quad \text{--- (i)}$$

$$SF_B = 0 \quad \text{--- (ii)}$$

$$\frac{1}{2} \cdot l \cdot \frac{(M_A + M_B)}{EI} \cdot \frac{l}{3} - \frac{M_A}{EI} \cdot l \cdot \frac{l}{2} = -\delta$$

$$M_B - 2M_A = -\delta \frac{6EI}{l^2} \quad \text{--- (A)}$$

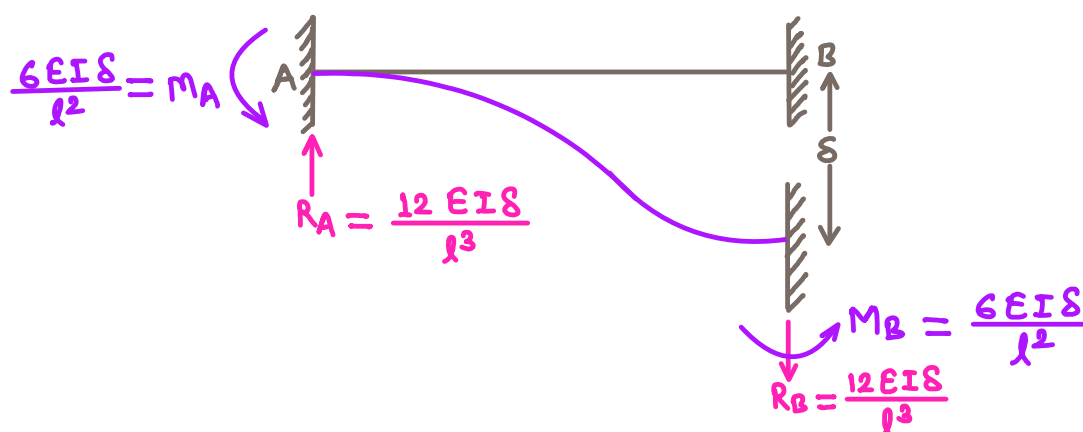
$$\frac{1}{2} \cdot l \cdot \frac{M_A + M_B}{EI} - \frac{M_A}{EI} \cdot l = 0$$

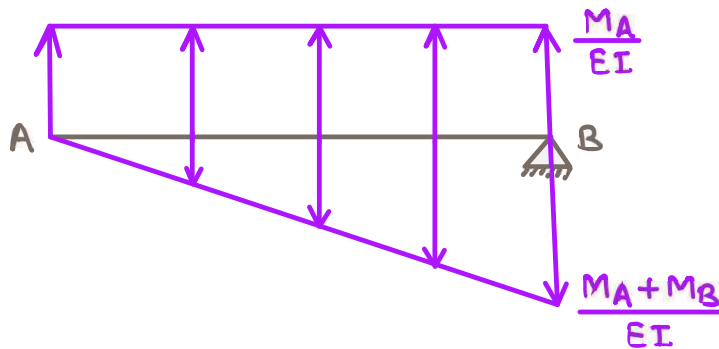
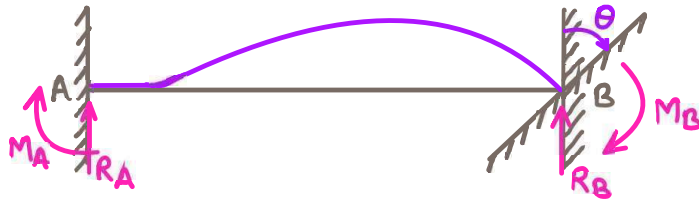
$$M_A = M_B \quad \text{--- (B)}$$

from (A) & (B)

$$M_A = M_B = \frac{6EI\delta}{l^2}$$

$R_A = \frac{2M_A}{l} = \frac{12EI\delta}{l^3}$	☆☆☆
$R_B = -\frac{12EI\delta}{l^3}$	





for Real beam :

$$R_A \times l = -(M_A + M_B)$$

$$R_A = -\frac{(M_A + M_B)}{l}$$

$$R_B = \frac{M_A + M_B}{l}$$

$$SF_B = -\theta$$

$$BM_B = 0$$

$$\frac{M_A \cdot l}{EI} - \frac{1}{2} \frac{(M_A + M_B)l}{EI} = -\theta$$

$$\frac{M_A}{EI} \cdot l \cdot \frac{l}{2} - \frac{1}{2} \cdot l \cdot \frac{(M_A + M_B)}{EI} \cdot \frac{l}{3} = 0$$

$$M_A - M_B = -2 \frac{\theta EI}{l} \quad \text{--- (i)}$$

$$2M_A = M_B \quad \text{--- (ii)}$$

from (i) & (ii)

$$M_A = \frac{2\theta EI}{l}, \quad M_B = \frac{4\theta EI}{l}$$

$$R_A = -\frac{6\theta EI}{l^2}, \quad R_B = \frac{6\theta EI}{l^2}$$



(E) Energy method

- Energy method is based upon the principle that any external work done on the structure get stored as strain energy in the structure.

Hence,

External work done = Strain energy stored.

$$U_e = U_i$$

★ External work done (U_e)

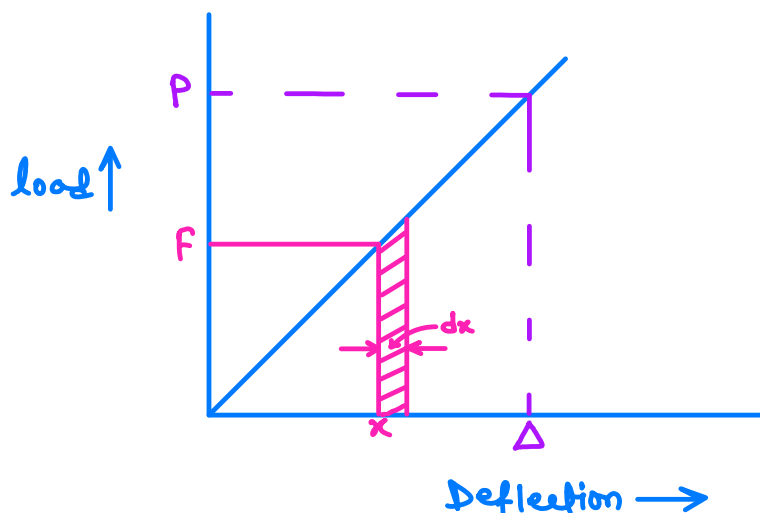
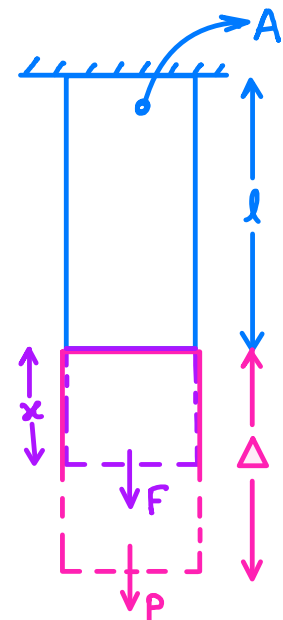
a. > Due to Axial force

External work done, $du_e = F dx$

$$U_e = \int du_e = \int_0^{\Delta} F dx = \int_0^{\Delta} \frac{P}{\Delta} x dx$$

$$U_e = \frac{P}{\Delta} \left| \frac{x^2}{2} \right|_0^{\Delta} = \frac{P \Delta^2}{2 \Delta}$$

$$U_e = \frac{P \Delta}{2} = \left[\text{Area of load-displacement curve} \right]$$

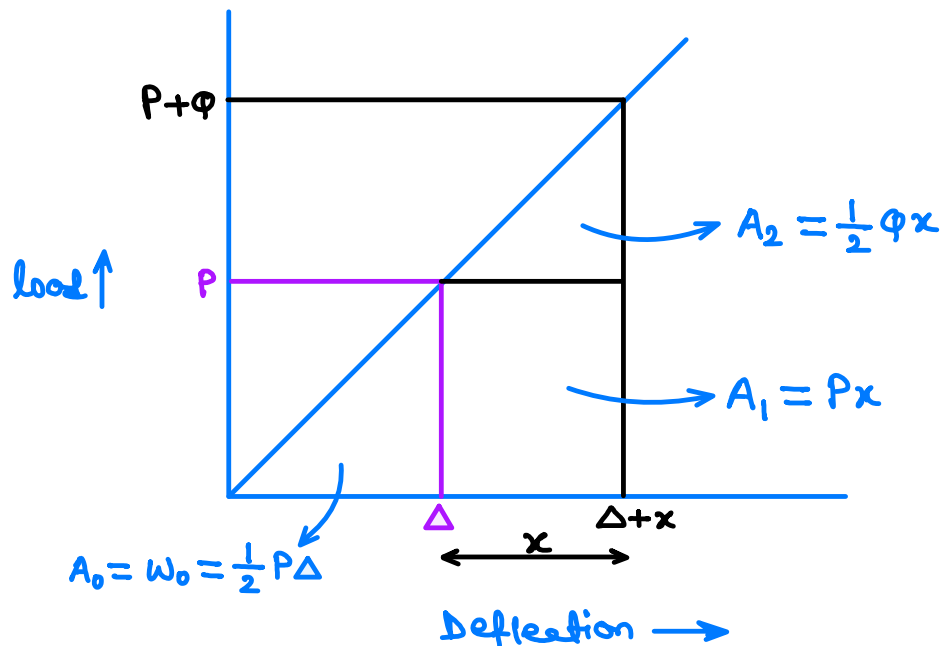


NOTE:-

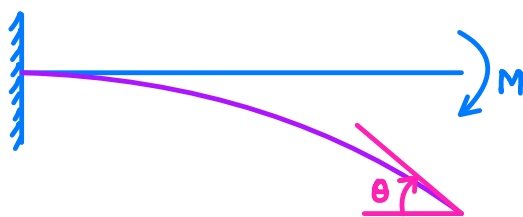
If force "P" is already applied to the bar and another force "Φ" is now applied so that bar deforms further by 'x', in this case

work done by "P" is $w_1 = A_1 = Px$

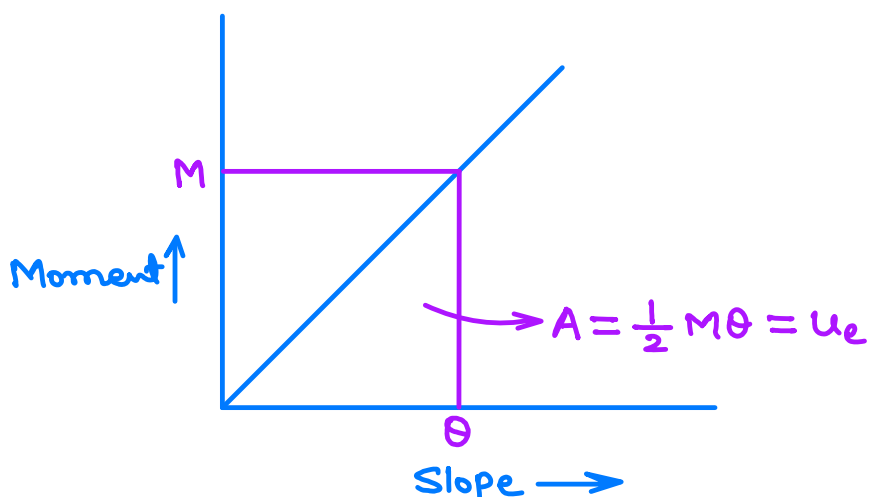
work done by "Φ" is $w_2 = A_2 = \Phi x$.



b> Due to moment



$$U_e = \int_0^{\theta} M d\theta = \frac{1}{2} M \theta = \text{Area Under Moment-Slope curve.}$$



Note :-

If moment M is already applied and another loading " N " deforms the structure by " α " then workdone by M , $w_1 = M\alpha$
and workdone by " N ", $w_2 = \frac{1}{2} N\alpha$

★ Strain Energy stored in structures (U_i)

a> Strain energy due to axial loading $= \int \frac{1}{2} P \cdot \frac{P dx}{AE} = \int \frac{P^2 dx}{AE}$

$AE \rightarrow$ Axial rigidity.

$\frac{AE}{L} \rightarrow$ Axial stiffness

b> Strain energy due to Bending moment $= \int \frac{M^2 dx}{2 EI}$

$EI \rightarrow$ Flexural rigidity

$\frac{EI}{L} \rightarrow$ Flexural stiffness.

c> Strain Energy due to Shear force $= K \int \frac{V^2 dx}{2 AG_s}$

$AG_s \rightarrow$ Shear rigidity

$\frac{AG_s}{L} \rightarrow$ Shear stiffness

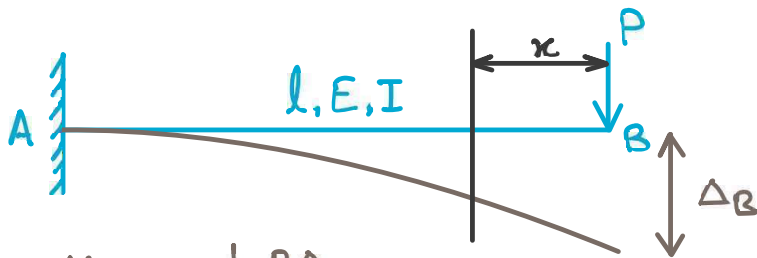
$K \rightarrow$ Form factor (depends on shape of cross-section)

d> Strain energy due to Torsional Moment $= \int \frac{T^2 dx}{2 GJ}$

$GJ \rightarrow$ Torsional rigidity

$\frac{GJ}{L} \rightarrow$ Torsional stiffness

Q. >



compute Δ_B

Solution

$$U_e = \frac{1}{2} P \Delta_B$$

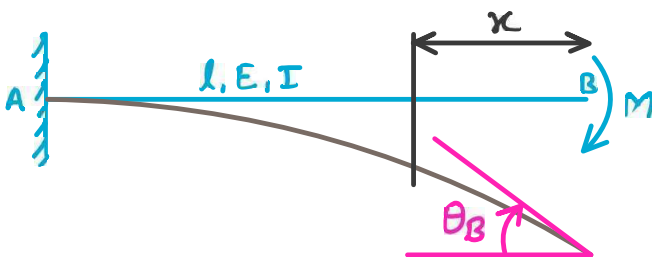
$$U_i = \int \frac{M^2 dx}{2EI} = \int_0^L \frac{(-Px)^2 dx}{2EI} = \int_0^L \frac{P^2 x^2 dx}{2EI} = \frac{P^2}{2EI} \left| \frac{x^3}{3} \right|_0^L$$

$$U_i = \frac{P^2 L^3}{6EI}$$

$$U_e = U_i$$

$$\frac{1}{2} P \Delta_B = \frac{P^2 L^3}{6EI} \Rightarrow \Delta_B = \frac{PL^3}{6EI}$$

Q. >



compute Δ_B

Solution

$$U_e = \frac{1}{2} M \theta_B$$

$$U_i = \int \frac{M^2 dx}{2EI} = \int_0^L \frac{(-M)^2 dx}{2EI} = \frac{M^2}{2EI} \int_0^L dx = \frac{M^2}{2EI} \left| x \right|_0^L$$

$$U_i = \frac{M^2 L}{2EI}$$

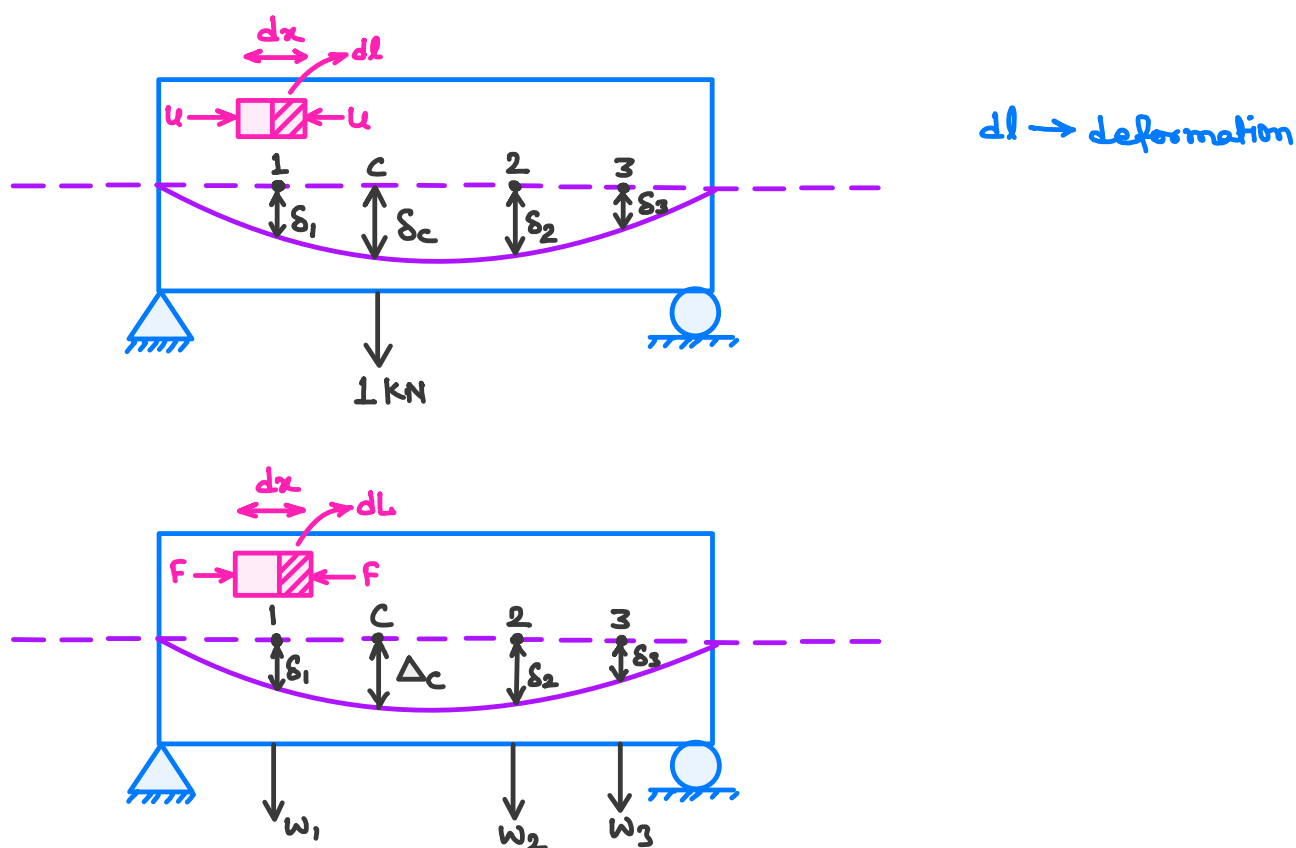
$$U_e = U_i$$

$$\frac{1}{2} M \theta_B = \frac{M^2 L}{2EI}$$

$$\theta_B = \frac{ML}{EI}$$

Note:-

- This method can be used only when load is acting and displacement is to be determined at the location of application of load.
- As if many loads are acting there would be unknown displacements at the location of application of the load. Hence, external workdone equation (U_e) is not applicable.

Method of virtual work (unit load method)

- $1 =$ Unit load applied at the point at which deflection is to be found out due to external loading (w_1, w_2, w_3).
- $\delta_1, \delta_2, \delta_3, \delta_c$ are deflection at point 1, 2, 3 & C due to unit load at point " C ".

- U = internal force developed at any point in the beam due to the effect of unit load.
- w_1, w_2, w_3 are external load applied at point 1, 2, 3.
- $\Delta_1, \Delta_2, \Delta_3, \Delta_c$ are deflection at point 1, 2, 3, c due to external load w_1, w_2, w_3 .
- F is the internal force developed in the differential element dx due to the effect of external load w_1, w_2, w_3 .
- Load " U " causes the displacement of " dl " in differential element " dx ".
- Load " F " causes the displacement of " dL " in differential element " dx ".

Total external workdone due to unit load and external load

$$U_e = U_{e_1} + U_{e_2}$$

$$U_{e_1} = \frac{1}{2} \times 1 \times \delta_c + 1 \times \Delta_c$$

$$U_{e_2} = \frac{1}{2} \times w_1 \times \Delta_1 + \frac{1}{2} \times w_2 \times \Delta_2 + \frac{1}{2} \times w_3 \times \Delta_3$$

$$U_e = \frac{1}{2} \times 1 \times \delta_c + 1 \times \Delta_c + \frac{1}{2} \times w_1 \times \Delta_1 + \frac{1}{2} \times w_2 \times \Delta_2 + \frac{1}{2} \times w_3 \times \Delta_3$$

Total Strain energy Stored (internal workdone)

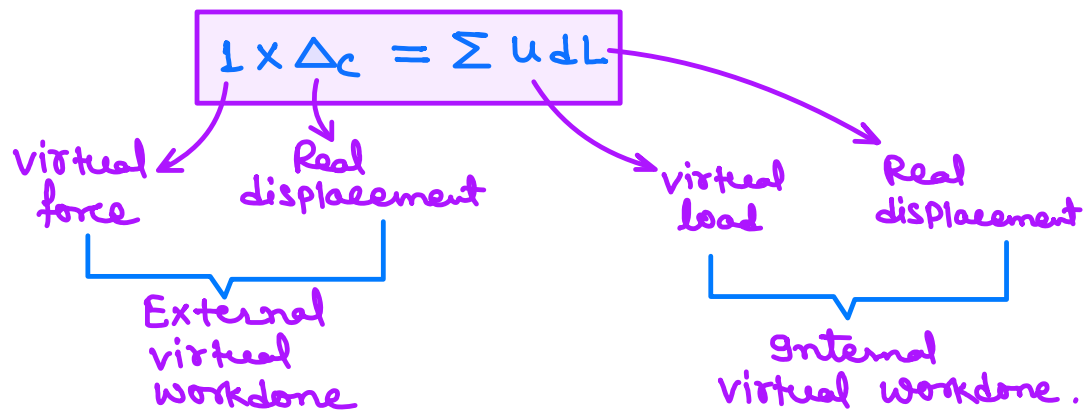
$$U_i = \frac{1}{2} \sum U dl + \frac{1}{2} \sum F dL + \sum U dL$$

Now, from energy principle,

$$U_e = U_i$$

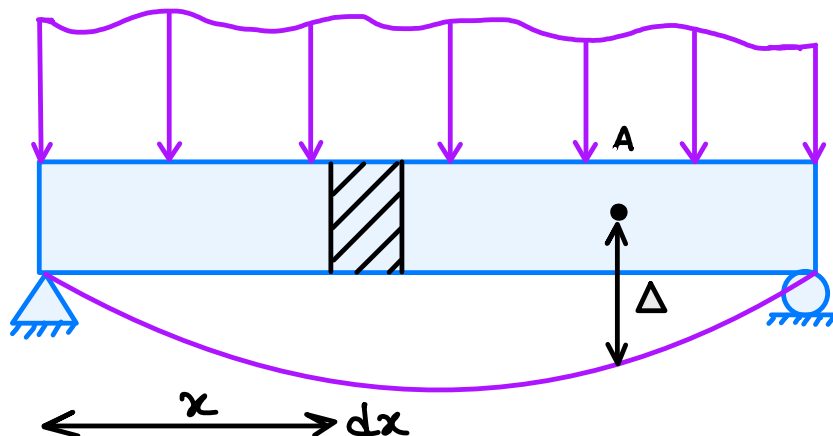
$$\frac{1}{2} \times 1 \times \delta_c = \frac{1}{2} \sum U \cdot dl$$

$$\frac{1}{2} \times w_1 \times \Delta_1 + \frac{1}{2} \times w_2 \times \Delta_2 + \frac{1}{2} \times w_3 \times \Delta_3 = \frac{1}{2} \sum F dL$$



- This method can be used to find out deflection of truss joint deflection and rotation in beams & frames due to external load.

★ Application of Unit load method/Method of virtual work in beams/frames



- Apply unit load at point "A" in the direction of desired deflection
- Due to this unit load calculate internal virtual moment "m" in the beam at any section x .
- When external load is applied now, point A moves by amount Δ & the element dx is deformed/rotates by an amount of $d\theta$ and has moment generated $= M$.

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \Rightarrow \frac{d\theta}{dx} = \frac{M}{EI} \Rightarrow d\theta = \frac{M}{EI} dx$$

Now, External virtual workdone, $U_e = 1 \times \Delta$

Internal virtual workdone, $U_i = \int_0^L m d\theta$

$$U_e = U_i$$

$$1 \times \Delta = \int_0^L m d\theta$$

$$\Delta = \int_0^L m \cdot \frac{M}{EI} dx$$

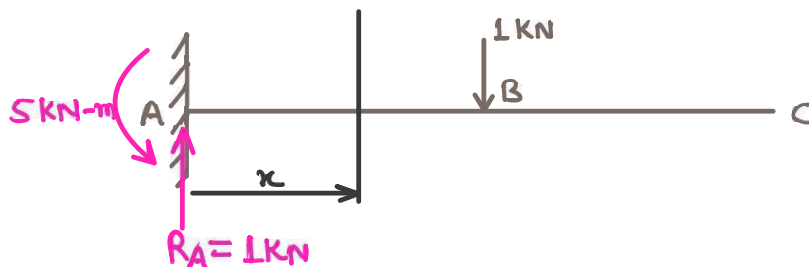
on similar line,

$$\theta = \int_0^L m \cdot \frac{M}{EI} dx$$

Q Compute θ_B & Δ_B



Solution a. for deflection at B



$$m = -1 \times (5 - x) = -(5 - x), \quad \text{for } 0 \leq x \leq 5$$

$$m = 0, \quad \text{for } x > 5$$



$$M = -3(10 - x) \quad , \quad \text{for } 0 \leq x \leq 10$$

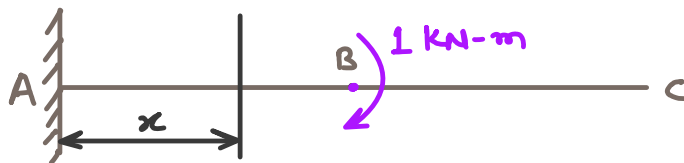
$$\Delta = \int_0^L m \cdot \frac{M}{EI} dx = \int_0^5 + (5-x) \frac{3(10-x)}{EI} dx + \int_5^{10} 0 \times \frac{-3(10-x)}{EI} dx$$

$$\Delta = \frac{3}{EI} \int_0^5 (50 - 15x + x^2) dx$$

$$\Delta = \frac{3}{EI} \left| 50x - \frac{15x^2}{2} + \frac{x^3}{3} \right|_0^5$$

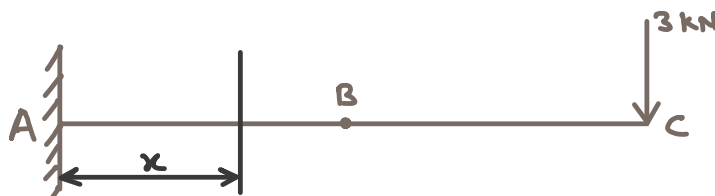
$$\Delta = \frac{312.5}{EI}$$

b.>



$$m = -1 \quad \text{for } 0 \leq x \leq 5$$

$$M = 0 \quad \text{for } 5 \leq x \leq 10$$



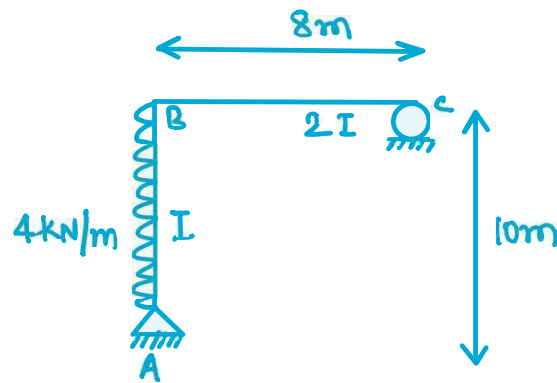
$$M = -3(10 - x) \quad , \quad \text{for } 0 \leq x \leq 10$$

$$1 \times \theta = \int_0^L m \frac{M}{EI} dx = + \int_0^5 3(10-x) dx + \int_5^{10} 0 \times 3(10-x) dx$$

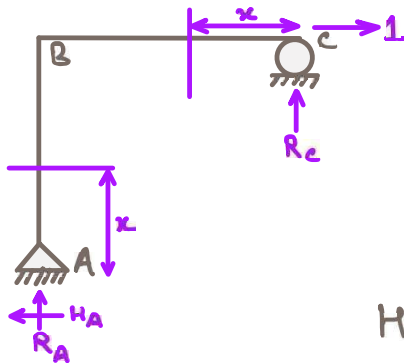
$$\theta = \frac{3}{EI} \left(10x - \frac{x^2}{2} \right)_0^5$$

$$\theta = \frac{112.5}{EI}$$

Q. > Compute Δ_{CH}



Solution



$$R_A + R_C = 0$$

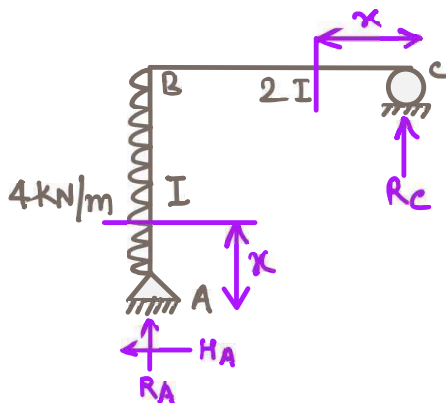
$$R_C \cdot 8 = 1 \times 10$$

$$R_C = \frac{10}{8}, \quad R_A = -\frac{10}{8}$$

$$H_A - 1 = 0 \Rightarrow H_A = 1. \quad (\text{for virtual unit load})$$

$$m_{AB} = 1 \times x \quad \text{for } 1 \leq x \leq 10$$

$$m_{CB} = \frac{10}{8} \cdot x \quad \text{for } 0 \leq x \leq 8$$



for Actual External load

$$R_A + R_C = 0$$

$$H_A = 4 \times 10 = 40$$

$$R_C \times 8 = 4 \times 10 \times 10/2$$

$$R_C = 25, \quad R_A = -25$$

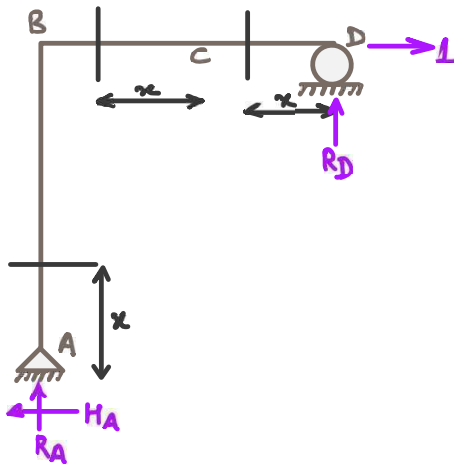
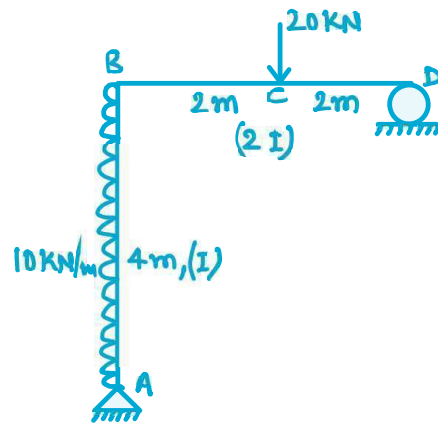
$$M_{AB} = 40x - \frac{4x^2}{2}, \quad \text{for } 0 \leq x \leq 10$$

$$M_{CB} = 25x, \quad \text{for } 0 \leq x \leq 8$$

$$\therefore 1 \cdot \Delta_{CH} = \int m \frac{M}{EI} dx = \int_0^{10} x \frac{(40x - 2x^2)}{EI} dx + \int_0^8 \frac{10}{8} x \frac{25x}{E(2I)} dx$$

$$\Delta_{CH} = \frac{11000}{EI} \rightarrow$$

Q. Compute Δ_{DH} & θ_D



$$R_A + R_D = 0, \quad H_A = 1$$

$$R_D \cdot 4 = 1 \cdot 4$$

$$R_D = 1, \quad R_A = -1$$

$$m_{AB} = H_A \cdot x = 1 \cdot x, \quad 0 \leq x \leq 4$$

$$m_{CD} = 1 \cdot x, \quad 0 \leq x \leq 2$$

$$m_{CB} = R_D \cdot (2+x) = 1 \cdot (2+x), \quad 0 \leq x \leq 2$$

$$H_A = 40$$

$$R_A + R_D = 20$$

$$R_D \cdot 4 = 20 \times 2 + 10 \times 4 \times \frac{4}{2}$$

$$R_D = 30, \quad R_A = -10$$

$$M_{AB} = 40x - 10 \times \frac{x^2}{2} = 40x - 5x^2$$

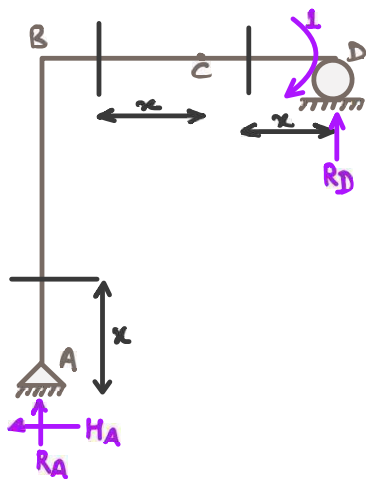
$$M_{DC} = 30x$$

$$M_{CB} = 30(x+2) - 20x = 60 + 10x$$

$$1. \Delta_{DH} = \int m \frac{M}{EI} dx = \int_0^4 x \cdot 1 \frac{(40x - 5x^2)}{EI} dx + \int_0^2 x \cdot \frac{30x}{E(2I)} dx + \int_0^2 \frac{1(2+x)}{E(2I)} (60 + 10x) dx$$

$$\Delta_{DH} = \frac{786.69}{EI} \rightarrow$$

b.)



$$R_A + R_D = 0, \quad H_A = 0$$

$$R_D \times 4 = 1 \Rightarrow R_D = \frac{1}{4}$$

$$R_A = -\frac{1}{4}$$

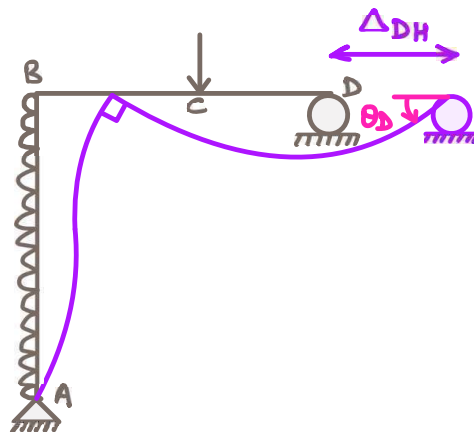
$$m_{AB} = 0$$

$$m_{DC} = \frac{1}{4} \cdot x - 1$$

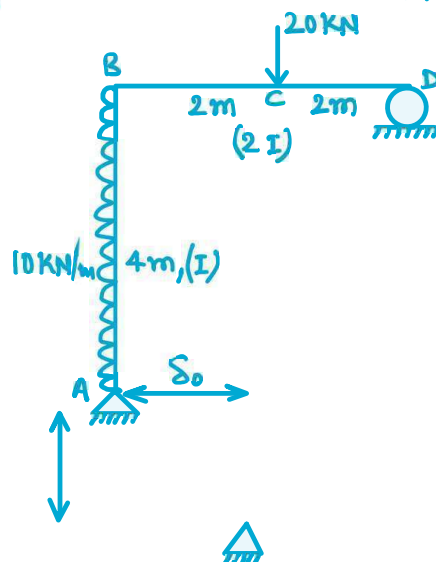
$$m_{CB} = \frac{1}{4}(x+2) - 1 = \frac{x}{4} - \frac{1}{2}$$

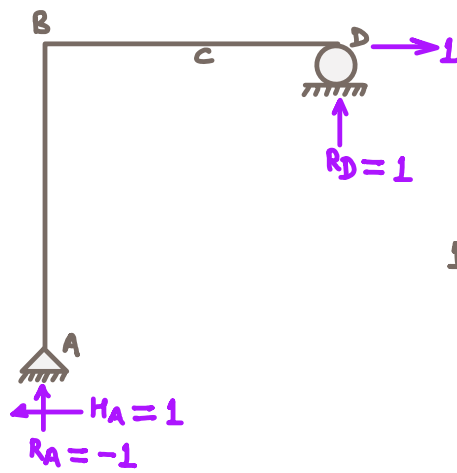
$$1. \theta_D = \int m \frac{M}{EI} dx = \int_0^2 0 \cdot \frac{(40x - 5x^2)}{EI} dx + \int_0^2 \left(\frac{1}{4}x - 1\right) \frac{30x}{2EI} dx + \int_0^2 \left(\frac{x}{4} - \frac{1}{2}\right) \frac{(60 + 10x)}{2EI} dx$$

$$\theta_D = -\frac{36.67}{EI}$$



Q. In the above case Support A settles down by δ_1 and moves to the right by δ_2 then find Δ_{DH} & θ_D .

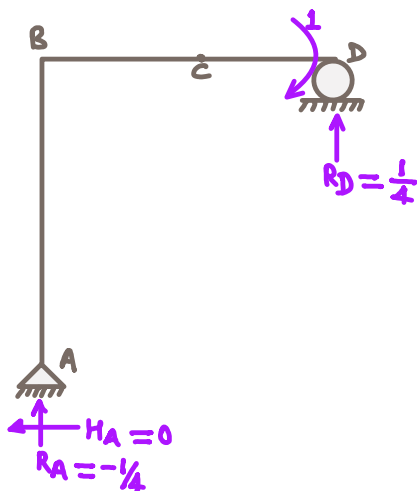


Solution

$$1 \times \Delta_{DH} + R_D \cdot 0 + (-R_A \cdot \delta_1) + (-H_A \times \delta_0) = \int_0^L m \frac{M}{EI} dx$$

$$\Delta_{DH} = \frac{786.67}{EI} + R_A \delta_1 + H_A \delta_0$$

$$\Delta_{DH} = \frac{786.67}{EI} - \delta_1 + \delta_0$$

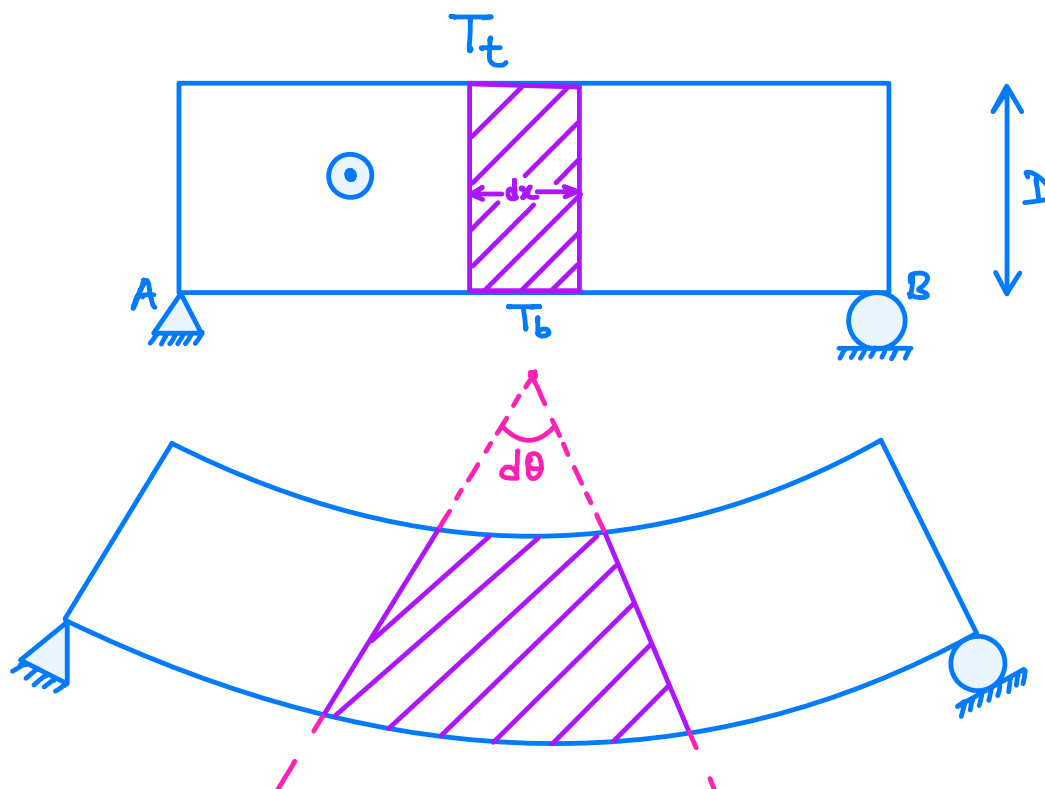


$$1 \cdot \theta_D + R_D \cdot 0 + (-H_A \cdot \delta_0) + (-R_A \cdot \delta_1) = \int_0^L m \cdot \frac{M}{EI} dx$$

$$\theta_D = \frac{-36.67}{EI} + H_A \delta_0 + R_A \delta_1$$

$$\theta_D = \frac{-36.67}{EI} - \frac{1}{4} \delta_1$$

Deflection of beam due to temperature changes



$$CD = EF = \frac{\{dx + \alpha dx (T_b - T_0)\} - \{dx + \alpha dx (T_t - T_0)\}}{2}$$

$$CD = EF = \frac{\alpha dx (T_b - T_t)}{2} = \frac{\alpha dx \Delta T}{2}$$

$$\tan \frac{d\theta}{2} = \frac{d\theta}{2} = \frac{\alpha dx \Delta T}{2 \cdot D}$$

$$d\theta = \frac{\alpha dx \Delta T}{D}$$

Here,

$$\Delta T = T_b - T_t, (\text{Assuming } T_b > T_t)$$

Note:-

$$\text{If } T_t > T_b \Rightarrow \Delta T = -ve.$$

$d\theta$ will have hogging moment.

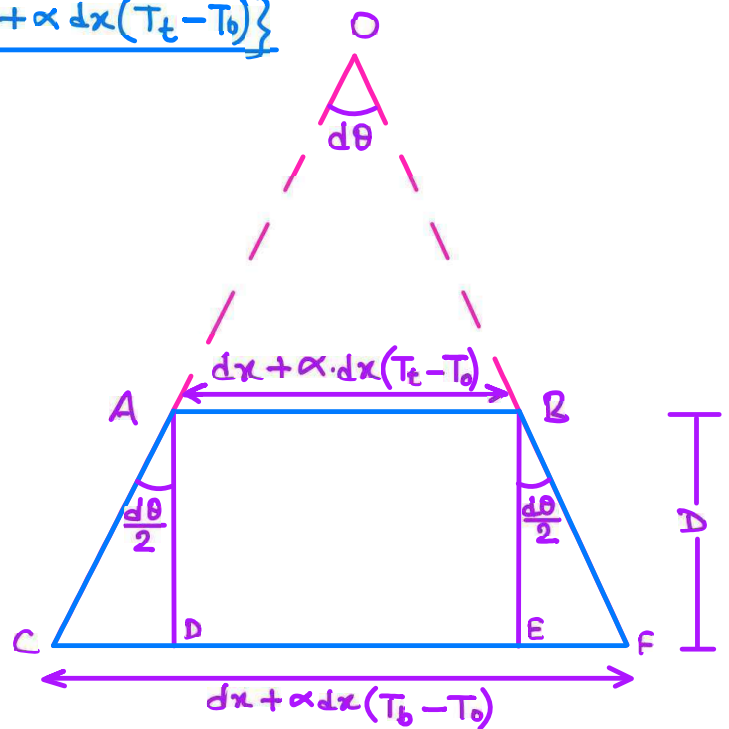
from principle of virtual workdone

$$1 \cdot \Delta = \int m \cdot d\theta$$

$$\text{Hence, } \Delta = \int m \left(\frac{\alpha \cdot \Delta T}{D} dx \right)$$

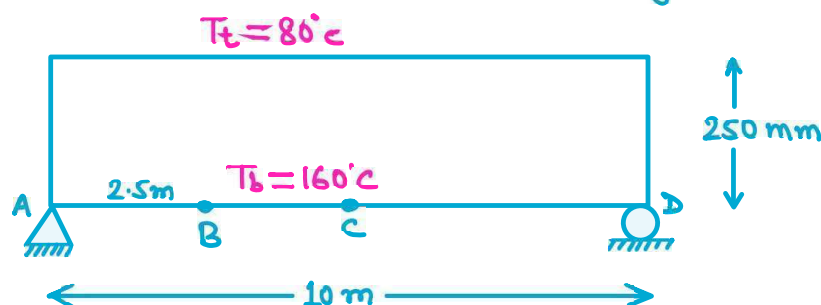
$$\text{Similarly, } 1 \cdot \theta = \int m \cdot d\theta$$

$$\theta = \int m \left(\frac{\alpha \Delta T}{D} dx \right)$$



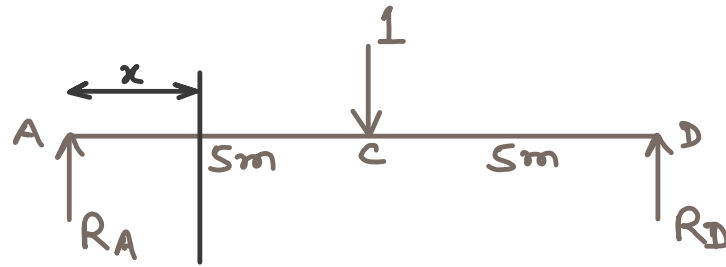
Q. Compute Δ_c , Δ_B for the given case. $\alpha = 6.5 \times 10^{-6}/^\circ\text{C}$

Also compute Δ_B if support A settles by 5mm.



Solution.

a.)



$$R_A = R_D = \frac{1}{2}$$

$$m = \frac{1}{2}x, \quad \text{for } 0 \leq x \leq 5$$

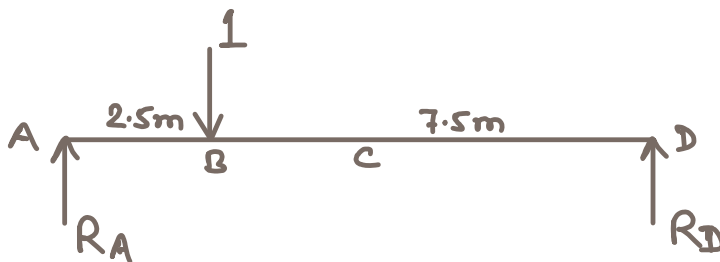
$$d\theta = \frac{\alpha \Delta T dx}{D} = \frac{6.5 \times 10^{-6} \times (160 - 80) dx}{250 \times 10^{-3}} = 2.08 \times 10^{-3} dx.$$

$$1. \Delta_c = 2 \cdot \int_0^5 \frac{x}{2} \times 2.08 \times 10^{-3} dx$$

$$\Delta_c = 2.08 \times 10^{-3} \left| \frac{x^2}{2} \right|_0^5$$

$$\Delta_c = 0.026 \text{ m} = 26 \text{ mm}.$$

b.)



$$R_A + R_D = 1 \quad \text{--- (i)}$$

$$R_A \cdot 10 = 1 \times 7.5 \Rightarrow R_A = \frac{3}{4} \text{ kN}, \quad R_D = \frac{1}{4} \text{ kN}.$$

$$m_{AB} = R_A \cdot x = \frac{3}{4}x, \quad 0 \leq x \leq 2.5$$

$$m_{BC} = \frac{3}{4}(x + 2.5) - 1 \cdot x \quad 2.5 \leq x \leq 7.5$$

External workdone = internal workdone.

$$1. \Delta_B = \int m d\theta = \int_0^{2.5} \frac{3}{4}x \times 2.08 \times 10^{-3} dx + \int_{2.5}^{7.5} \left(\frac{15}{8} - \frac{x}{4} \right) 2.08 \times 10^{-3} dx$$

$$\Delta_B = \frac{3}{4} \times 2.08 \times 10^{-3} \left| \frac{x^2}{2} \right|_0^{2.5} + 2.08 \times 10^{-3} \left| \frac{15}{8}x - \frac{x^2}{8} \right|_{2.5}^{7.5}$$

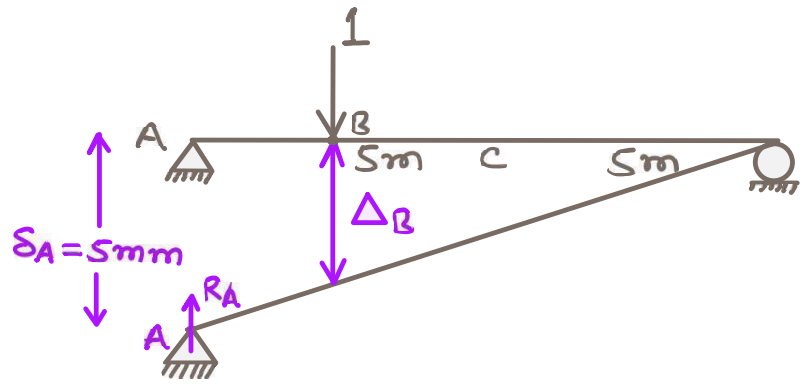
$$\Delta_B = 19.5 \times 10^{-3} \text{ m} = 19.5 \text{ mm}.$$

c>

$$1 \cdot \Delta_B - R_A \cdot \delta_A = \int m \, d\theta$$

$$\Delta_B = 19.5 + \frac{3}{4} \times 5$$

$$\Delta_B = 23.25 \text{ mm.}$$



Castigliano's theorem (Method of least work)

— As per this theorem

a> Partial differential of Strain energy of the structure with respect to load at any point is equal to deflection at that point.

$$\frac{\partial u}{\partial P_j} = \Delta_j$$

b> Partial differential of Strain energy of the structure with respect to couple/moment at any point is equal to slope at that point

$$\frac{\partial u}{\partial M_j} = \theta_j$$

★ Assumptions in this theorem

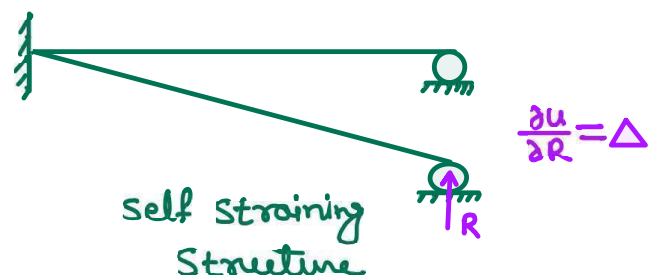
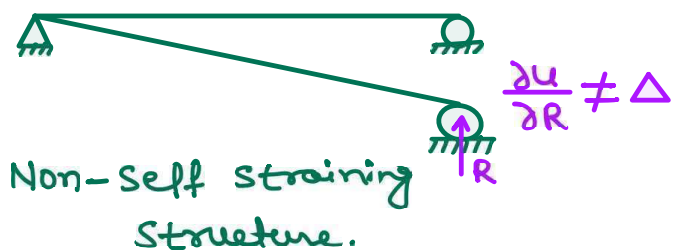
This theorem is applied to the structure that has constant temperature, unyielding support, linear elastic material response.

Note:-

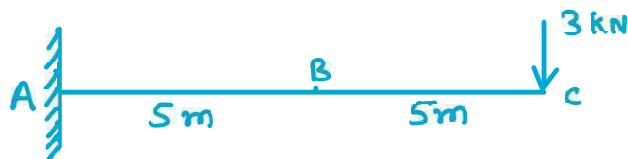
a> In unit load method these assumptions were not required.

b> Castigliano's theorem however can be applied in case of support settlement provided that the structure is self straining.

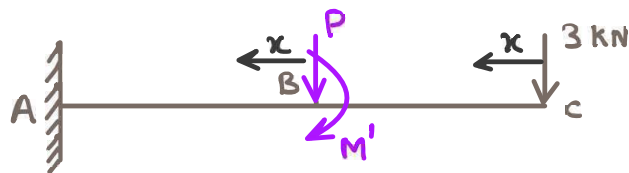
- c> If in any structure all support reactions can be found out using equation of static equilibrium, the structure is said to be externally determinate otherwise externally indeterminate.
- d> All externally indeterminate structures are self straining.



Q> Compute Slope and deflection at Point "B".



Solution



Section	M	$\frac{\partial M}{\partial P}$	$\frac{\partial M}{\partial M'}$	x (Range)
Bc	$-3x$	0	0	(0-5)
AB	$-3(x+5) - Px - M'$	$-x$	-1	(0-5)

$$\Delta = \frac{\partial u}{\partial P} = \frac{\partial}{\partial P} \left[\int \frac{M^2}{2EI} \cdot dx \right] = \int_0^L 2M \frac{\partial M}{\partial P} \cdot \frac{1}{2EI} dx = \int_0^L \frac{M \frac{\partial M}{\partial P}}{EI} dx$$

$$\Delta = \int_0^5 -3x \times \frac{0}{EI} dx + \int_0^5 \frac{(3x+15)x}{EI} dx$$

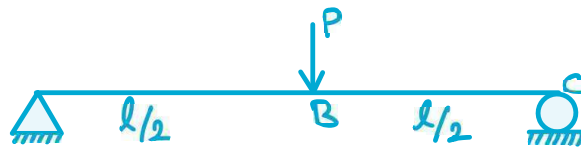
$$\Delta = \frac{312.5}{EI} \downarrow$$

$$\theta = \frac{\partial u}{\partial M'} = \frac{\partial}{\partial M'} \left[\int \frac{M^2}{2EI} dx \right] = \int_0^L 2M \frac{\partial M}{\partial M'} \cdot \frac{1}{2EI} dx = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M'} dx$$

$$\theta = -3x \times \frac{0}{EI} dx + \int_0^5 \frac{(3x+15)}{EI} dx$$

$$\theta = \frac{112.5}{EI}$$

Q> Compute Δ_B



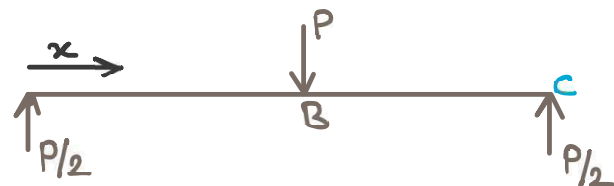
Solution:-

$$M = \frac{P}{2} x$$

$$\Delta = \int \frac{M}{EI} \frac{\partial M}{\partial P} dx$$

$$= 2 \int_0^{l/2} \frac{P}{2} x \cdot \frac{x}{2} \frac{1}{EI} dx = \frac{P}{2EI} \int_0^{l/2} x^2 dx = \frac{P}{2EI} \left| \frac{x^3}{3} \right|_0^{l/2}$$

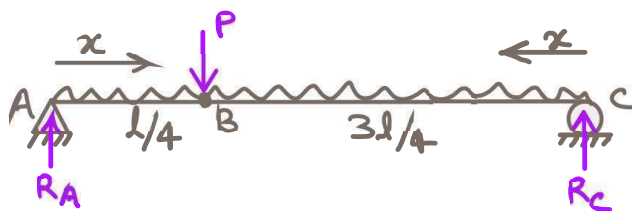
$$\Delta = \frac{Pl^3}{48EI}$$



Q> Compute Δ_B



Solution



$$R_A + R_C = P + wl$$

$$R_A \cdot l = P \cdot \frac{3l}{4} + wl \cdot \frac{l}{2}$$

$$R_A = \frac{3P}{4} + \frac{wl}{2}, \quad R_C = \frac{P}{4} + \frac{wl}{2}$$

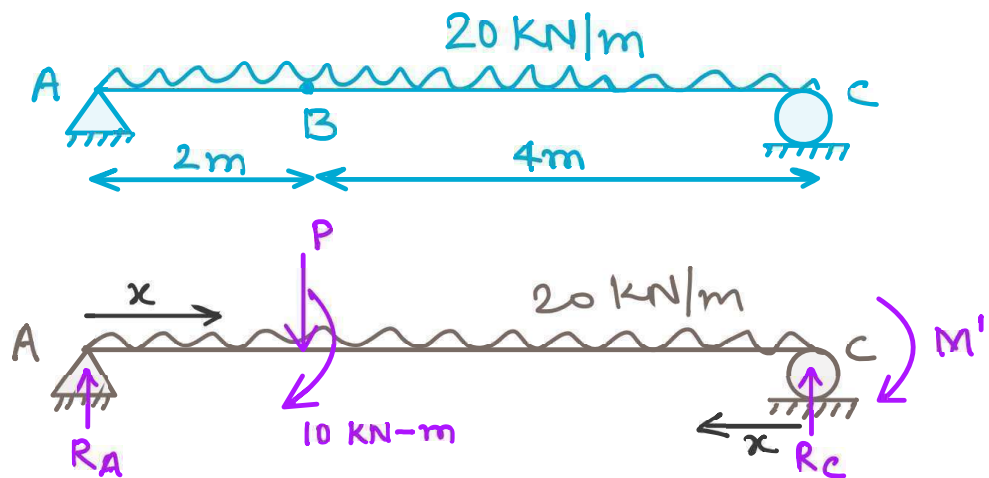
Section	M	$\frac{\partial M}{\partial P}$	x
AB	$\left(\frac{3P}{4} + \frac{wl}{2}\right)x - \frac{wx^2}{2}$	$\frac{3x}{4}$	$(0 - l/4)$
BC	$\left(\frac{P}{4} + \frac{wl}{2}\right)x - \frac{wx^2}{2}$	$\frac{x}{4}$	$(0 - \frac{3l}{4})$

$$\Delta = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P} dx \Big|_{P=0}$$

$$\Delta = \int_0^L \frac{\left(\frac{wlx}{2} - \frac{wx^2}{2}\right) \left(\frac{3x}{4}\right)}{EI} dx + \int_0^L \frac{\left(\frac{wlx}{2} - \frac{wx^2}{2}\right) \frac{x}{4}}{EI} dx$$

$$\Delta = \underline{\hspace{2cm}}$$

Φ> Compute Δ_B & θ_C



$$R_A + R_C = P + 20 \times 6 \quad \text{--- (i)}$$

$$R_A \cdot 6 = -10 - M' + 20 \times \frac{6^2}{2} + P \times 4$$

$$R_A = \frac{350 - M' + 4P}{6}, \quad R_C = \frac{370 + M' + 2P}{6}$$

Section	M	$\frac{\partial M}{\partial P}$	$\frac{\partial M}{\partial M'}$	x
AB	$\left(\frac{350 - M' + 4P}{6}\right)x - 20 \frac{x^2}{2}$	$\frac{2}{3}x$	$-\frac{x}{6}$	$(0 - 2)$
440 BC	$\left(\frac{370 + M' + 2P}{6}\right)x - \frac{20x^2}{2} - M'$	$\frac{1}{3}x$	$\frac{x}{6} - 1$	$(0 - 4)$

$$\Delta_B = \frac{\partial U}{\partial P} = \int m \frac{\partial M}{\partial P} dx \Big|_{P=0, M'=0}$$

$$= \int_0^2 \frac{(350 \frac{x}{6} - 10x^2) \frac{2}{3} x dx}{EI} + \int_0^4 \frac{(370 \frac{x}{6} - 10x^2) (\frac{x}{3})}{EI} dx$$

$$\Delta_B = \frac{1}{EI} \left[\left| \frac{350}{6} \cdot \frac{2}{3} \cdot \frac{x^3}{3} - 10 \cdot \frac{2}{3} \cdot \frac{x^4}{4} \right|_0^2 + \frac{370}{6} \cdot \frac{1}{3} \cdot \frac{x^3}{3} - \frac{10}{3} \cdot \frac{x^4}{4} \right]_0^4$$

$$\Delta_B = \frac{302.218}{EI}$$

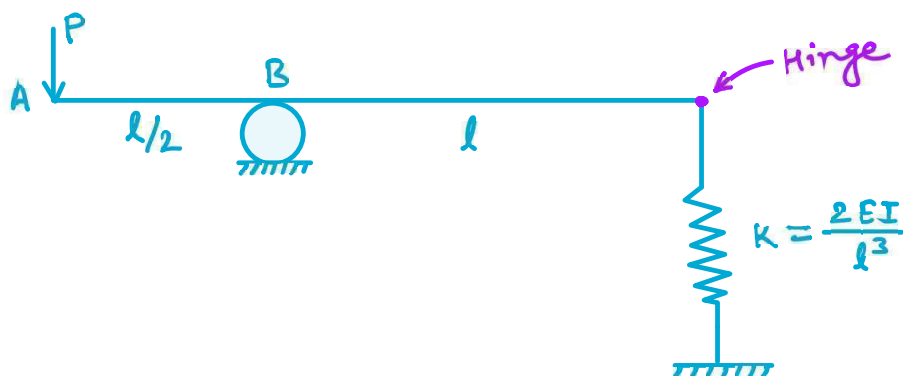
$$\theta_C = \int m \frac{\partial M}{\partial M'} dx \Big|_{P=0, M'=0}$$

$$\theta_C = \int_0^2 \frac{(350 \frac{x}{6} - 10x^2) (-\frac{x}{2}) dx}{EI} + \int_0^4 \frac{(370 \frac{x}{6} - 10x^2) (\frac{x}{6} - 1) dx}{EI}$$

$$\theta_C = \frac{1}{EI} \left[\left| -\frac{350}{6} \cdot \frac{1}{2} \cdot \frac{x^3}{3} + 10 \cdot \frac{2}{3} \cdot \frac{x^4}{4} \right|_0^2 + \frac{370}{6} \cdot \frac{1}{6} \cdot \frac{x^3}{3} - \frac{370}{6} \cdot \frac{x^2}{2} - \frac{10}{6} \cdot \frac{x^4}{4} + \frac{10x^3}{3} \right]_0^4$$

$$\theta_C = \frac{-186.66}{EI} \quad (\curvearrowright)$$

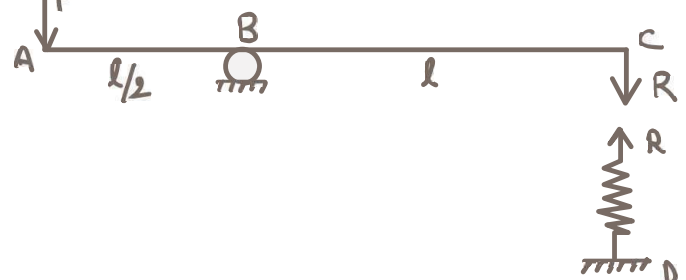
Q. >



Solution

$$U = U_{\text{beam}} + U_{\text{spring}}$$

$$P \cdot \frac{l}{2} = R \cdot L \Rightarrow R = \frac{P}{2}$$



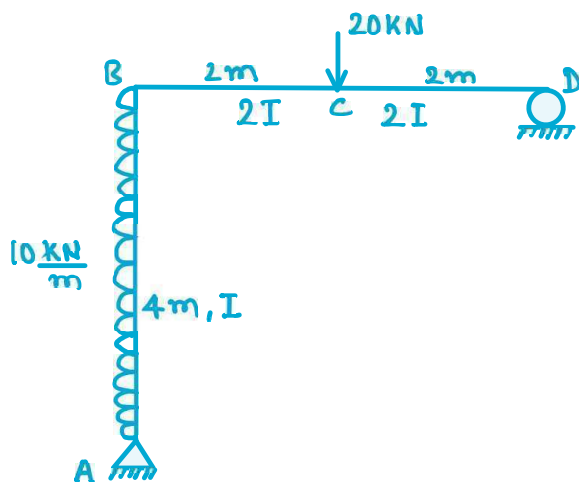
$$U_{beam} = \int_0^{l/2} \frac{(-Px)^2}{2EI} dx + \int_0^l \frac{(-Rx)^2}{2EI} dx$$

$$= \frac{P^2}{2EI} \left| \frac{x^3}{3} \right|_0^{l/2} + \frac{P^2}{8EI} \left| \frac{x^3}{3} \right|_0^l = \frac{P^2 l^3}{16EI}$$

$$U_{spring} = \frac{1}{2} \cdot R \cdot \delta = \frac{1}{2} R \cdot \frac{R}{k} = \frac{1}{2} \left(\frac{P}{2} \right)^2 \cdot \frac{l^3}{2EI} = \frac{P^2 l^3}{16EI}$$

$$U = \frac{P^2 l^3}{8EI}$$

$$\Delta_A = \frac{\partial U}{\partial P} = \frac{2Pl^3}{8EI} = \frac{Pl^3}{4EI}$$



Solution :

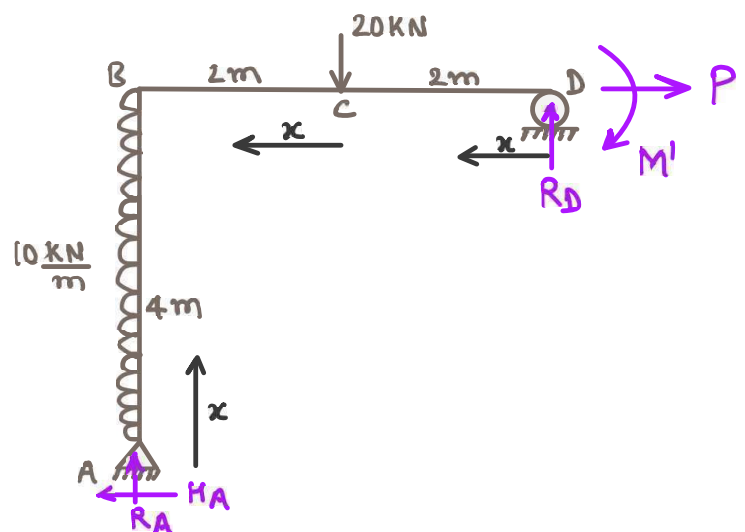
$$H_A = 10 \times 4 + P = 40 + P$$

$$R_A + R_D = 20$$

$$R_D \cdot 4 = M' + P \cdot 4 + 20 \cdot 2 + 10 \cdot \frac{4^2}{2}$$

$$R_D = 30 + P + \frac{M'}{4}$$

$$R_A = -\left(10 + \frac{M'}{4} + P\right)$$



Section	M	$\frac{\partial M}{\partial P}$	$\frac{\partial M}{\partial M'}$	x
AB	$(40+P)x - \frac{10x^2}{2}$	x	0	(0-4)
BC	$-20x - M' + (30+P+\frac{M'}{4})(x+2)$	$\frac{x}{2}$	$-1 + \frac{(x+2)}{4}$	(0-2)
CD	$(30+P+\frac{M'}{4})x - M'$	x	$\frac{x}{4} - 1$	(0-2)

$$\Delta_{DH} = \int \frac{M}{EI} \frac{\partial M}{\partial P} dx \Big|_{\substack{P=0 \\ M=0}} =$$

$$= \int_0^4 \left(\frac{40x - 5x^2}{EI} \right) x dx + \int_0^2 \frac{(-20x + 30x + 60)}{2EI} (x+2) dx + \int_0^2 \frac{30x \cdot x}{2EI} dx$$

$$\Delta_{DH} = \frac{1}{EI} \left| \frac{40x^3}{3} - \frac{5x^3}{3} \right|_0^4 + \frac{1}{2EI} \left| -\frac{20x^3}{3} + \frac{30x^3}{3} + \frac{60x^2}{2} - \frac{40x^2}{2} + \frac{60x^2}{2} + 120x \right|_0^2 + \frac{1}{2EI} \left| \frac{30x^3}{3} \right|_0^2$$

$$\Delta_{DH} = \frac{786.67}{EI}$$

$$\theta_D = \int \frac{M}{EI} \frac{\partial M}{\partial M'} dx$$

$$= \int_0^4 \left(\frac{40x - 5x^2}{EI} \right) \cdot 0 dx + \int_0^2 \frac{(10x+10)}{2EI} \left(\frac{x}{4} - \frac{1}{2} \right) dx + \int_0^2 \frac{30x}{2EI} \left(\frac{x}{4} - 1 \right) dx$$

$$= \frac{-36.67}{EI}$$

Q. >



Compute Reaction at Propped end.

Solution $\frac{\partial u}{\partial R} = \Delta_A = 0$

$$\int_0^l \frac{M}{EI} \frac{\partial M}{\partial R} dx = 0$$



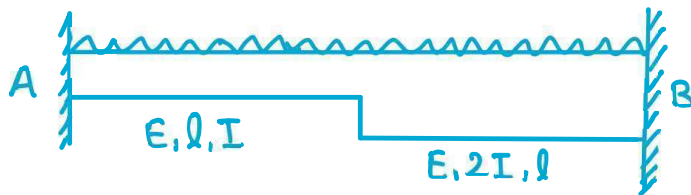
$$\int_0^l \frac{(Rx - \frac{wx^2}{2})x}{EI} dx = 0$$

$$\frac{1}{EI} \left(R \frac{x^3}{3} - \frac{wx^4}{2 \cdot 4} \right)_0^l = 0$$

$$\frac{Rl^3}{3} - \frac{wl^4}{8} = 0$$

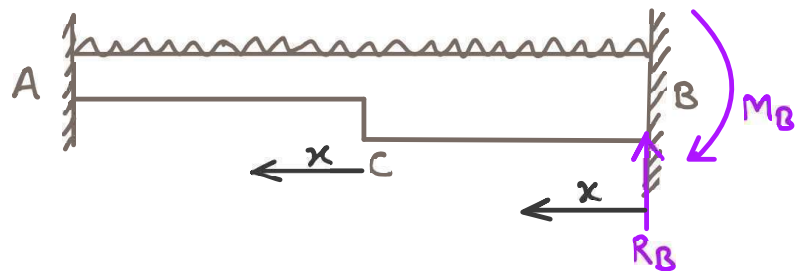
$$R = \frac{3}{8} wl.$$

Q7

Solution

$$\frac{\partial u}{\partial R_B} = 0 \quad \text{--- (i)}$$

$$\frac{\partial u}{\partial M_B} = 0 \quad \text{--- (ii)}$$

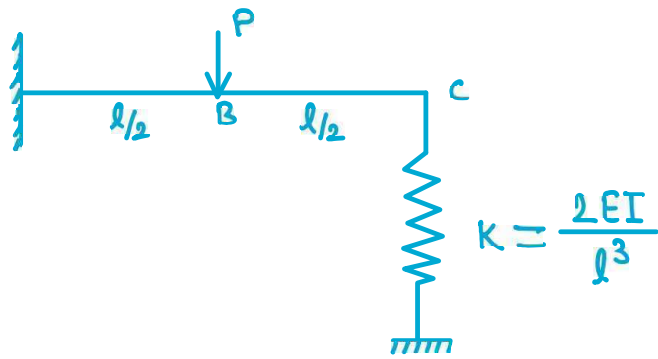


$$\frac{\partial u}{\partial R_B} = \int_0^l \frac{(R_B x - M_B - \frac{wx^2}{2})x}{2EI} dx + \int_0^l \left\{ R_B(l+x) - M_B - \frac{w(x+l)^2}{2} \right\} \frac{(l+x)}{EI} dx$$

$$= 40 R_B \cdot l - 30 wl^2 - 28 M_B = 0 \quad \text{--- (i)}$$

$$\frac{\partial u}{\partial M_B} = \frac{(R_B x - M_B - \frac{wx^2}{2})}{2EI} (-1) dx + \int_0^l \left\{ R_B(l+x) - M_B - \frac{w(x+l)^2}{2} \right\} (-1) dx = 0$$

=



Compute force in spring.

Solution

$$\Delta_{C \text{ Beam}} = \Delta_{C \text{ Spring}}$$

$$\frac{\partial u}{\partial R} = -\Delta_{C \text{ Spring}} = -\frac{R}{k} = -\frac{R \cdot l^3}{2EI}$$

$$-\frac{Rl^3}{2EI} = \int_0^{l/2} \frac{(Rx)x \, dx}{EI} + \int_0^{l/2} \frac{(R(x + \frac{l}{2}) - Px)(x + \frac{l}{2}) \, dx}{EI}$$

$$-\frac{Rl^3}{2EI} = \frac{R}{EI} \left| \frac{x^3}{3} \right|_0^{l/2} + R \left| \frac{(x + \frac{l}{2})^3}{3} \right|_0^{l/2} - P \left| \frac{x^3}{3} + \frac{x^2}{2} \cdot \frac{l}{2} \right|_0^{l/2}$$

$$-\frac{Rl^3}{2EI} = \frac{Rl^3}{EI} - \frac{5}{48} \frac{Pl^3}{EI}$$

$$\frac{R}{2} + \frac{R}{3} = \frac{5}{48} P$$

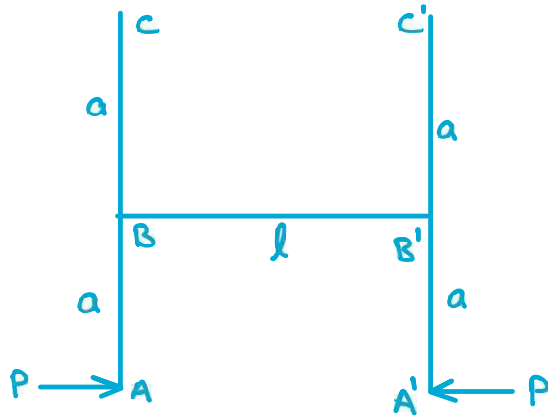
$$\underline{\underline{R = \frac{P}{8}}}$$

(OR)

$$\frac{P(\frac{l}{2})^3}{3EI} + \frac{P(\frac{l}{2})^2 \cdot (l/2)}{2EI} - \frac{Rl^3}{3EI} = \frac{Rl^3}{2EI}$$

$$\underline{\underline{R = \frac{P}{8}}}$$

Q. >

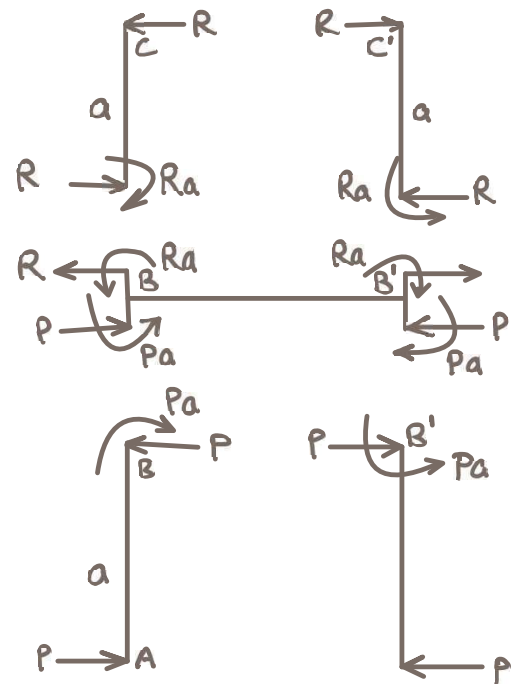
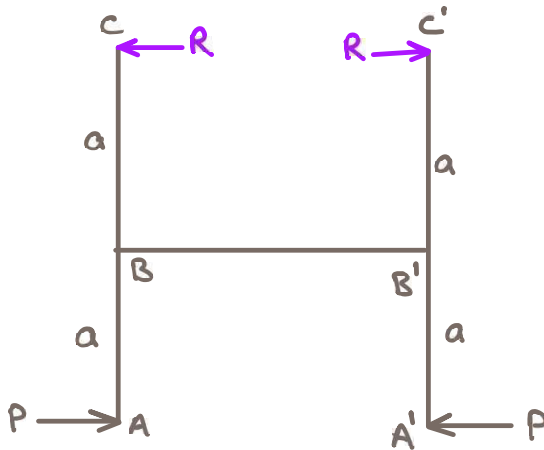


By what distance CC' moves apart?

a) If BB' is axially rigid

b) If BB' is axially flexible.

Solution



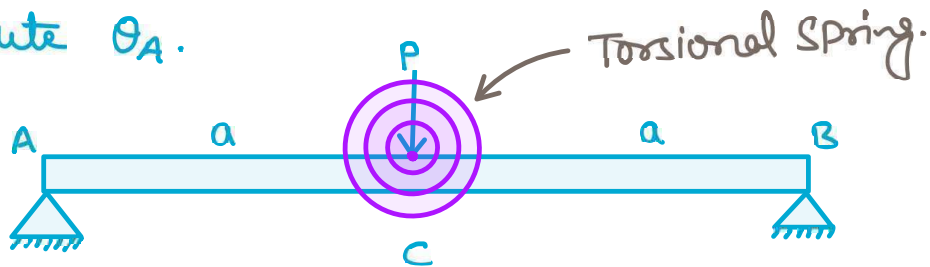
$$\frac{\partial U}{\partial R} = \int M \frac{\partial M}{\partial R} \frac{dx}{EI} \bigg|_{R=0}$$

$$\begin{aligned} \text{a)} \quad \Delta_{CC'} &= 2 \int_0^a \frac{(-Px) \cdot 0}{EI} dx + \int_0^l \frac{+(R+P)a \cdot a}{EI} dx + 2 \int_0^a \frac{(Rx) \cdot x}{EI} dx \\ &= \frac{(R+P)a^2 x}{EI} \bigg|_0^l + \frac{2R}{EI} \bigg| \frac{x^3}{3} \bigg|_0^a \bigg|_{R=0} \\ &= \frac{(R+P)a^2 l}{EI} + \frac{2Ra^3}{3EI} \bigg|_{R=0} \end{aligned}$$

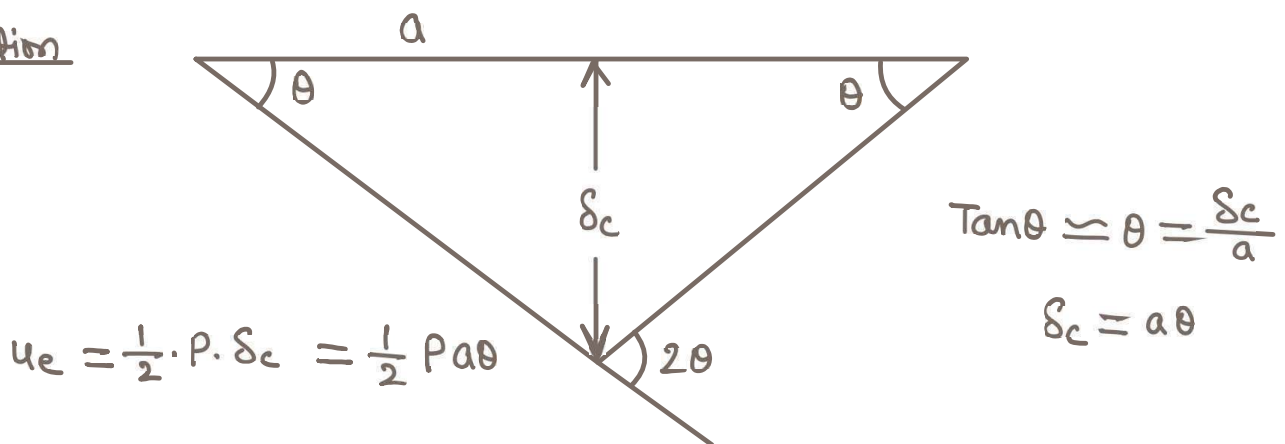
$$\Delta_{CC'} = \frac{Pa^2 l}{EI}$$

$$\begin{aligned}
 b.> \quad \Delta_{cc'} &= \frac{Pa^2l}{EI} + \int_0^l \frac{(P-R)(-1)dx}{AE} \Big|_{R=0} \\
 &= \frac{P^2la}{EI} - \frac{Pl}{AE}
 \end{aligned}$$

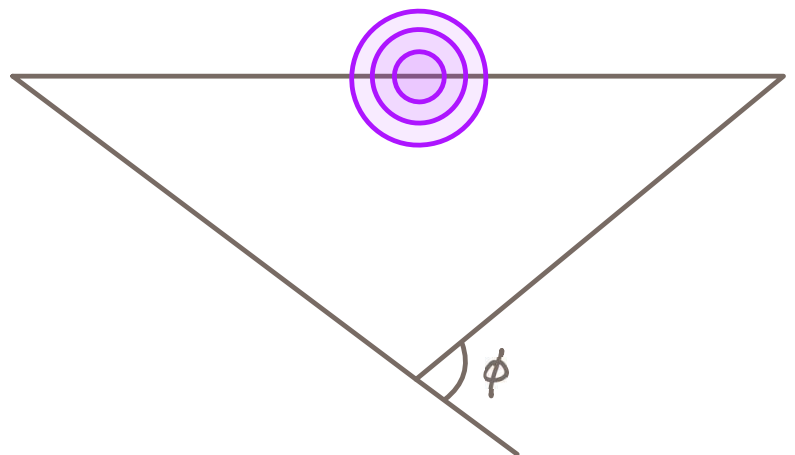
Q. compute θ_A .



Solution



$$u_j = \frac{1}{2} M\phi = \frac{1}{2} K(2\theta)(2\theta)$$



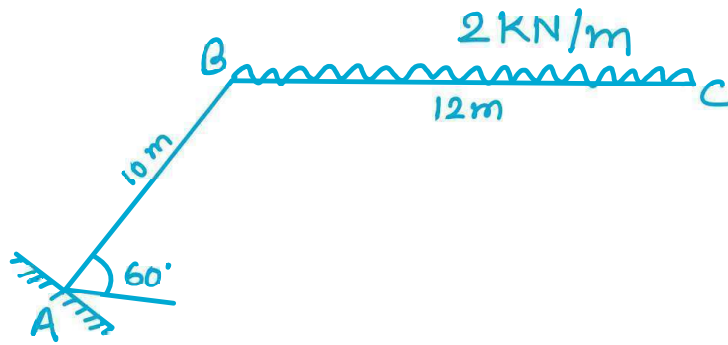
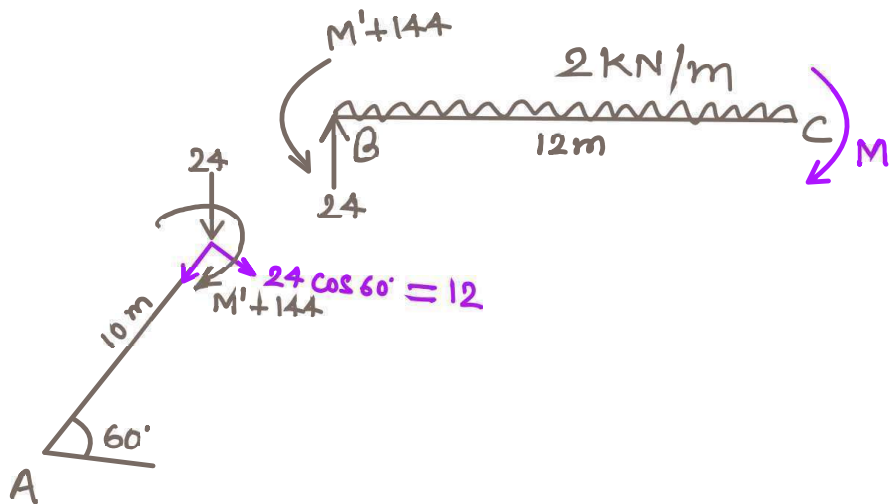
$$u_j = u_e$$

$$\frac{1}{2} K \cdot 4\theta^2 = \frac{1}{2} Pa\theta$$

$$\theta_A = \frac{Pa}{4K}$$

$$\theta_c = 2\theta.$$

Q. >

Compute θ_C .Solution

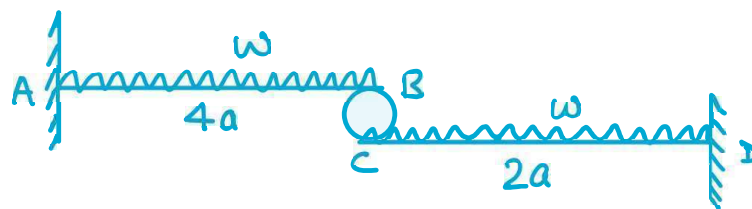
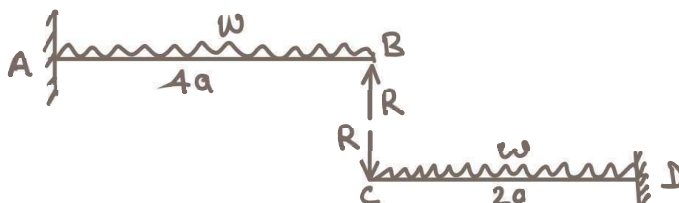
$$\theta = \int \frac{M}{EI} \frac{\partial M}{\partial M'} dx \Big|_{M'=0}$$

$$\theta = \int_0^{12} \frac{\left(\frac{2x^2}{2} + M' \right) \cdot 1}{EI} dx + \int_0^{10} \frac{(12 \cdot x + M' + 144) \cdot 1}{EI} dx$$

$$\theta = \frac{1}{3EI} \Big| x^3 \Big|_0^{12} + \frac{6}{EI} \Big| x^2 \Big|_0^{10} + \frac{144}{EI} \Big| x \Big|_0^{10}$$

$$\theta = \frac{2616}{EI}$$

Q. >

Compute Δ_B Solution:

$$\left. \frac{\partial U}{\partial R} \right|_{AB} = - \left. \frac{\partial U}{\partial R} \right|_{CD}$$

$$\int_0^{4a} \frac{\left(Rx - \frac{Wx^2}{2}\right) x dx}{EI} = - \int_0^{2a} \frac{(-) \left(Rx + \frac{Wx^2}{2}\right) x dx}{EI}$$

$$\left| R \frac{x^3}{3} - \frac{Wx^4}{8} \right|_0^{4a} = - \left| \frac{Rx^3}{3} + \frac{Wx^4}{8} \right|_0^{2a}$$

$$R = \frac{5}{4} Wa$$

$$\Delta_B = \Delta_C = \left. \frac{\partial U}{\partial R} \right|_{CD} = \left. \left| \frac{Rx^3}{3} + \frac{Wx^4}{8} \right|_0^{2a} \right\}_{R=\frac{5}{4}Wa}$$

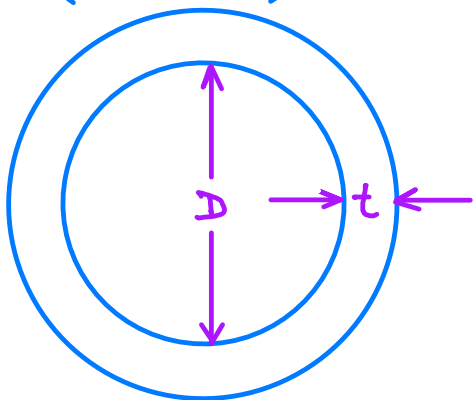
$$\Delta_B = \Delta_C = \frac{16}{3} \frac{Wa^4}{EI}$$

=

THICK AND THIN VESSELS

(Cylindrical and Spherical)

- The effect of fluid pressure inside thick or thin vessel is analysed in this case.
- A vessel/Shell is called thin if its thickness is less than $\left(\frac{D}{10} \text{ to } \frac{D}{15}\right)$ or else it is termed as thick shell/vessel.

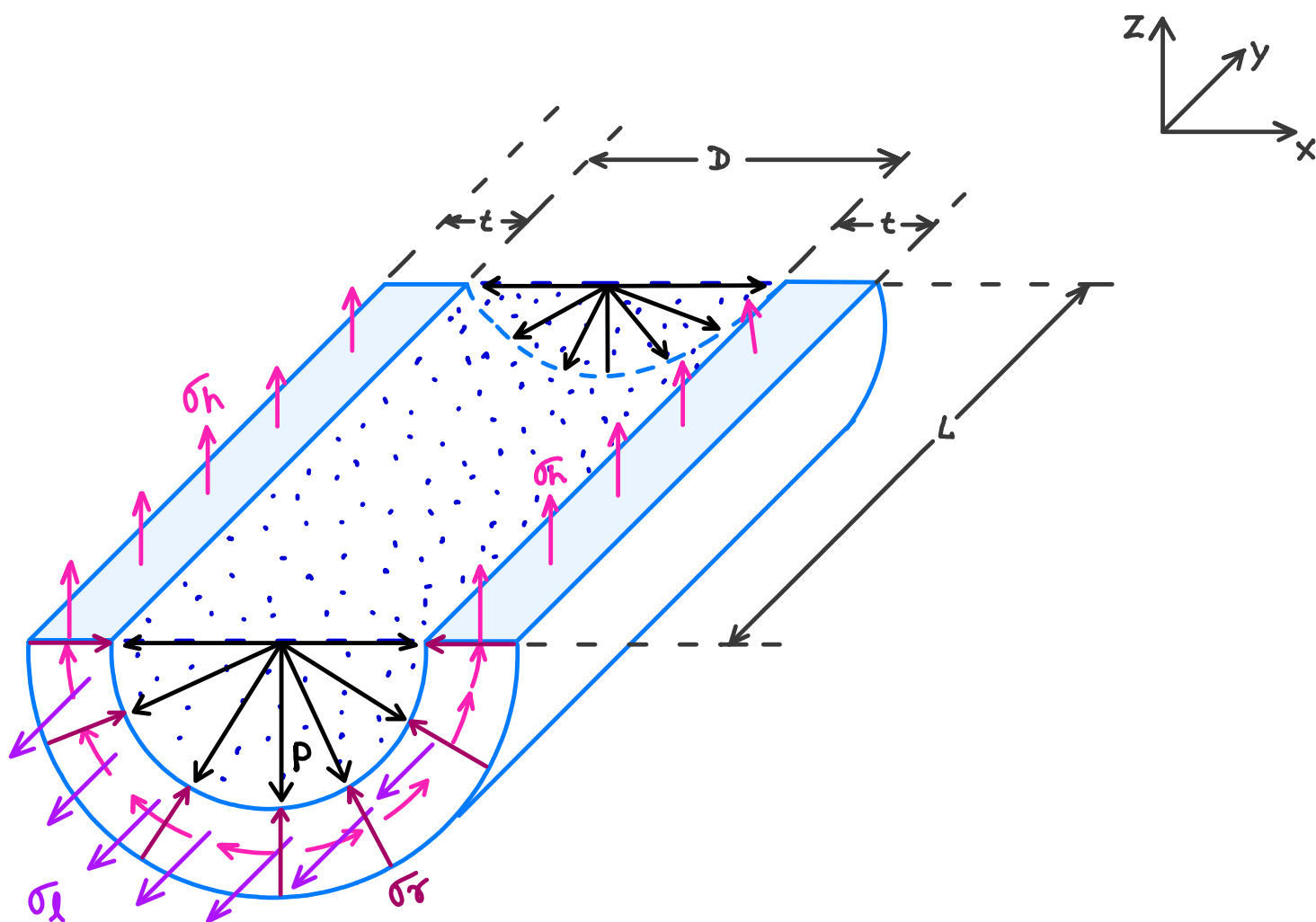
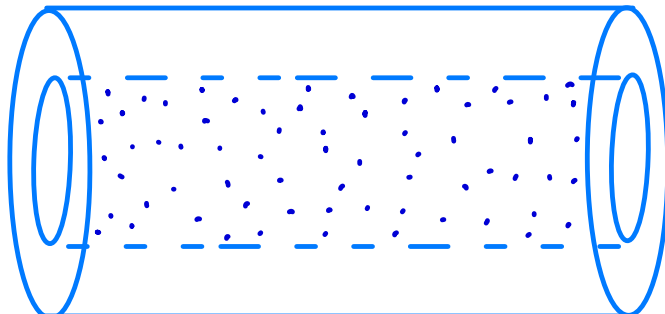
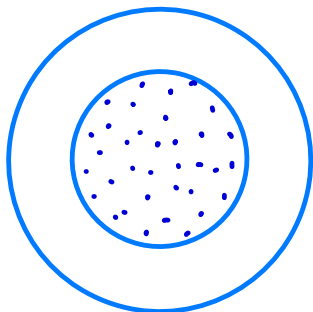


t = thickness of shell/vessel.
 D = internal dia of shell/vessel.

- The fluid pressure inside the Shell/vessel produces.
 - a> Hoop stress/Meridional stress/Circumferential stress (σ_h)
 - b> Longitudinal stress (σ_l)
 - c> Radial stress (σ_r)
- Hoop stress is constant throughout the thickness in thin shell, however it is maximum at inner face and minimum at outer face in thick shell.
- Longitudinal stress is constant across the thickness in both thick and thin shells.
- Radial stress is maximum inside & minimum outside (0) but is very less in comparison to hoop & longitudinal stress, hence is neglected. $(\sigma_h > \sigma_l > \sigma_r)$

(I) Thin Shells

a> cylindrical shell



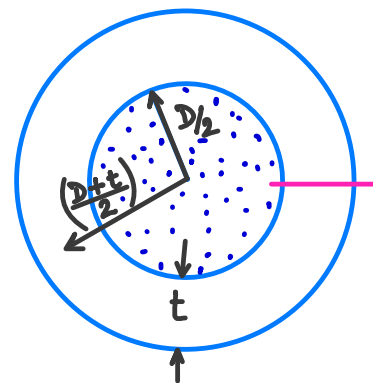
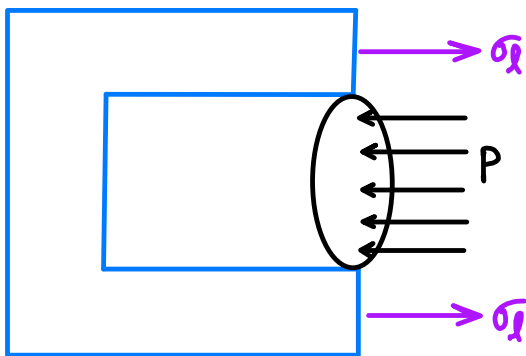
from static equilibrium

$$\sum f_z = 0$$

$$2 \sigma_h (L \cdot t) = P D L$$

$$\sigma_h = \frac{PD}{2t} \quad (\text{Tensile})$$

- As no shear acts on plane of hoop stress it is principal stress.
- Hoop stress is always tensile both in thick and thin shells due to fluid pressure from inside.
- If fluid pressure acts from outside, it is compressive.



Now,

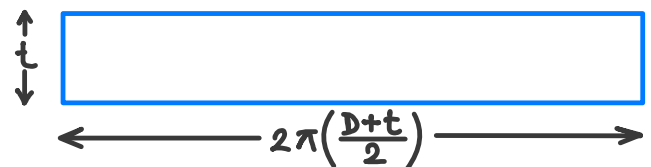
$$\sum f_y = 0$$

$$\sigma_l \cdot 2\pi \left(\frac{D+t}{2} \right) t = \frac{P \pi D^2}{4}$$

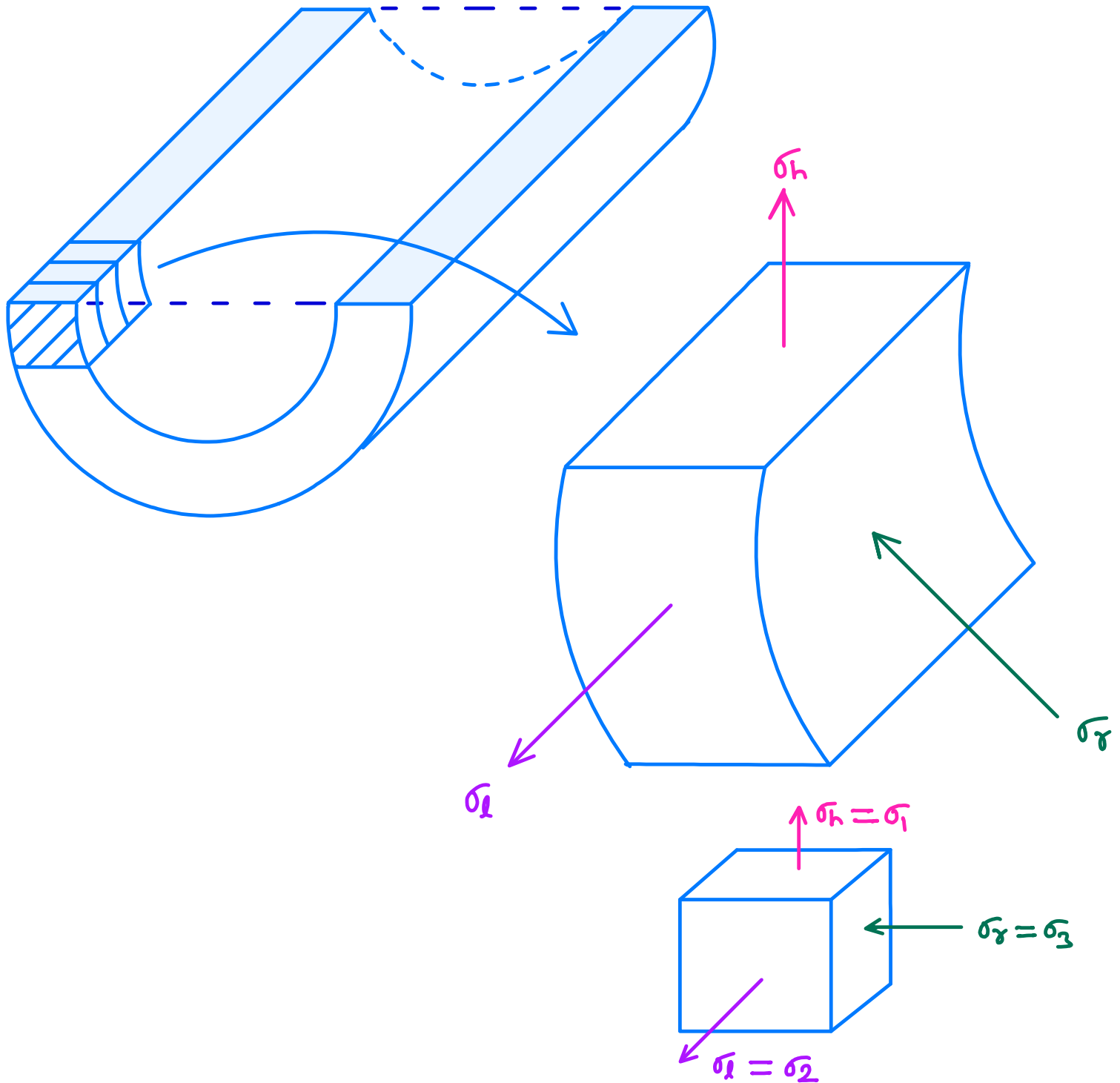
$$\sigma_l \left[\pi D t + \cancel{\pi t^2}^0 \right] = \frac{P \pi D^2}{4}$$

$$\sigma_l \pi D t = \frac{P \pi D^2}{4}$$

$$\sigma_l = \frac{PD}{4t}$$



- As no shear stress acts on the plane of longitudinal stress it is principal stress.
- It is always tensile due to fluid pressure from inside.
- Also, $\sigma_r = P$, (Compressive from inside) & ($\sigma_r = 0$ from outside).



- Since radial stress is less, it is Neglected.

Hooke Strain

$$\epsilon_h = \frac{\Delta D}{D} = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E} - \mu \frac{\sigma_3}{E}$$

$$\epsilon_h = \frac{PD}{2tE} - \mu \frac{PD}{4tE} - \mu \frac{(-P)}{E}$$

$$\epsilon_h = \frac{PD}{4tE} [2 - \mu] + \frac{\mu P}{E}$$

if radial stress is neglected.

$$\epsilon_h = \frac{PD}{4tE} (2 - \mu)$$

Longitudinal Strain

$$\epsilon_l = \frac{\Delta L}{L} = \frac{\sigma_2}{E} - \mu \frac{\sigma_3}{E} - \mu \frac{\sigma_1}{E}$$

$$\epsilon_l = \frac{PD}{4tE} - \mu \frac{(-P)}{E} - \mu \frac{PD}{2tE}$$

$$\epsilon_l = \frac{PD}{4tE} (1 - 2\mu) + \frac{\mu P}{E}$$

if radial stress is neglected.

$$\epsilon_l = \frac{PD}{4tE} (1 - 2\mu)$$

Volumetric Strain

$$V = \frac{\pi D^2}{4} \cdot L$$

$$\Delta V = \frac{\pi}{4} [2D \cdot dD \cdot L + D^2 dL]$$

$$\epsilon_v = \frac{\Delta V}{V} = \frac{\frac{\pi}{4} (2D \cdot dD \cdot L + D^2 dL)}{\frac{\pi}{4} D^2 \cdot L} = \left(\frac{dL}{L} + \frac{2dD}{D} \right)$$

$$\epsilon_v = \epsilon_l + 2\epsilon_h$$

$$\epsilon_v = \frac{PD}{4tE} (1-2\mu) + \frac{\mu P}{E} + 2 \left[\frac{PD}{4tE} (2-\mu) + \frac{\mu P}{E} \right]$$

$$\epsilon_v = \frac{PD}{4tE} [1-2\mu+4-2\mu] + \frac{3\mu P}{E}$$

$$\epsilon_v = \frac{PD}{4tE} (5-4\mu) + \frac{3\mu P}{E}$$

If radial stress is Neglected.

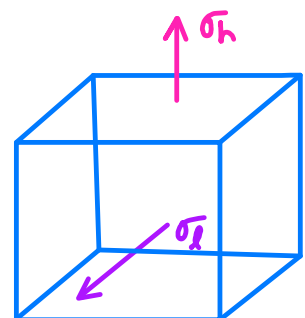
$$\epsilon_v = \frac{PD}{4tE} (5-4\mu)$$

Shear Stress

$$\tau_{\max} (\text{in plane}) = \left| \frac{\sigma_1 - \sigma_2}{2} \right| = \frac{PD}{8t}$$

$$\tau_{\max} (\text{absolute}) = \max \left[\left| \frac{\sigma_1 - \sigma_2}{2} \right|, \left| \frac{\sigma_1}{2} \right|, \left| \frac{\sigma_2}{2} \right| \right]$$

$$\tau_{\max} (\text{absolute}) = \frac{PD}{4t}$$



Q. > A copper tube of 50 mm dia & 120 mm length has a thickness of 1.2 mm with closed ends. It is filled with water at atmospheric pressure. Find the increase in pressure when an additional volume of 32 cc of water is pumped into the tube.

$$E_{\text{copper}} = 100 \text{ GPa}, \quad \mu = 0.3, \quad K_{\text{water}} = 2000 \text{ N/mm}^2$$

Solution:

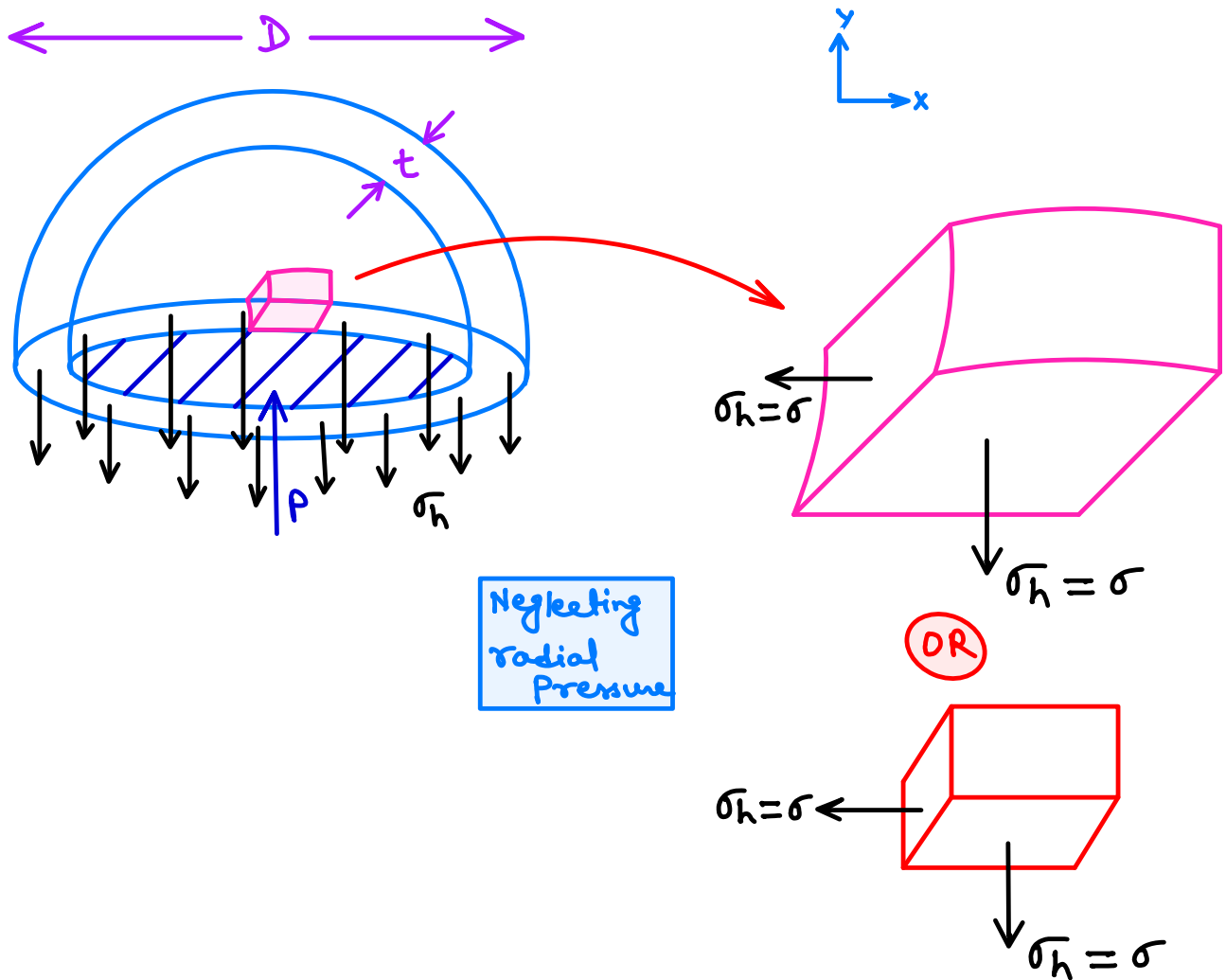
$$\begin{aligned} \text{Additional volume pumped in} &= \text{Increase in Volume of Cylinder} + \text{Decrease in Volume of Water} \\ &= \epsilon_v \cdot V + \frac{dP}{K} V \end{aligned} \quad \left\{ K = -\frac{dP}{\frac{dV}{V}} \right\}$$

$$= \left[\frac{dP D}{4 t E} (5 - 4 \mu) + \frac{dP}{K} \right] \frac{\pi D^2}{4} \cdot l$$

$$32 \times 10^3 = dP \left[\frac{50}{4 \times 1.2 \times 100 \times 10^5} (5 - 4 \times 0.3) + \frac{1}{2000} \right] \frac{\pi (50)^2}{4} \times 120$$

$$dP = 151.6 \text{ mpa.}$$

b. > Thin Sphere

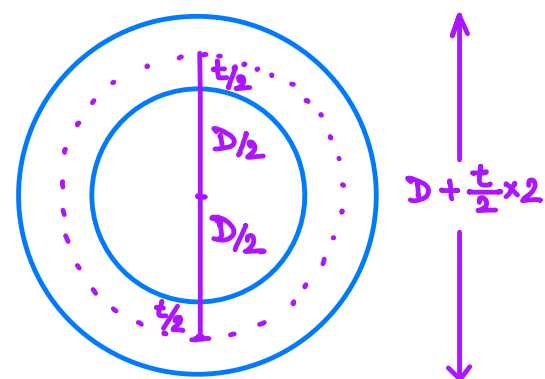


from static equilibrium condition

$$\sum f_y = 0$$

$$P\left(\frac{\pi D^2}{4}\right) - \sigma 2\pi \left(\frac{D}{2} + \frac{t}{2} \times 2\right) t = 0$$

$$\sigma = \frac{PD}{4t}$$



Note:-

- In sphere hoop stress and longitudinal stress is same.
- Since no shear acts on plane of hoop stress, so it is principal stress (Both Major and Minor).

$$\epsilon_h = \frac{\Delta D}{D} = \frac{\sigma_1}{E} - \mu \frac{\sigma_2}{E} - \mu \frac{\sigma_3}{E}$$

$$\frac{\Delta D}{D} = \frac{\sigma_h}{E} - \mu \frac{\sigma_h}{E} - \frac{\mu(-P)}{E}$$

$$\epsilon_h = \frac{\sigma_h}{E} (1-\mu) + \frac{\mu P}{E}$$

$$\epsilon_h = \frac{PD}{4tE} (1-\mu) + \frac{\mu P}{E}$$

if radial pressure is Neglected.

$$\epsilon_h = \frac{PD}{4tE} (1-\mu)$$

$$\epsilon_v = \frac{dV}{V} = \frac{1}{V} \left[\pi \cdot 3D^2 \frac{dD}{6} \right]$$

$$\left[Vol. = \frac{\pi D^3}{6} \right]$$

$$\epsilon_v = \frac{\pi 3D^2}{6} \frac{dD}{\pi D^3} \cdot 6 = 3 \frac{dD}{D}$$

$$\epsilon_v = 3 \epsilon_h$$

$$\epsilon_v = \frac{3PD}{4tE} (1-\mu)$$

(Neglecting radial pressure)

$$\tau_{\max} (\text{in plane}) = \left| \frac{\sigma_1 - \sigma_2}{2} \right| = \left| \frac{\sigma - \sigma}{2} \right| = 0$$

$$\begin{aligned} \tau_{\max} (\text{absolute}) &= \left\{ \frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_1}{2}, \frac{\sigma_2}{2} \right\} \\ &= \left\{ 0, \frac{\sigma}{2}, \frac{\sigma}{2} \right\} \\ &= \frac{\sigma}{2} \end{aligned}$$

$$\tau_{\max} (\text{absolute}) = \frac{PD}{8t}$$

(II) Thick Shells

a) Thick Cylinder

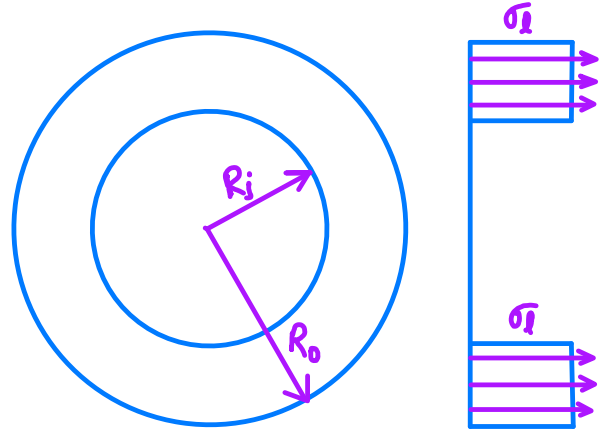
- In this case,

Longitudinal stress due to internal pressure is constant through out the thickness and is tensile.

from static equilibrium condition,

$$P \pi R_i^2 = \sigma_l \pi (R_o^2 - R_i^2)$$

$$\sigma_l = \frac{P R_i^2}{R_o^2 - R_i^2}$$

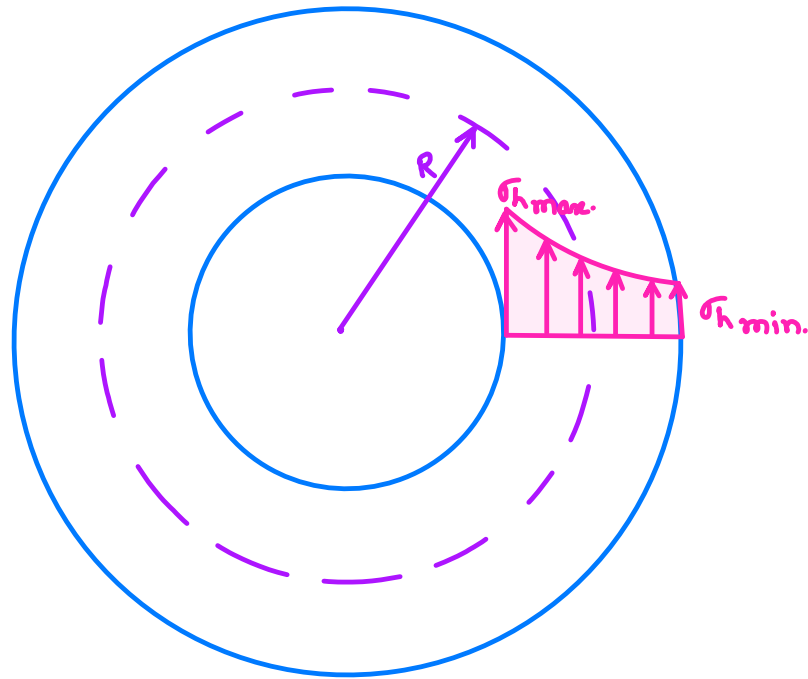


- Hoop stress due to internal pressure is maximum inside, minimum outside and is tensile throughout.
- Hoop stress due to external pressure is maximum inside, minimum outside and is compressive throughout.
- Hoop stress at any radius "R" from centre is given by **LAMES Theory**, which assumes material to be homogenous and isotropic.

$$\sigma_h = \frac{B}{R^2} + A \quad (\text{Tensile})$$

where,

A & B are Lames' Constant.



- Radial pressure is maximum inside, minimum outside and is compressive throughout.
- Radial pressure at any radius R from centre is given by

$$P_R = \frac{B}{R^2} - A \quad (\text{compressive})$$

where, A & B are Lame's constants.

- If $R = R_i$, $P_R = P$ and if $R = R_o$, $P_R = 0$

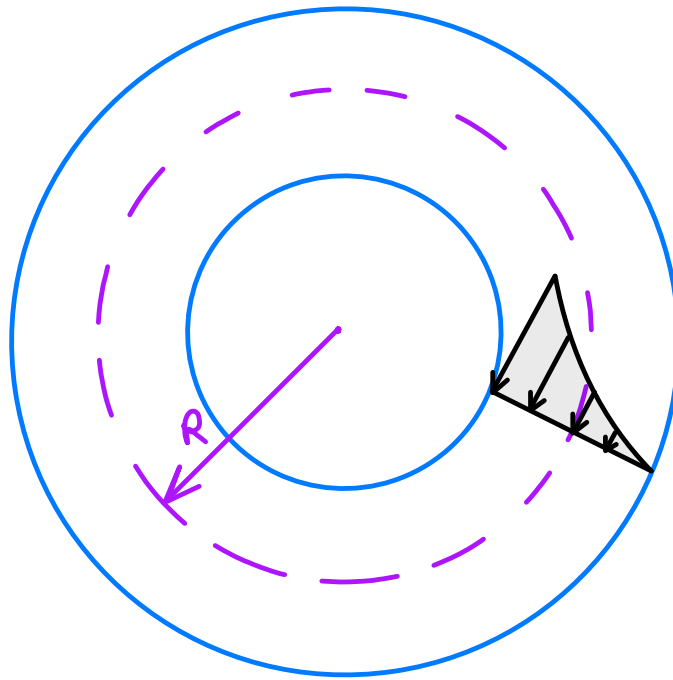
$$0 = \frac{B}{R_o^2} - A$$

$$A = \frac{B}{R_o^2} \quad \text{--- (i)}$$

$$P = \frac{B}{R_i^2} - A$$

$$= \frac{B}{R_i^2} - \frac{B}{R_o^2}$$

$$B = \frac{P}{\left(\frac{1}{R_i^2} - \frac{1}{R_o^2}\right)} \quad \text{--- (ii)}$$



b) Thick Sphere

- Here hoop stress and longitudinal stress are same and varies from maximum at inner face to minimum at outer face and is tensile throughout.
- Hoop stress/longitudinal stress at any radius "R" is given by

$$\sigma_l = \sigma_h = \frac{B}{R^3} + A \quad (\text{tensile})$$

where, A & B are Lame's constants.

- Radial pressure is maximum at inner face and minimum at outer face & is compressive throughout.
- Radial pressure is given by

$$P_R = \frac{2B}{R^3} - A \quad (\text{compressive})$$

A & B are Lame's constants.

$$\text{If } R = R_i, \quad P_R = P$$

$$R = R_o, \quad P_R = 0$$

Note:

Lamé's constants A & B both are positive for internal pressure and negative for external pressure.

Q. > A cylinder of hydraulic press has an internal diameter of 0.3 m and is to be designed to withstand a pressure of 10 MN/m^2 without the material being stressed over 20 MN/m^2 . Determine the thickness of the metal & the hoop stress on the outer side of the cylinder.

Solution: $R_i = \frac{0.3}{2} = 0.15 \text{ m}$

$$P_R = \frac{B}{R^2} - A$$

If $R = R_i, \quad P_R = P$

$$10 = \frac{B}{0.15^2} - A \quad \text{--- (1)}$$

also,

$$\sigma_h = \frac{B}{R^2} + A$$

If $R = R_i, \quad \sigma_h = \sigma_{h \text{ max.}}$

$$20 = \frac{B}{0.15^2} + A \quad \text{--- (ii)}$$

from (i) & (ii)

$$A = 5, \quad B = 0.3375$$

Also, at $R = R_o$, $P_R = 0$

$$0 = \frac{B}{R_o^2} - A$$

$$R_o = \left(\frac{B}{A}\right)^{1/2} = 0.26 \text{ m}$$

Also, $\sigma_{h \min} = \frac{B}{R_o^2} + A$

$$= \frac{0.3375}{(0.26)^2} + 5 = 10 \text{ MN/m}^2$$

$$t = R_o - R_i = 0.26 - 0.15$$

$$t = 0.11 \text{ m.}$$

SPRINGS

- The primary function of the spring is to deflect or distort under load and to recover its original shape after the removal of the load.
- During deflection or distortion, it absorbs the energy and releases the same after the removal of the load.
- Springs are generally classified into two
 - (a) Helical Spring (b) Leaf Spring.

(a) Helical Spring

- Helical spring are formed by coiling thick spring wires into helix.



- Helical Springs are further classified into two :-
 - i> closed coiled helical spring
 - ii> open coiled helical spring.

closed & open coiled helical springs

- when the helix angle is less than/equal to 10° , it is termed as closed coiled helical spring.
- In such springs, the wire undergoes too little bending or direct stress and their effect is neglected.
- Torsional stress in these springs are pre-dominant, hence, these are also termed as "Torsional Spring".
- If however helix angle is greater than 10° , then spring experiences both torsional and Bending stresses and are termed as open-coiled helical spring or "Bending Spring".



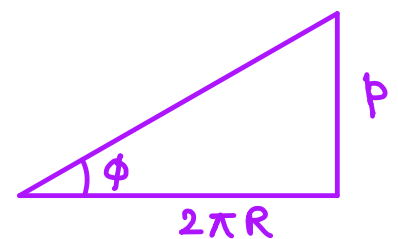
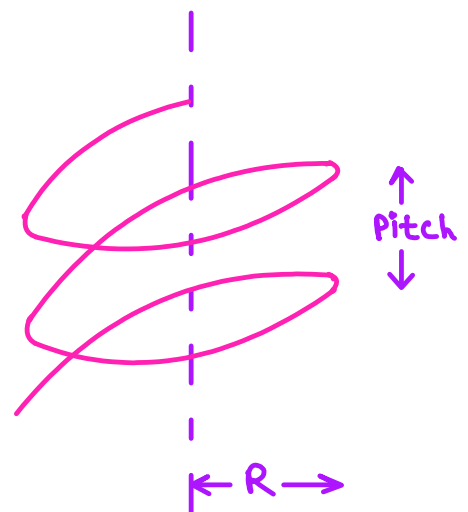
closed Coiled

Helix Angle $\phi \leq 10^\circ$



open Coiled.

Helix Angle $\phi > 10^\circ$



$\phi = \text{Helix Angle}$



Closed coil helical spring

Open coiled helical spring



Open Coil



Closed Coil

Note:-

i> Proof load

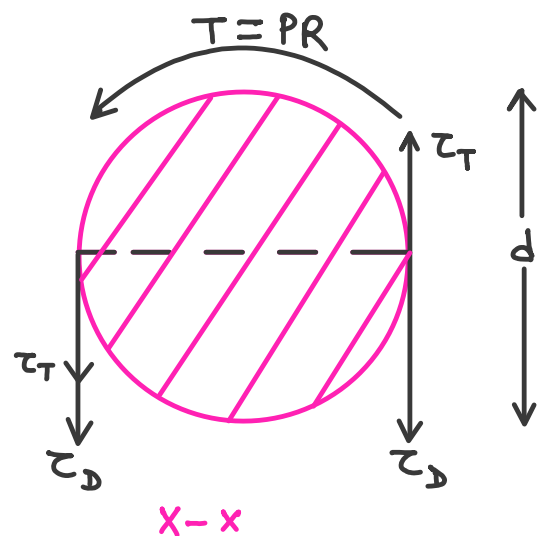
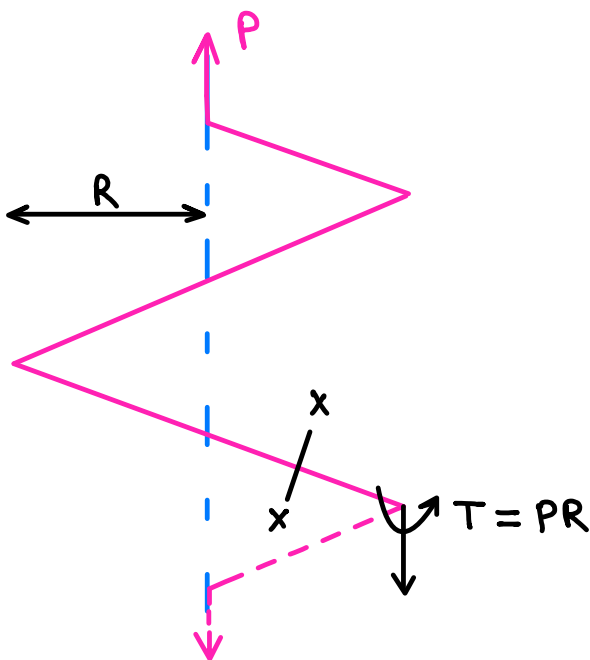
It is the maximum load that can be carried by the spring without getting permanently deformed.

ii> Proof stress

- It is the maximum stress in the spring when subjected to proof load and the strain energy stored in the spring corresponding to this is termed as **Proof Resilience/energy**.

i> closed coiled helical spring

In this, since helix angle is very small, each turn can be regarded as practically lying in plane perpendicular to axis of the helix/spring.



$$\begin{aligned}
 \tau_{\text{Total}} &= \tau_{\text{Direct stress}} \pm \tau_{\text{Torsional stress}} = \tau_D \pm \tau_T \\
 &= \frac{4}{3} (\tau_{\text{avg}}) \pm \frac{T r}{J} \quad \{r = \text{wire radius.}\} \\
 &= \frac{4}{3} \left(\frac{P}{\frac{\pi d^2}{4}} \right) \pm \frac{PR(d/2)}{\frac{\pi d^4}{32}}
 \end{aligned}$$

$$\tau_{\text{max}} = \frac{16 PR}{\pi d^3} \left[1 + \frac{d}{3R} \right], \quad \tau_{\text{min}} = \frac{16 PR}{\pi d^3} \left[1 - \frac{d}{3R} \right]$$

$$\text{Here, } \frac{d}{3R} \ll 1$$

$$\tau_{\text{max}} / \tau_{\text{min}} = \frac{16 PR}{\pi d^3}$$

→ Now, Strain energy stored in the Spring (u)

$$u = \frac{T^2 L}{2 G J}$$

$$T = PR$$

$$L = 2 \pi R \cdot n$$

$$J = \frac{\pi d^4}{32}$$

$$u = \frac{(PR)^2 \cdot 2 \pi R n}{2 G \frac{\pi d^4}{32}}$$

$$u = \frac{32 P^2 R^3 n}{G d^4}$$

→ Axial deflection of Spring (δ)

$$\delta = \frac{\partial u}{\partial P} = \frac{\partial}{\partial P} \left[\frac{32 P^2 R^3 n}{G d^4} \right]$$

$$\delta = \frac{64 P R^3 n}{G d^4}$$

→ Stiffness of the spring (k)

$$k = \frac{P}{\delta} = \frac{G d^4}{64 R^3 n}$$

→ proof load (W_{max})

$$f_s = \frac{16 P R}{\pi d^3} = \frac{16 W_{max} R}{\pi d^3}$$

$$W_{max} = \frac{\pi d^3 \cdot f_s}{16 R}$$

for proof load

$\tau_{max} = f_s = \text{Permissible Stress}$

$P = W_{max}$.

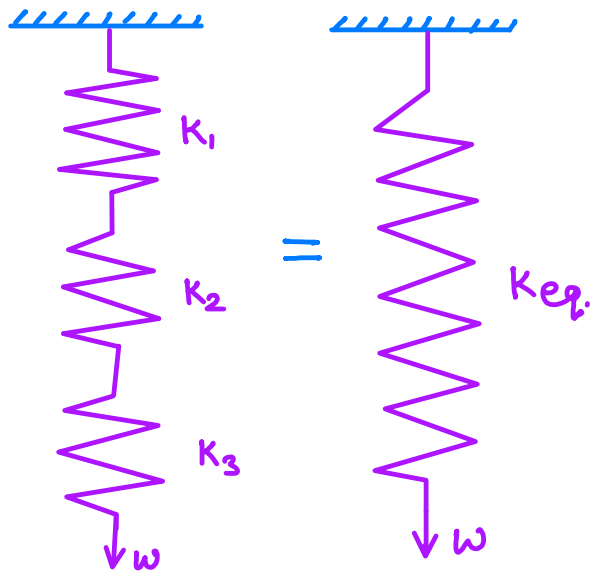
Note :-

If spring having total number of coil " n " is cut into " m " equal parts then stiffness of each part is given by

$$k' = \frac{G d^4}{64 R^3 \left(\frac{n}{m} \right)} = m k$$

Series and parallel Combination of Spring

a> Series Combination



for Series Combination,

$$W_1 = W_2 = W_3 = W_n = W$$

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \dots + \Delta_n$$

$$\frac{W}{K_{eq}} = \frac{W}{K_1} + \frac{W}{K_2} + \frac{W}{K_3} + \dots + \frac{W}{K_n}$$

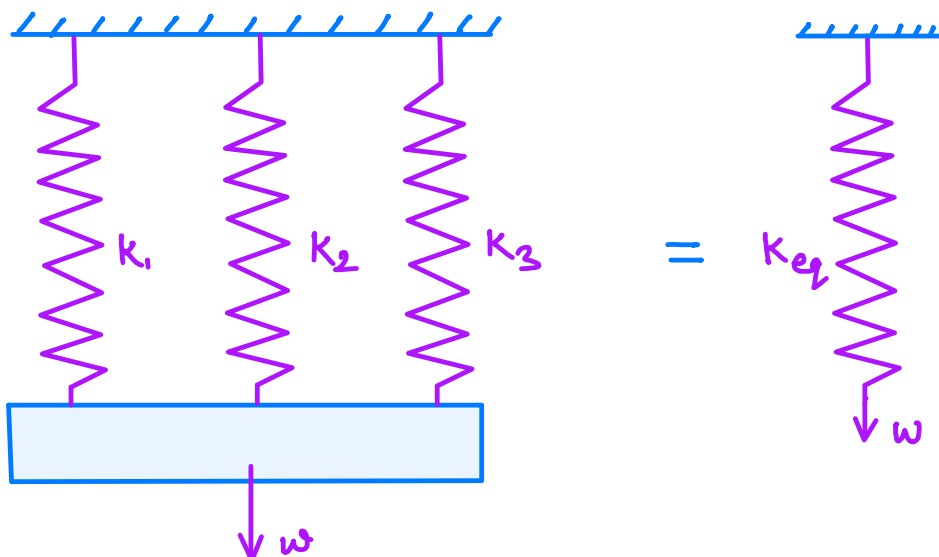
$$K_{eq} = \frac{1}{\sum_{j=1}^n \left(\frac{1}{K_j} \right)}$$

Similarly,

$$f_{eq} = \sum f_j$$

Here, f = flexibility

b> parallel Combination



$$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_n = \Delta$$

$$W = W_1 + W_2 + W_3 - \dots - W_n.$$

$$K_{eq} \Delta = K_1 \Delta + K_2 \Delta + K_3 \Delta - \dots - K_n \Delta$$

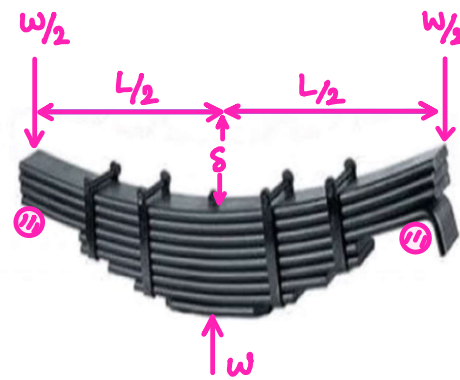
$$K_{eq} = \sum K_i$$

Similarly,

$$f_{eq} = \frac{1}{\sum (\frac{1}{f_i})}$$

(B) Leaf Spring

- These springs are commonly used in Carriages such as cars, Railway wagons etc. hence are also termed as laminated/Carriage Springs.
- It is made up of number of leaves of equal width & thickness, but varying length placed in laminations and are loaded as beam
- The length of these plates/leaves is designed such that maximum bending stress remains same in every plate i.e it behaves like beam of uniform strength.
- It is assumed that the plates are bent to same radius so that they contact only at their edges.



$$BM_{\max} = \frac{WL}{4}$$

from Bending theory, $\frac{M}{I} = \frac{\sigma}{y}$

$$M = \sigma Z_i$$

$$\frac{WL}{4} = \sigma \left(\frac{bt^3}{6} \right)$$

$Z_i =$ Section
Modulus of
Single Plate

$$\sigma = \frac{3WL}{2nbt^2}$$

Also,

$$M = \frac{EI}{R} = \frac{E}{R} \left(\frac{bt^3}{12} \right) n$$

$$\frac{WL}{4} = \frac{E}{R} \left(\frac{bt^3}{12} \right) n$$

$$R = \frac{Ebt^3n}{3WL}$$

Using the property of circle

$$\frac{L}{2} \cdot \frac{L}{2} = (2R - \delta) \delta$$

$$\frac{L^2}{4} = 2R\delta - \delta^2 \rightarrow 0$$

$$\delta = \frac{L^2}{8R}$$

$$\delta = \frac{L^2}{8} \frac{3WL}{Ebt^3n}$$

$$\delta = \frac{3}{8} \frac{WL^3}{Ebt^3n}$$

★ Stiffness of leaf spring

$$k = \frac{W}{\delta} = \frac{W}{\frac{3}{8} \frac{WL^3}{Ebt^3n}}$$

$$k = \frac{8Ebt^3n}{3WL^3}$$

Q. > A helical spring in which slope of helix may be assumed small, is required to transmit a maximum pull of 1 kN and to extend 10 mm for 200 N load. If the mean dia of the coil is 80 mm. Find the suitable diameter for the wire and Number of coils required. $G = 80 \text{ GPa}$ and Allowable stress is 100 MPa.

Solution: a) $f_s = \frac{16WR}{\pi d^3} = \frac{8W\phi}{\pi d^3}$

$$100 = \frac{80 \times 1 \times 10^3 \times 80}{\pi d^3}$$

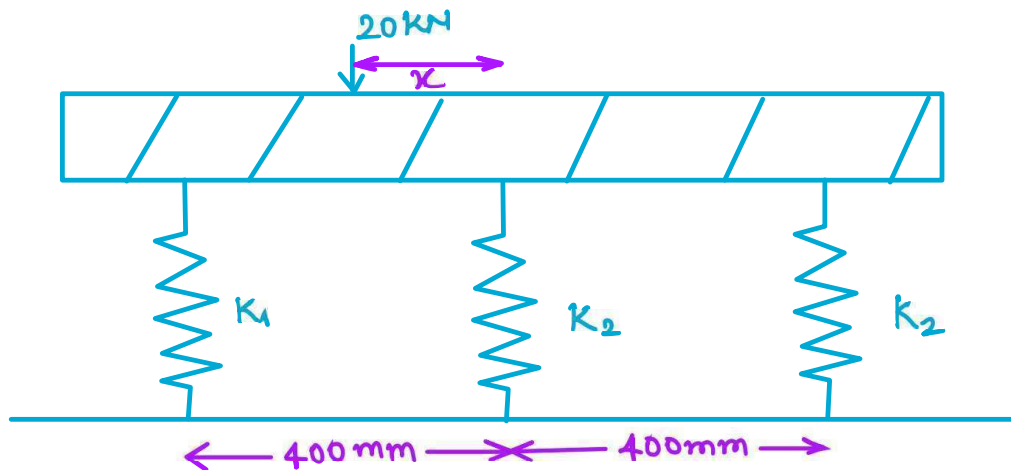
$$d = 12.68 \approx 13 \text{ mm.}$$

b) $\delta = \frac{64WR^3n}{Gd^4} \Rightarrow n = \frac{8Gd^4}{64WR^3}$

$$= \frac{16 \times 80 \times 10^3 \times (12.678)^4}{64 \times 200 \times \left(\frac{80}{2}\right)^3}$$

$$n = 25.2 \approx 26$$

Q. > A rigid bar weighing 5 kN and carrying a load of 20 kN is supported by 3 springs as shown having spring constant $k_1 = 30 \text{ kN/m}$, $k_2 = 18 \text{ kN/m}$, $k_3 = 12 \text{ kN/m}$. If the unloaded springs are of same length find the distance "x" such that bar is horizontal.



Solution

$$\Delta_1 = \Delta_2 = \Delta_3$$

$$\frac{W_1}{k_1} = \frac{W_2}{k_2} = \frac{W_3}{k_3}$$

$$\frac{W_1}{30} = \frac{W_2}{18} = \frac{W_3}{12} \quad \text{--- (i)}$$

$$W_1 + W_2 + W_3 = 20 + 5 = 25 \quad \text{--- (ii)}$$

from (i) & (ii)

$$W_1 = 12.5 \text{ kN}$$

$$W_2 = 7.5 \text{ kN}$$

$$W_3 = 5 \text{ kN}$$

from Static equilibrium Condition :

$$\sum M_{\text{2nd Spring}} = 0$$

$$W_1 \times 400 - 20 \times x - W_3 \times 400 = 0$$

$$12.5 \times 400 - 20 \times x - 5 \times 400 = 0$$

$$\underline{\underline{x = 150 \text{ mm.}}}$$

Q.7 A Composite Spring has two close helical springs Connected in Series, each spring has 12 coils at a mean diameter of 25 mm. Find the diameter of the wire in one of the spring if the dia of the other spring is 2.5 mm and the stiffness of the Composite spring is 700 N/m. Estimate the greatest load that can be carried by the Composite Spring for a maximum Shearing Stress of 160 MPa, $G = 80 \text{ GPa}$.

Solution

$$\Delta = \Delta_1 + \Delta_2$$

$$\frac{W}{K_{eq}} = \frac{W}{K_1} + \frac{W}{K_2}$$

$$\delta = \frac{64 WR^3 n}{G d^4}$$

$$\frac{1}{K} = \frac{\delta}{W} = \frac{64 R^3 n}{G d^4}$$

$$\frac{1}{\left(\frac{700}{10^3}\right)} = \frac{64 \times (25/2)^3 \times 12}{80 \times 10^3 \times d_1^4} + \frac{64 \times (2.5/2)^3 \times 12}{80 \times 10^3 \times (2.5)^4}$$

$$d_1 = 2.108 \text{ mm}$$

$$b) \quad f_s = \frac{8 W D}{\pi d^3} \Rightarrow W = \frac{f_s \pi d^3}{8 D}$$

$$W_1 = 180 \times \pi \frac{(2.108)^3}{8 \times 25} = 26.48 \text{ N}$$

$$W_2 = 180 \times \pi \frac{(2.5)^3}{8 \times 25} = 44.3 \text{ N}$$

Maximum load to be carried by spring 26.48 N.

Q. > A truck weighing 30 kN and moving at 6 km/hr has to be brought to rest by a buffer. Find how many springs each of 20 coils will be required to store the energy of motion during a compression of 200 mm. The spring is made out of 20 mm dia steel rod coiled to a mean diameter of 200 mm, $G = 100$ GPa.

Solution Energy of truck = Energy stored in Springs

$$\text{Energy of truck} = \frac{1}{2} mv^2 = \frac{1}{2} \times \frac{30 \times 10^3}{9.81} \times \left(\frac{6 \times 10^3}{3600} \right)^2$$

$$= 4248 \text{ N-m}$$

$$\delta = \frac{64 WR^3 n}{G d^4}$$

$$200 = \frac{64 \times W \times (200/2)^3 \times 20}{100 \times 10^3 \times 20^4}$$

$$W = 2500 \text{ N} \quad \text{————— (i)}$$

$$\text{Energy of Springs} = \frac{1}{2} W \delta N$$

$$= \frac{1}{2} \times 2500 \times 200 \times 10^{-3} \text{ N}$$

$$= 250 \text{ N} \xrightarrow{\text{No. of Springs.}} \text{N-m} \quad \text{————— (ii)}$$

from (i) & (ii)

$$4248 = 250 N$$

$$N = \frac{4248}{250} =$$

$$N = 17$$

COLUMNS

- Columns are vertical members carrying compressive load and moments.
- On the basis of mode of failure they are classified as
 - Short Column: which fails due to crushing.
 - Long Column: which fails due to Buckling
 - Intermediate Column: which fails due to both crushing and Buckling.

Note:-

Generally, if the unsupported length of column $<$ 10 times least lateral dimension
 $\rightarrow$ It is termed as **Short Column**.

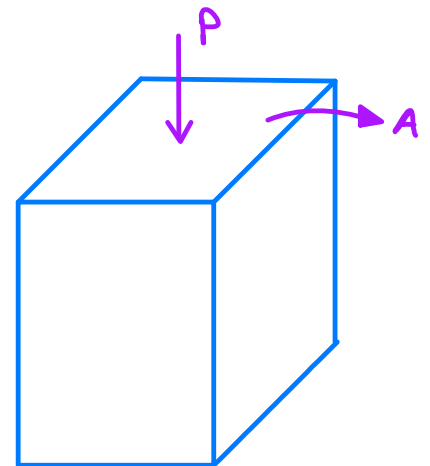
1) Short Column

$$\sigma = \frac{P}{A}$$

for design purpose, Stress developed (σ) must be less than Permissible Stress (σ_p).

$$\frac{P}{A} \leq \sigma_p$$

$$A \geq \frac{P}{\sigma_p}$$



ii> Long Column

— Analysis of long column is done as per Euler's theory which has following assumptions :-

- i> It is valid for long columns
- ii> Material is homogeneous, isotropic and prismatic.
- iii> Load is purely axial.
- iv> Axis of the column is perfectly straight when loaded.

Note :-

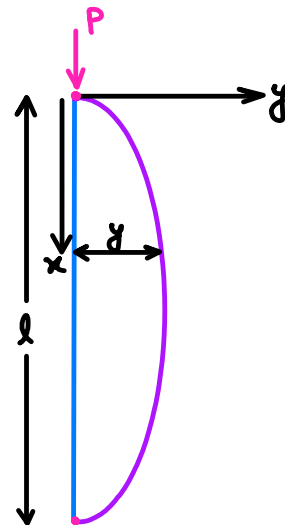
Euler's theory can not be applied practically as pure axial loading is not possible.

Case a> Pin Connected Column

$$EI \frac{d^2 y}{dx^2} = M = -Py$$

$$\frac{d^2 y}{dx^2} = -\frac{Py}{EI}$$

$$\frac{d^2 y}{dx^2} + \frac{Py}{EI} = 0$$



Solution of this equation is

$$y = A \sin\left(\sqrt{\frac{P}{EI}} \cdot x\right) + B \cos\left(\sqrt{\frac{P}{EI}} \cdot x\right)$$

$$\text{At } x=0, y=0 \Rightarrow 0 = 0 + B \cdot 1.$$

$$B = 0$$

$$\text{At } x=l, y=0 \Rightarrow 0 = A \sin \sqrt{\frac{P}{EI}} \cdot l$$

$$\sqrt{\frac{P}{EI}} \cdot l = n\pi \quad (n = 0, 1, 2, \dots)$$

$$\frac{P}{EI} \cdot l^2 = n^2 \pi^2$$

$$P = \frac{n^2 \pi^2 EI}{l^2}$$

The above load is called critical buckling load.

Hence, for no buckling $P \leq P_{\text{critical}}$

For minimum load causing buckling, $n=1$, $I = I_{\text{min}}$

$$P_{cr} = \frac{\pi^2 EI_{\text{min}}}{l^2}$$

Also,

$$I = A r^2$$

$$P_{cr} = \frac{\pi^2 E A r_{\text{min}}^2}{l^2}$$

$$P_{cr} = \frac{\pi^2 E A}{\left(\frac{l}{r_{\text{min}}}\right)^2}$$

Here, $\frac{l_{\text{eff.}}}{r_{\text{min}}} = \lambda = \text{Slenderness Ratio}$

$$P_{cr} = \frac{\pi^2 E A}{\lambda^2}$$

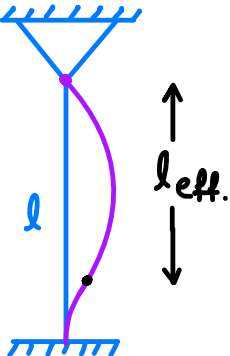
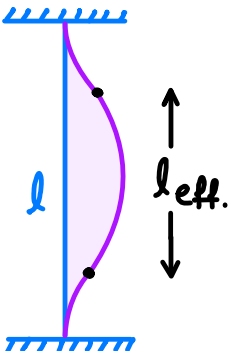
$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{\lambda^2}$$

$$p_{cr} = \frac{\pi^2 E}{\lambda^2}$$

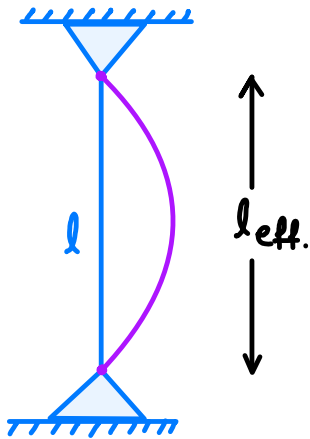
$$\left[\frac{P_{cr}}{A} = p_{cr} = \begin{array}{l} \text{Critical} \\ \text{buckling} \\ \text{Stress} \\ \text{OR} \\ \text{Euler's} \\ \text{Stress.} \end{array} \right]$$

Also, here I_{min} is minimum moment of inertia/minor moment of inertia.

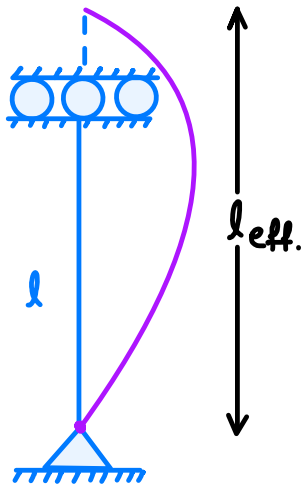
- If section has atleast one axis of symmetry then this axis and axis perpendicular to this axis are principal axis.
- Hence, moment of inertia along these axis would be major and minor moment of inertia.
- Also effective depends upon the support conditions as follows :-

Description	Effective length (l_{eff})	Effective length as Per Code
a.> 	$\frac{l}{\sqrt{2}}$ (or) $0.707l$	$0.8l$
b.> 	$\frac{l}{2}$ (or) $0.5l$	$0.65l$

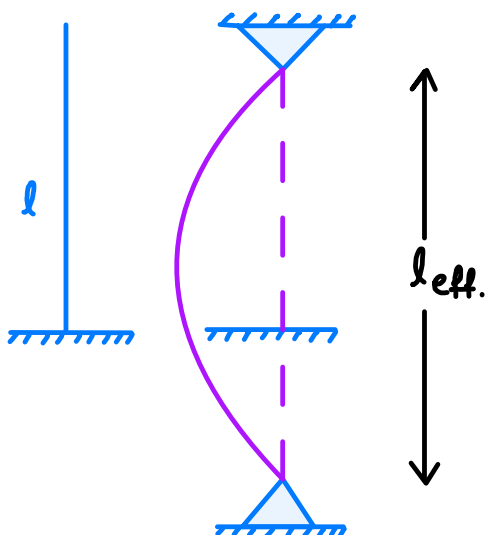
c.)

 l l

d.)

 $2l$ $2l$

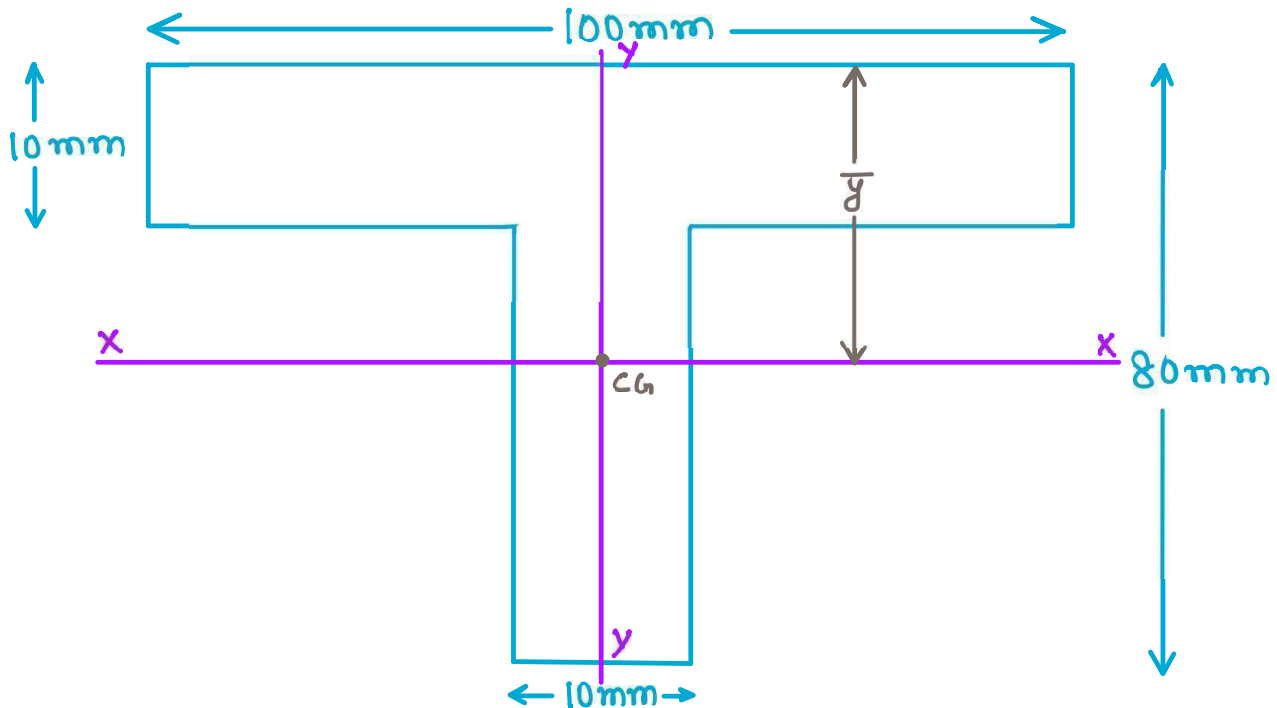
e.)

 $2l$ $2l$ Note:-

- l_{eff} is the length of column which is effectively restrained
- It is the distance between point of contraflexure of buckled

480 Column.

Q. > compute the buckling load for a strut of T-section, flange width being 100 mm, overall depth 80 mm & both flange and stem are 10 mm thick. The strut is 3m long and is hinged at both ends, $E = 200 \text{ GN/m}^2$



Solution

$$I_y = \frac{1}{12} \times 10 \times 100^3 + \frac{1}{12} \times 70 \times 10^3 = 839166.67 \text{ mm}^4$$

$$\bar{y} = \frac{10 \times 100 \times 5 + 70 \times 10 \times (35 + 10)}{10 \times 100 + 70 \times 10} = 21.47 \text{ mm}$$

$$I_x = \left[\frac{1}{12} \times 100 \times 10^3 + (21.47 - 5)^2 \right] + \left[\frac{1}{12} \times 10 \times 70^3 + 10 \times 70 \times (45 - 21.47)^2 \right]$$

$$I_x = 952990.19 \text{ mm}^4$$

$$I_{\min} = I_y = 839166.67 \text{ mm}^4$$

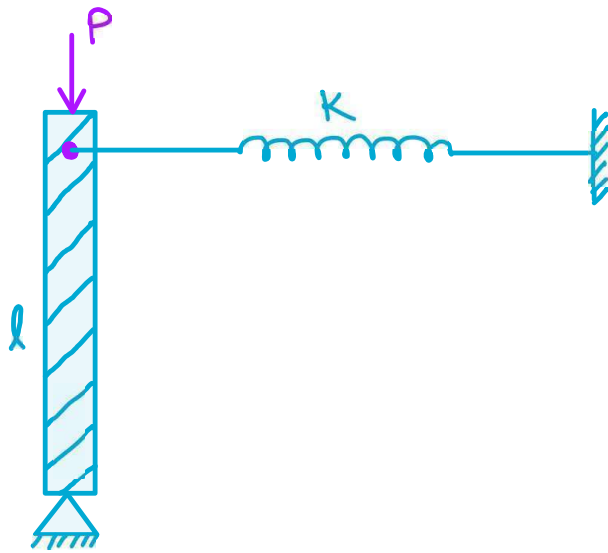
$$l_{\text{eff}} = l = 3 \text{ m}$$

$$P_{cr} = \frac{\pi^2 EI}{l^2} = \frac{3.14^2 \times 200 \times 10^3 \times 839166.67}{(3 \times 10^3)^2}$$

$$P_{cr} = 184 \text{ kN.}$$

NOTE :-

The buckling load obtained by the use of Euler's theory is more than actual buckling load, as it takes the effect of direct compression also.

Q. >

Compute Buckling load..

Solution

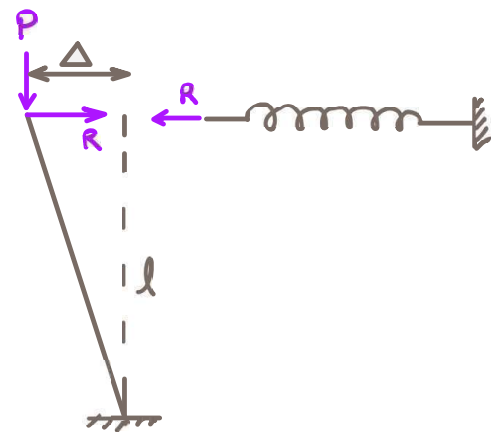
$$Rl = P\Delta \quad \text{--- (i)}$$

$$R = K\Delta \quad \text{--- (ii)}$$

from (i) & (ii)

$$K\Delta l = P\Delta$$

$$P = K \cdot l$$



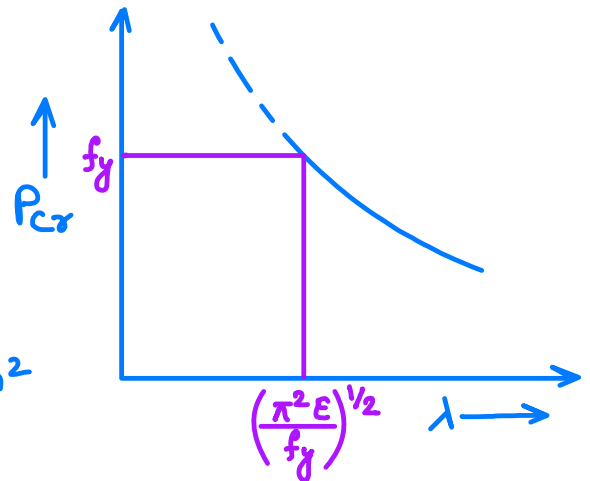
— Euler's formula can't be applied if $\lambda < \sqrt{\frac{\pi^2 E}{f_y}}$ as in such case, stress becomes more than permissible stress " f_y " and crushing will occur.

— As we know, $P_{cr} = \frac{\pi^2 E}{\lambda^2}$

- Crushing is characteristics of Short Column & Euler's theory is only for long column. hence, $\lambda > \sqrt{\frac{\pi^2 E}{f_y}}$

- For $f_y = 250$ & $E = 2 \times 10^5 \text{ N/mm}^2$

$$\lambda = \sqrt{\frac{(3.14)^2 \times 2 \times 10^5}{250}} = 88.86$$



Rankine theory

- This theory is valid for both Short & long column, hence failure by both crushing & buckling has been accounted in this case.

- If P = failure load

$$P_c = \text{Crushing load} = f_c \cdot A$$

$$P_E = \text{Buckling load} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 EA}{\lambda^2}$$

Crushing stress in compression.

then, $\boxed{\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_E}}$ — (Rankine's formula)

$$P = \frac{P_c \cdot P_E}{P_c + P_E} = \frac{P_c}{1 + \frac{P_c}{P_E}}$$

$$P = \frac{f_c A}{1 + \frac{f_c A}{\frac{\pi^2 EA}{\lambda^2}}} = \frac{f_c A}{1 + \left(\frac{f_c}{\pi^2 E}\right) \lambda^2}$$

$\Rightarrow$

$$\boxed{P = \frac{f_c A}{1 + a \lambda^2}}$$

Here, $\frac{f_c}{\pi^2 E} = a$ = Rankine's constant found by performing experiments.

— Long Column Under eccentric loading is analysed using "Secant formula".

$$EI \frac{d^2 y}{dx^2} = M_x$$

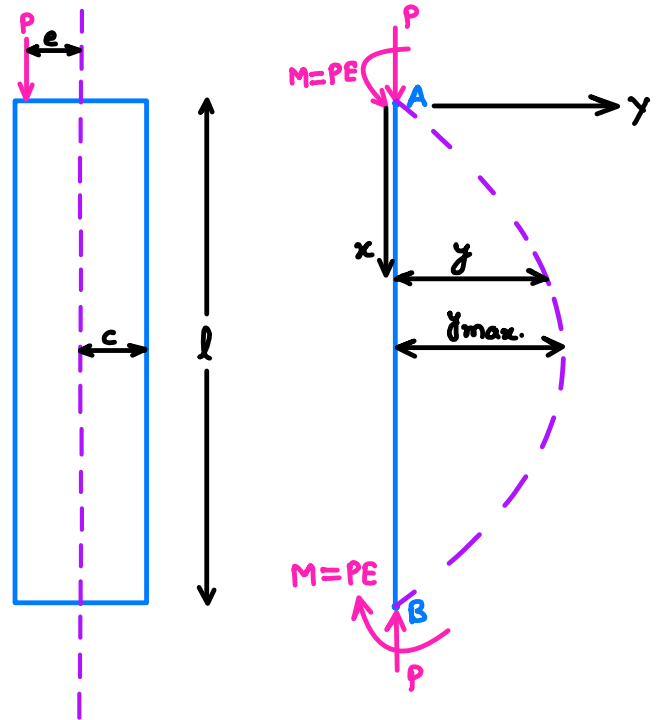
$$EI \frac{d^2 y}{dx^2} = -Pe - Py$$

$$\frac{d^2 y}{dx^2} = -\frac{Pe}{EI} - \frac{Py}{EI}$$

$$\left[\frac{P}{EI} = \omega^2 \right]$$

$$\frac{d^2 y}{dx^2} = -\omega^2 y - \omega^2 e$$

$$\frac{d^2 y}{dx^2} + \omega^2 y = -\omega^2 e$$



Solution of above equation is $y = A \sin \omega x + B \cos \omega x - e$

$$\text{At } x=0, y=0 \Rightarrow 0 = 0 + B - e$$

$$\Rightarrow B = e$$

$$\text{At } x=l, y=l \Rightarrow 0 = A \sin \omega l + e \cos \omega l - e$$

$$A \sin \omega l = e [1 - \cos \omega l]$$

$$2A \sin \frac{\omega l}{2} \cos \frac{\omega l}{2} = e \cdot 2 \sin^2 \frac{\omega l}{2}$$

$$A = e \tan \frac{\omega l}{2}$$

$$y = e \tan \frac{\omega l}{2} \sin \omega x + e \cos \omega x - e$$

$$\text{At } x = \frac{l}{2}, y = y_{\max}$$

$$\therefore y_{\max} = e \tan \frac{\omega l}{2} \sin \omega \frac{l}{2} + e \cos \omega \frac{l}{2} - e$$

$$\begin{aligned} y_{\max} &= e \left[\frac{\sin^2 \frac{\omega l}{2}}{\cos \frac{\omega l}{2}} + \cos \frac{\omega l}{2} - 1 \right] \\ &= e \left[\frac{\sin^2 \frac{\omega l}{2} + \cos^2 \frac{\omega l}{2}}{\cos \frac{\omega l}{2}} - 1 \right] \end{aligned}$$

$$y_{\max} = e \left[\sec \frac{\omega l}{2} - 1 \right]$$

$$\rightarrow \text{If } \frac{\omega l}{2} = \frac{\pi}{2} \Rightarrow y_{\max} = \infty$$

$$\sqrt{\frac{P}{EI}} \frac{l}{2} = \frac{\pi}{2}$$

$$P = \frac{\pi^2 EI}{l^2} = \text{Critical buckling load/Euler's load}$$

$$P_{cr} = \frac{\pi^2 EI}{l^2}$$

Hence,

$$y_{\max} = e \left[\sec \frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} - 1 \right]$$

— Maximum stress will occur at section where Bending moment is maximum.

$$\sigma_{\max} = \frac{P}{A} + \frac{M_{\max} \cdot c}{I}$$

$$M_{\max} = Pe + Py_{\max}$$

$$M_{\max} = P(y_{\max} + e)$$

$$\sigma_{\max} = \frac{P}{A} + \frac{P(y_{\max} + e)c}{A r^2}$$

$$\left[\begin{aligned} \frac{M}{I} &= \frac{\sigma}{y} \\ \Rightarrow \sigma &= \frac{My}{I} \end{aligned} \right]$$

$$\sigma_{max} = \frac{P}{A} \left[1 + \frac{P(y_{max} + e)c}{\gamma^2} \right]$$

$$\sigma_{max} = \frac{P}{A} \left[1 + \frac{(e \cdot \sec \frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} - e + e) \cdot c}{\gamma^2} \right]$$

$$\sigma_{max} = \frac{P}{A} \left[1 + \frac{ec}{\gamma^2} \sec \frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right]$$

$$\frac{P}{A} = \frac{\sigma_{max}}{1 + \frac{ec}{\gamma^2} \cdot \sec \frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}}}$$

← Secant formula.

⊙ OR

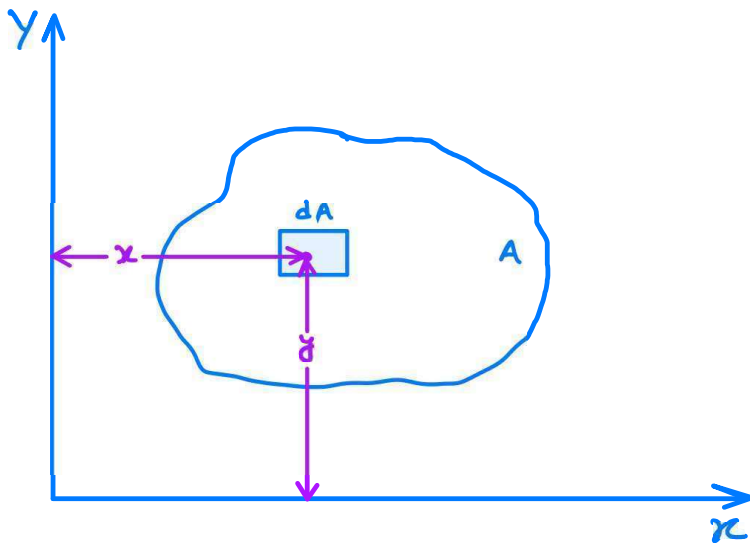
$$\frac{P}{A} = \frac{\sigma_{max}}{1 + \frac{ec}{\gamma^2} \cdot \sec \frac{1}{2} \sqrt{\frac{P}{EA} \frac{leff.}{\gamma}}}$$

MOMENT OF INERTIA

- For design of any structure its geometrical properties are required like Centroid, area, Moment of inertia, Polar moment of inertia, Product of inertia.

i) Centroid

- It is the point at which total area of a plane figure like circle, rectangle, triangle etc. is assumed to be concentrated.
- It may also be termed as centre of gravity.



$$\bar{x}y = \int x dA$$

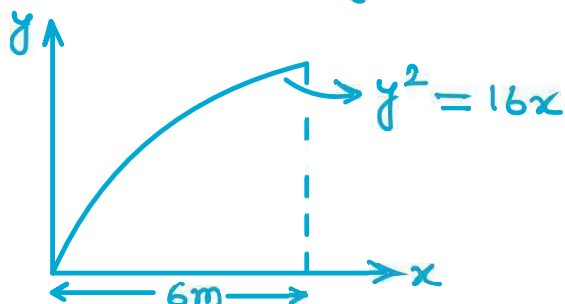
$$\bar{x} = \frac{\int x dA}{A}$$

$$\bar{y}A = \int y dA$$

$$\bar{y} = \frac{\int y dA}{A}$$

$$\text{Here, } A = \int dA = \int y dx.$$

Q. Find centroid for the given plane surface, Here $y^2 = 16x$.

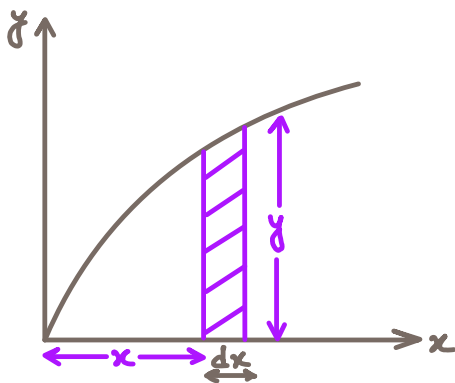


Solution

$$\bar{x} = \frac{\int x dA}{A}$$

$$\bar{y} = \frac{\int y dA}{A}$$

$$A = \frac{2}{3} \text{ Area of Rect.}$$



$$A = \int dA = \int y dx = \int_0^6 4\sqrt{x} dx$$

$$A = 4 \left| \frac{x^{3/2}}{3/2} \right|_0^6 = \frac{8}{3} \times 6^{3/2}$$

$$A = 39.19 \text{ m}^2$$

$$\bar{x} = \int \frac{x dA}{A} = \int \frac{x \cdot y}{A} dx$$

$$\bar{x} = \int_0^6 \frac{x \cdot 4\sqrt{x}}{A} dx = \frac{4}{39.19} \left| \frac{x^{5/2}}{5/2} \right|_0^6$$

$$\bar{x} = 3.6 \text{ m.}$$

$$\bar{y} = \int \frac{y \cdot dA}{A} = \int \frac{y}{2} \cdot \frac{y dx}{A}$$

$$= \frac{16}{2 \times 39.19} \int_0^6 x dx$$

$$\bar{y} = \frac{8}{39.19} \left| \frac{x^2}{2} \right|_0^6$$

$$\bar{y} = 3.67 \text{ m.}$$

C.G. of Rectangular
Element = $y/2$
from x-Axis

ii> Second Moment of Plane Area.

— Second moments of area "A" about "x" & "y" axis are given by,

$$I_{xx} = \int_A y^2 dA$$

$$I_{yy} = \int_A x^2 dA$$

— Second moment of area can not be negative unlike first moment of area.

- If at a single point entire area is concentrated then second moment of area is given by,

$$I_{xx} = A r_x^2 = \int_A y^2 dA$$

$$r_x = \sqrt{\frac{\int_A y^2 dA}{A}}$$

$$I_{yy} = A r_y^2 = \int_A x^2 dA$$

$$r_y = \sqrt{\frac{\int_A x^2 dA}{A}}$$

iii> Product of inertia

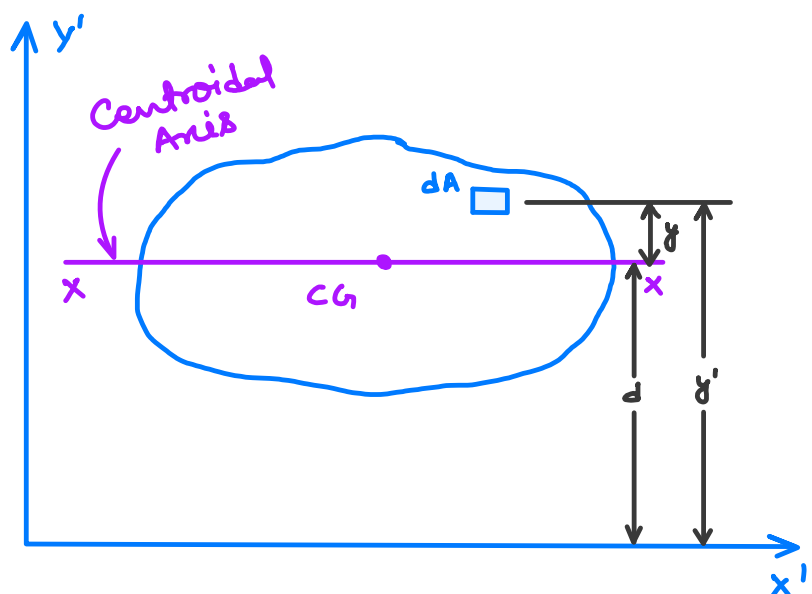
$$I_{xy} = \int_A x y dA$$

- It relates the area to a set of axis (x & y).
- It can be negative.

Transfer theorem

- If $x'x'$ is axis parallel to and is at a distance " d " from an axis xx passing through the centroid of area.

then,

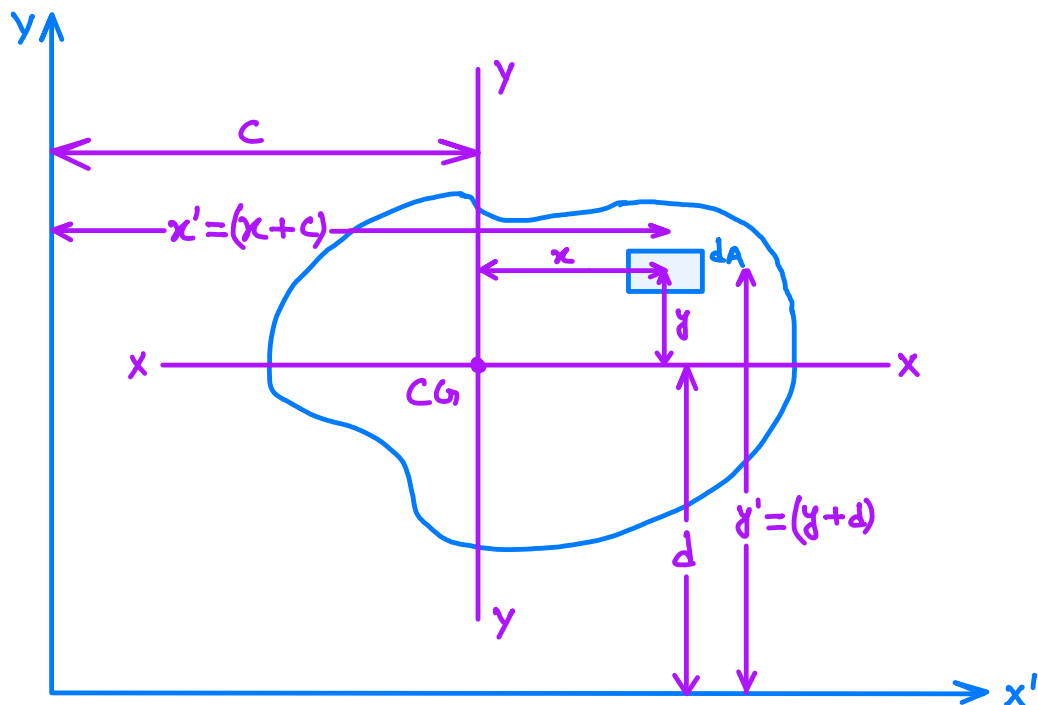


$$\begin{aligned}
 I_{xx'} &= \int_A y'^2 dA = \int_A (y+d)^2 dA \\
 &= \int_A y^2 dA + 2d \int_A y dA + d^2 \int_A dA
 \end{aligned}$$

$$I_{\text{about any Parallel Axis}} = I_{\text{about Centroidal Axis}} + 0 + d^2 A$$

$$I_{xx'} = I_{C.G} + d^2 A \Rightarrow \text{Parallel Axis theorem}$$

→ On the same lines product of inertia for parallel axis is:-



$$\begin{aligned}
 I_{x'y'} &= \int_A x' y' dA = \int_A (x+c)(y+d) dA \\
 &= \int_A xy dA + d \int_A x dA + c \int_A y dA + cd \int_A dA
 \end{aligned}$$

$$I_{x'y'} = I_{xy} + 0 + 0 + cdA$$

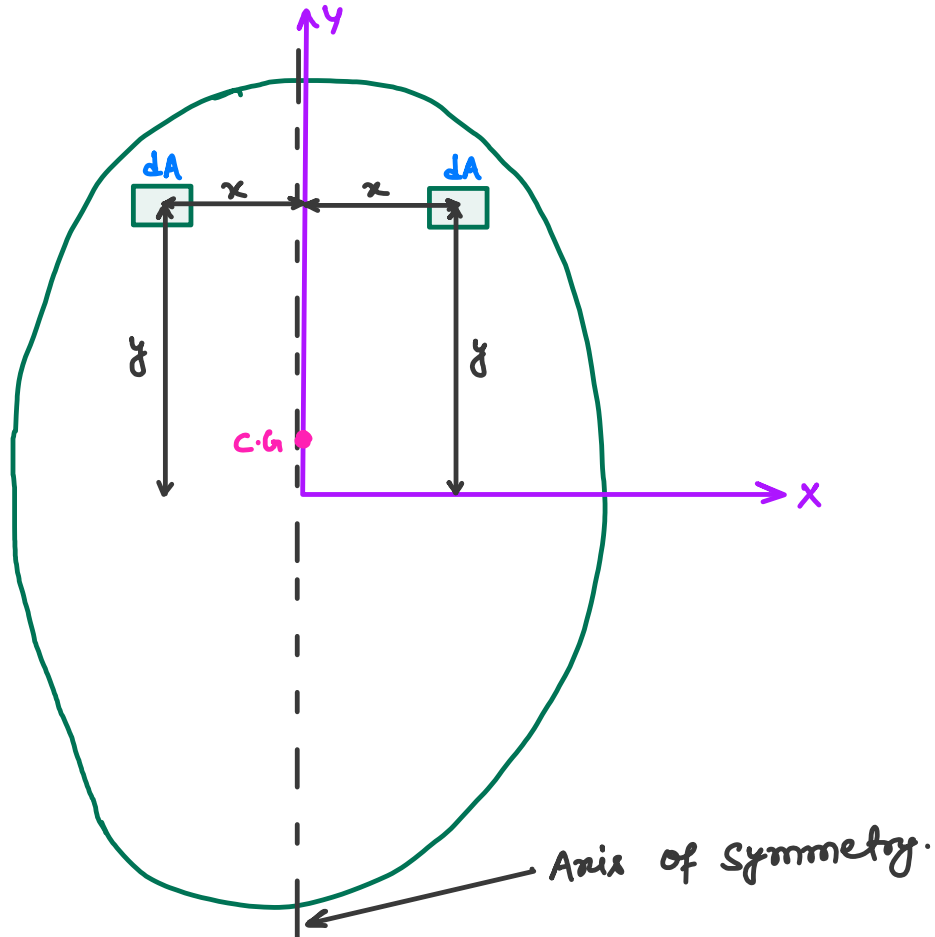
$$I_{x'y'} = I_{xy} + C \cdot d \cdot A$$

$$I_{x'y'} \text{ for any Parallel axes} = I_{xy} \text{ about Centroidal Axis} + C \cdot d \cdot A$$

↓
Parallel Axis theorem

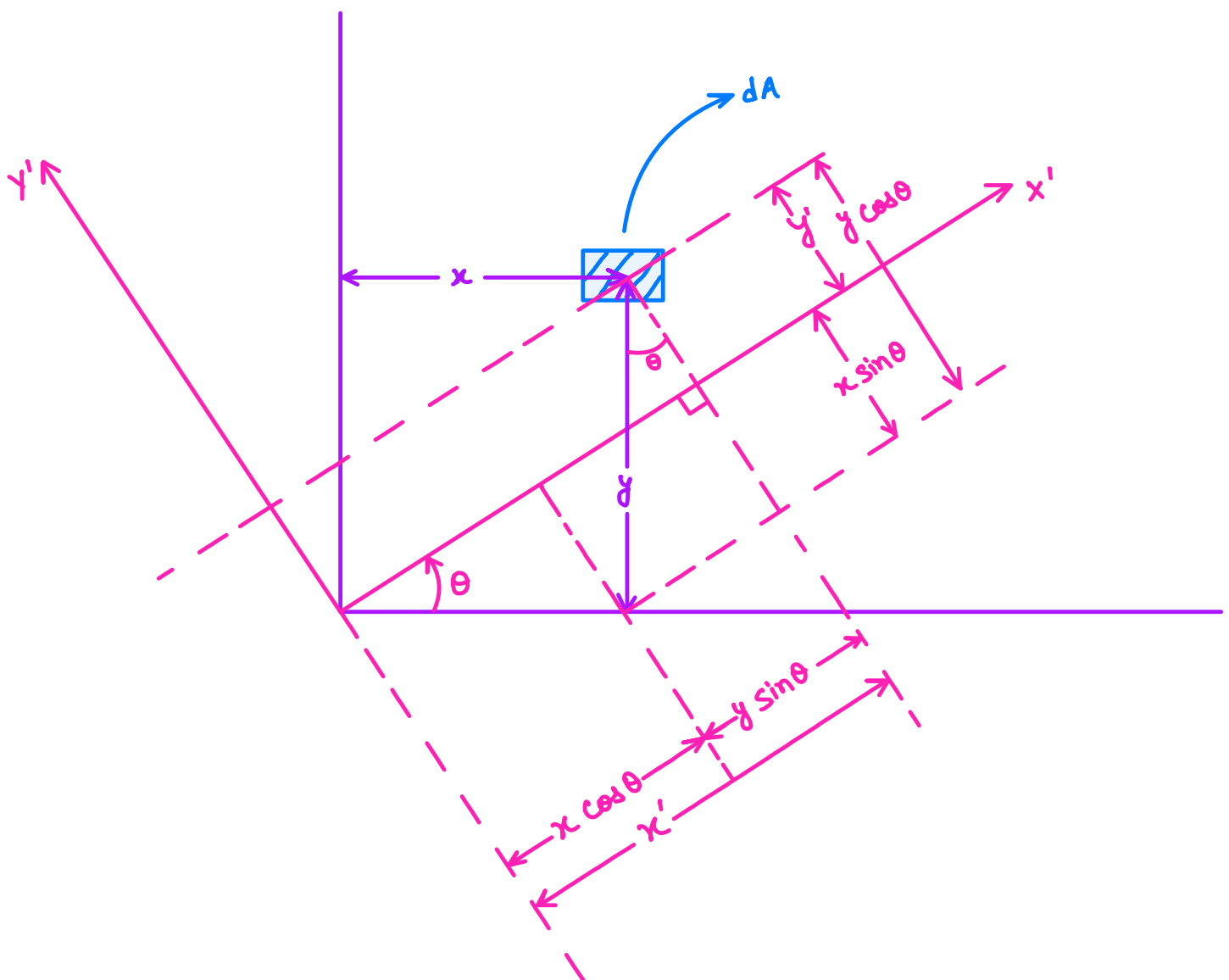
Note:-

- If the area under consideration has an axis of symmetry then the product of inertia for this axis and any axis orthogonal to this axis must be zero.



Relation between moments and product of inertia

- Let I_{xx} , I_{yy} and I_{xy} be the second moment of area and product of inertia about $x-x$ and $y-y$ axis.
- with the help of this moment of area and product of inertia about any other $x'-x'$ & $y'-y'$ passing through same point at an angle θ (Anti-clockwise) can be found as follows:-



$$x' = x \cos \theta + y \sin \theta$$

$$y' = -x \sin \theta + y \cos \theta$$

$$I_{x'x'} = \int_A y'^2 dA = \int_A (-x \sin \theta + y \cos \theta)^2 dA$$

$$I_{x'x'} = \sin^2 \theta \int_A x^2 dA - 2 \sin \theta \cos \theta \int_A xy dA + \cos^2 \theta \int_A y^2 dA$$

$$I_{x'x'} = I_{yy} \sin^2 \theta - I_{xx} \cos^2 \theta - I_{xy} \sin 2\theta$$

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2} \cos 2\theta - I_{xy} \sin 2\theta \quad **$$

for $I_{yy'}$, put $\theta = 90^\circ + \theta$

$$I_{yy'} = \frac{I_{xx} + I_{yy}}{2} - \frac{I_{xx} - I_{yy}}{2} \cos 2\theta + I_{xy} \sin 2\theta \quad **$$

Also,

$$I_{xx} + I_{yy} = I_{xx'} + I_{yy'}$$

$$I_{x'y'} = \int_A x' y' dA = \int_A (x \cos \theta + y \sin \theta) (-x \sin \theta + y \cos \theta) dA$$

$$I_{x'y'} = -\sin \theta \cos \theta \int_A x^2 dA + \cos^2 \theta \int_A xy dA - \sin^2 \theta \int_A xy dA + \cos \theta \sin \theta \int_A y^2 dA$$

$$I_{x'y'} = \sin \theta \cos \theta (I_{xx} - I_{yy}) + I_{xy} (\cos^2 \theta - \sin^2 \theta)$$

$$I_{x'y'} = \left(\frac{I_{xx} - I_{yy}}{2} \right) \sin 2\theta + I_{xy} \cos 2\theta$$

- Hence, if I_{xx} , I_{yy} & I_{xy} are known then moment and product of Inertia about any other axis at an angle of θ (anti-clockwise) can be computed.

→ for $I_{x'x'}$ to be minimum/maximum, $\frac{dI_{x'x'}}{d\theta} = 0$ or $I_{x'y'} = 0$

$$I_{x'y'} = 0 \Rightarrow \left(\frac{I_{xx} - I_{yy}}{2} \right) \sin 2\theta + I_{xy} \cos 2\theta = 0$$

$$\left(\frac{I_{xx} - I_{yy}}{2} \right) \sin 2\theta = -I_{xy} \cos 2\theta$$

$$\tan 2\theta = \frac{-2I_{xy}}{I_{xx} - I_{yy}}$$

- from where two values of θ , separated by 90° would be obtained for which maximum & minimum value of $I_{x'x'}$ can be computed and is termed as maximum/MAJOR principle Moment of inertia (I_{uu}) & minimum/minor principal Moment of inertia (I_{vv})

$$I_{x'x'_{\max}} = I_{uu} = \frac{I_{xx} + I_{yy}}{2} + \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

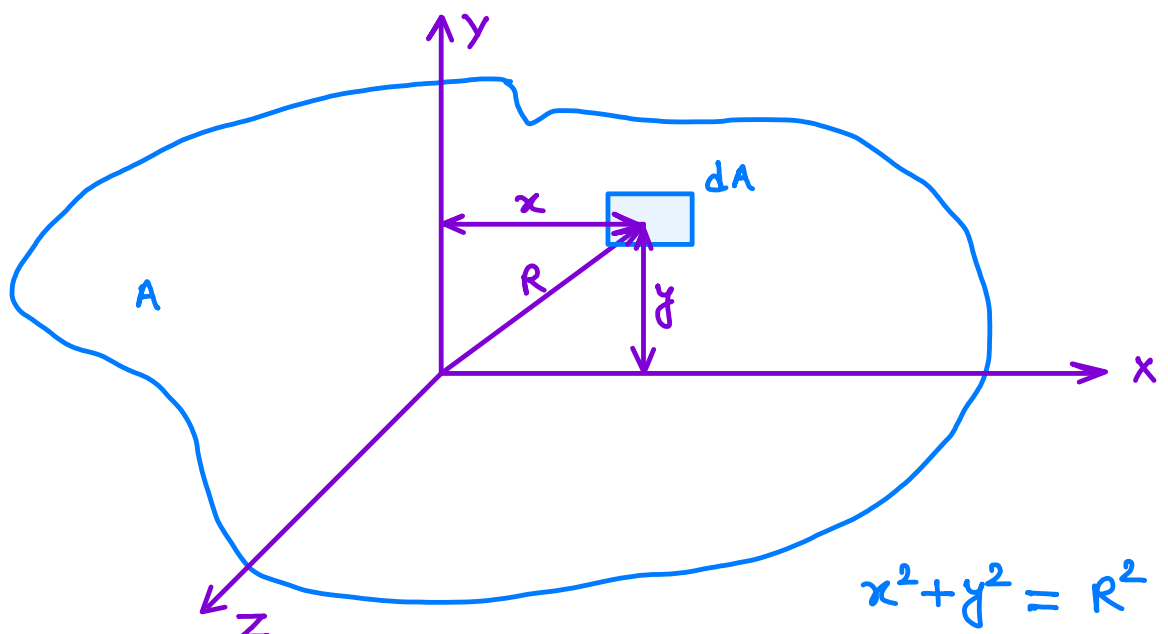
$$I_{y'y'_{\max}} = I_{vv} = \frac{I_{xx} + I_{yy}}{2} - \sqrt{\left(\frac{I_{xx} - I_{yy}}{2}\right)^2 + I_{xy}^2}$$

Note:-

- The above analysis can also be done by use of MOHR circle.

Polar Moment of Inertia

- Moment of inertia of given plane section about z-axis is called "Polar moment of inertia".

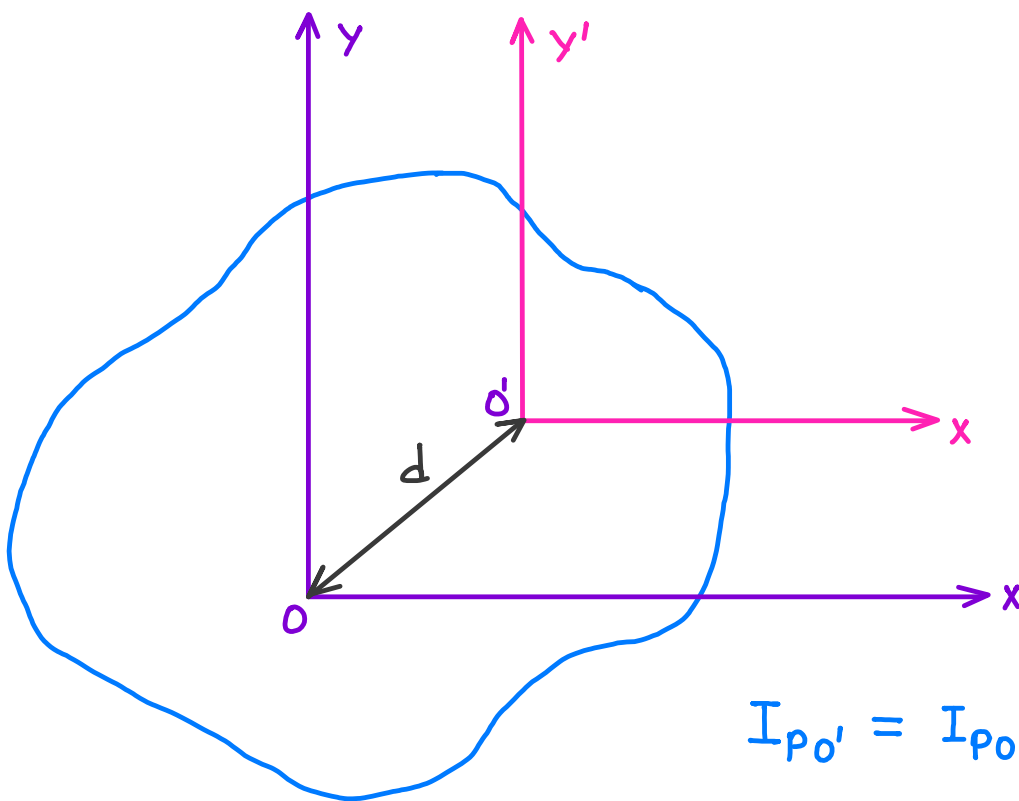


$$I_{xx} + I_{yy} = \int y^2 dA + \int x^2 dA = \int (x^2 + y^2) dA$$

$$\underbrace{I_{xx} + I_{yy}}_{\text{Perpendicular Axis theorem.}} = \int R^2 dA = I_{zz} = I_p$$

Perpendicular Axis theorem.

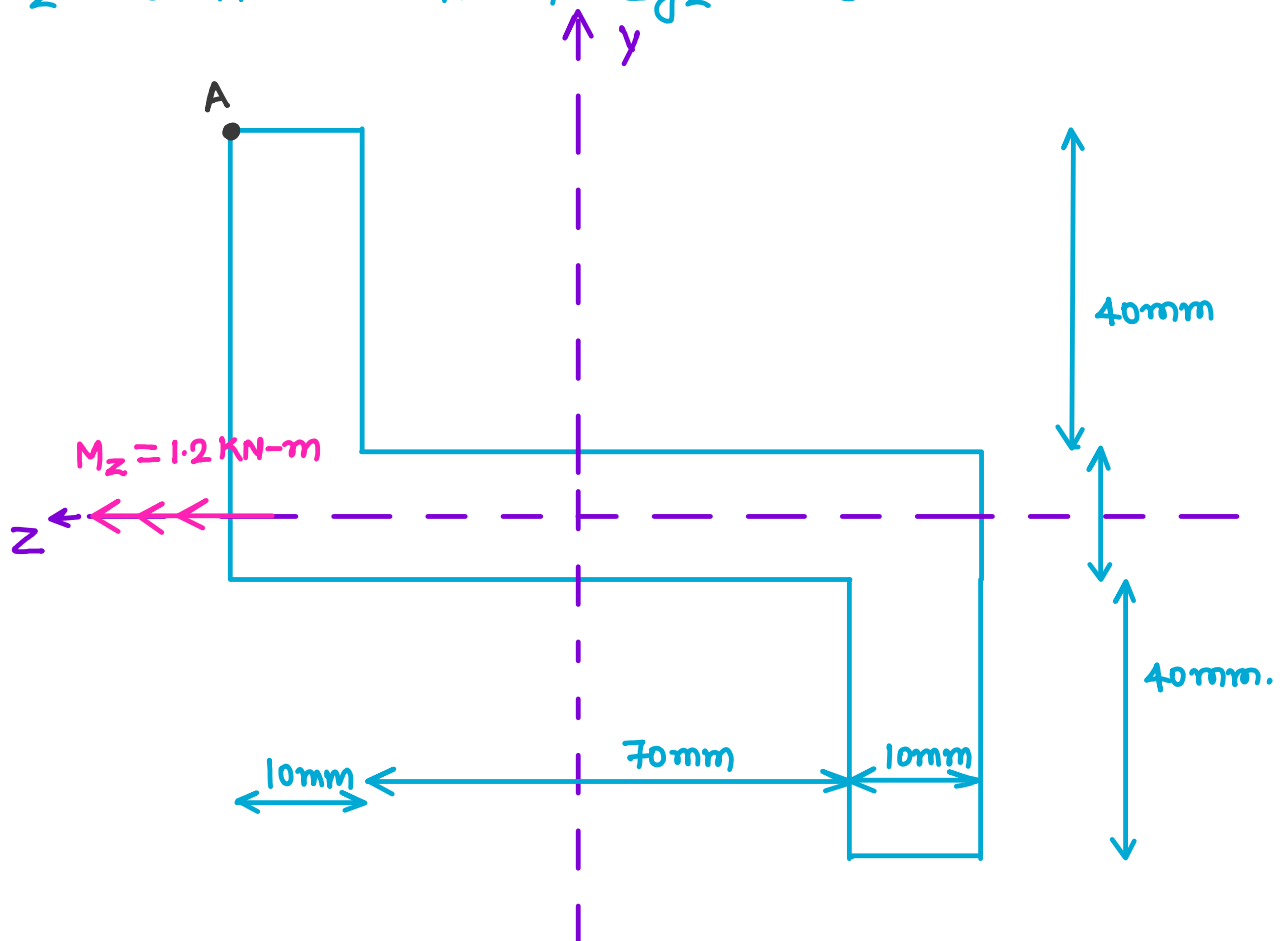
- Hence, Sum of moment of inertia about orthogonal axis (which is function of only position is termed as "Polar moment of inertia".
- Here also parallel axis theorem can be applied.



$$I_{pO'} = I_{pO} + Ad^2$$

$$I_{pO} = I_{xx} + I_{yy}$$

Q. A couple M_z acts in the vertical plane and is applied to a beam of cross-section as shown. Determine the stress at point A, $I_y = 1.894 \times 10^6 \text{ mm}^4$, $I_z = 0.614 \times 10^6 \text{ mm}^4$, $I_{yz} = 0.8 \times 10^6 \text{ mm}^4$



Solution

$$I_{uu} = \frac{I_{zz} + I_{yy}}{2} + \sqrt{\left(\frac{I_{zz} - I_{yy}}{2}\right)^2 + I_{yz}^2}$$

$$= \frac{(1.894 + 0.614) \times 10^6}{2} + \sqrt{\left(\frac{1.894 \times 10^6 - 0.614 \times 10^6}{2}\right)^2 + (0.8 \times 10^6)^2}$$

$$= 2.278 \times 10^6 \text{ mm}^4$$

$$I_{vv} = \frac{I_{zz} + I_{yy}}{2} - \sqrt{\left(\frac{I_{zz} - I_{yy}}{2}\right)^2 + I_{yz}^2}$$

$$= \frac{(1.894 + 0.614) \times 10^6}{2} - \sqrt{\left(\frac{1.894 \times 10^6 - 0.614 \times 10^6}{2}\right)^2 + (0.8 \times 10^6)^2}$$

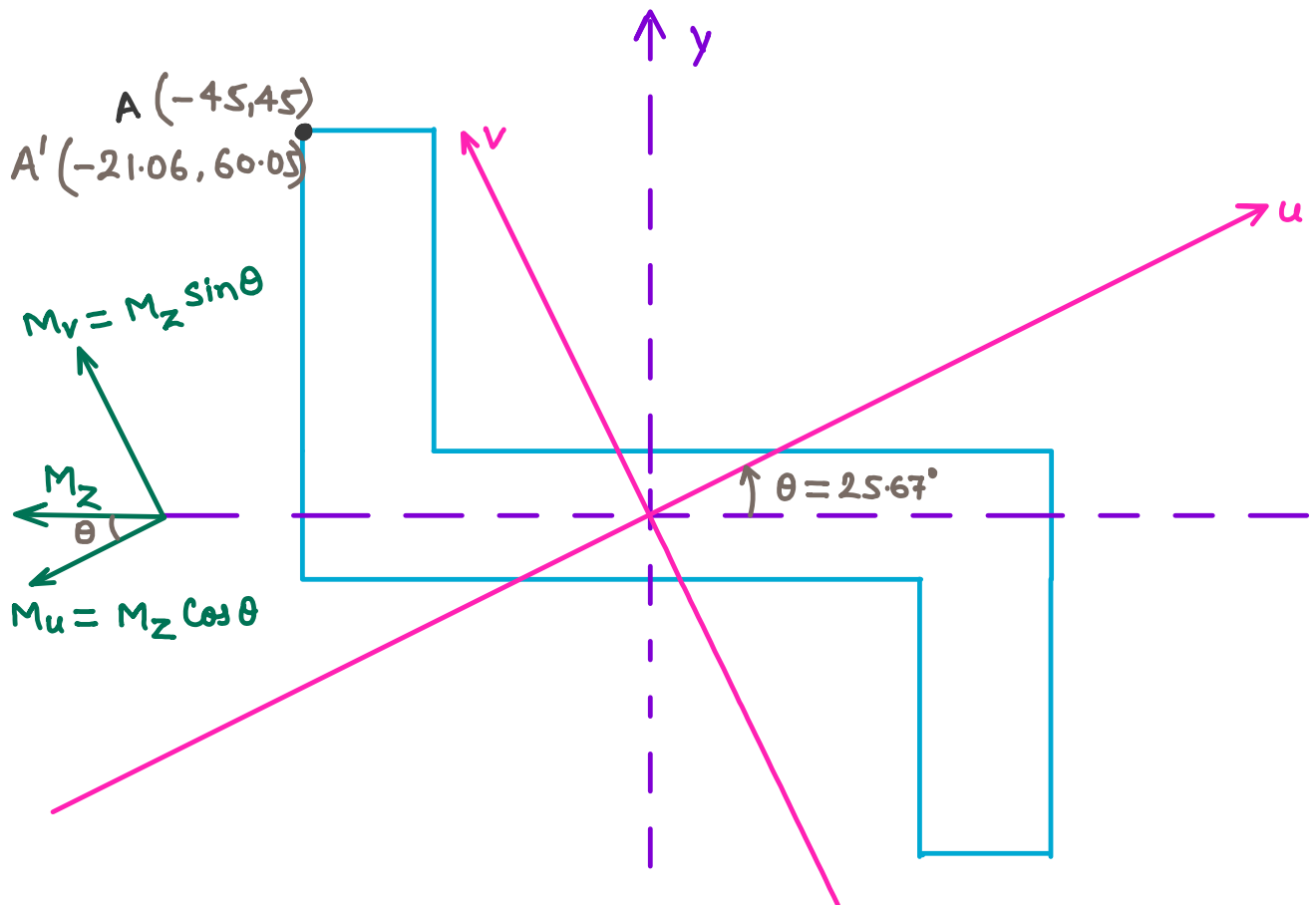
$$= 0.229 \times 10^6 \text{ mm}^4$$

$$\tan 2\theta = \frac{-2 I_{zy}}{I_{zz} - I_{yy}} = \frac{-2 \times 0.8 \times 10^6}{(0.614 - 1.894) \times 10^6} = \frac{5}{4}$$

$$\theta = 25.67^\circ$$

$$u' = z \cos \theta + y \sin \theta = -(35+10) \cos 25.67^\circ + (5+40) \sin 25.67^\circ = -21.06$$

$$v' = -z \sin \theta + y \cos \theta = +45 \sin 25.67^\circ + 45 \cos 25.67^\circ = 60.05$$



$$\begin{aligned} \sigma &= \frac{M_u v'}{I_{uu}} + \frac{M_v u'}{I_{vv}} \\ &= \frac{-1.2 \cos 25.67^\circ \times 60.05}{2.278 \times 10^6} + \frac{1.2 \sin 25.67^\circ (-21.06)}{0.229 \times 10^6} \\ &= -76.31 \text{ N/mm}^2 \text{ (comp)} \end{aligned}$$

